

ON THE STRUCTURE OF OPEN EQUIVARIANT TOPOLOGICAL CONFORMAL FIELD THEORIES

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Abstract

A classification of open equivariant topological conformal field theories in terms of Calabi-Yau A_∞ -categories endowed with a group action is presented.

1. Introduction

One can think of 2-dimensional topological conformal field theories (TCFTs for short) as an extension of topological quantum field theories in dimension 2 (henceforth TQFTs), since one works not with surfaces but with Riemann surfaces.

The first studies concerning TCFTs were made by Moore and Segal, in [11], [12], who first gave a precise definition for TCFTs and suggested the importance of their study.

For a finite set, whose elements are called D -branes, let us consider \mathbf{OC}_Λ , the category whose class of objects are 1-manifolds (disjoint unions of circles and intervals) with boundary labeled by D -branes and with the class of morphisms given by singular chains on moduli spaces of Riemann surfaces. Given a field \mathbb{K} of characteristic zero, it makes sense to consider differential graded symmetric monoidal functors of the form $\mathfrak{F}: \mathbf{OC}_\Lambda \rightarrow \mathbf{Comp}_{\mathbb{K}}$, where $\mathbf{Comp}_{\mathbb{K}}$ is the category of chain complexes over \mathbb{K} . Such a functor \mathfrak{F} , satisfying certain conditions, is called a 2-dimensional TCFT. Depending on the boundary components of the Riemann surfaces one works with one gets open, closed, or open-closed TCFTs.

Relying on a ribbon graph decomposition of the moduli space of marked Riemann surfaces, open TCFTs were classified by Costello [4] in terms of A_∞ -categories satisfying a Calabi-Yau condition. Costello also gave a universal extension from open TCFTs to open-closed TCFTs and proved that the homology associated with the closed part of an open-closed TCFT is described in terms of the Hochschild homology of the Calabi-Yau A_∞ -category associated to its open part.

Inspired by Costello and Braun [1], the author developed a full generalization of Costello's results in the unoriented setting replacing Riemann surfaces with Klein surfaces [5].

For a discrete group G and moduli spaces of geometric structures over Riemann surfaces, one can talk about equivariant TCFTs. As before, depending on the boundary components of our Riemann surfaces, one can talk about open, closed, or open-closed equivariant TCFTs. By extending the techniques developed in [5] to the equivariant setting, the research developed here studies the classification of open equivariant TCFTs. The main results are:

THEOREM 1.1. *For a discrete group G , there is a homotopy equivalence (see [5, §1.7] for a definition) between open equivariant TCFTs and Calabi-Yau A_∞ -categories endowed with group action.*

A unital A_∞ -category is a category endowed with an identity element $1 \in \text{Hom}$ satisfying certain conditions that can be found in §3.3 in this paper. The following result holds:

THEOREM 1.2. *There exists a homotopy equivalence between the following categories:*

- (1) *unital extended equivariant A_∞ -categories;*
- (2) *unital equivariant A_∞ -categories; and*
- (3) *unital equivariant DG categories.*

2. Moduli spaces of geometric structures

Moduli spaces of Riemann surfaces equipped with a principal G -bundle are important in the present classification. This section, whose main reference is [6], introduces the main definitions and results related to moduli spaces of geometric structures on surfaces. Open equivariant TCFTs and the categories governing such theories together with its description in terms of generators and relations are also introduced. This description will lead to the equivalence between open equivariant theories and Calabi-Yau categories.

2.1. Strict 2-categories and functors

A *strict 2-category* is a category \mathbf{A} enriched over \mathbf{Cat} , the category of categories and functors, such that:

- (1) There is a class of objects $\text{Obj}(\mathbf{A})$;
- (2) for objects $a_1, a_2 \in \text{Obj}(\mathbf{A})$, a category $\text{Hom}_{\mathbf{A}}(a_1, a_2)$, whose objects are called 1-morphisms and whose morphisms are called 2-morphisms. They are assumed to be isomorphisms.

- (3) composition functors $\mathrm{Hom}_{\mathbf{A}}(a_1, a_2) \times \mathrm{Hom}_{\mathbf{A}}(a_2, a_3) \rightarrow \mathrm{Hom}_{\mathbf{A}}(a_1, a_3)$ are strictly associative and have a unit in $\mathrm{Hom}_{\mathbf{A}}(a, a)$.

REMARK 2.1. Let \mathbf{Cat} be the strict 2-category whose objects are small categories and whose Hom categories are the categories of functors and natural transformations.

For 2-categories \mathbf{A} and \mathbf{B} , a 2-functor $\mathfrak{F}: \mathbf{A} \rightarrow \mathbf{B}$ is a map $F: \mathrm{Obj}(\mathbf{A}) \rightarrow \mathrm{Obj}(\mathbf{B})$ together with a functor $\mathrm{Hom}_{\mathbf{A}}(a_1, a_2) \rightarrow \mathrm{Hom}_{\mathbf{B}}(F(a_1), F(a_2))$ such that it is strictly compatible with the composition functors.

2.2. Surfaces with collars

Let Σ be a surface with boundary partitioned into intervals and locally isomorphic to the Cartesian product $[0, +\infty) \times [0, +\infty)$. For a boundary interval $J \subset \Sigma$, a *collar* of J is a smooth embedding ϖ of $(-1, 0] \times [0, 1]$ into Σ such that

- (1) it preserves boundaries, and
- (2) is a diffeomorphism of $\{0\} \times [0, 1]$ onto J .

A surface with a finite set of disjoint boundary intervals equipped with disjoint collars, labeled by a set I , is called *I-collared*. A diffeomorphism of *I-collared* surfaces $\Sigma_1 \rightarrow \Sigma_2$ is a diffeomorphism of the underlying surfaces such that it respects the labeling and the parametrization of the collars.

Given a surface Σ with disjoint boundary intervals J_1, J_2 equipped with disjoint collars ϖ_1, ϖ_2 respectively, two boundary intervals can be glued together and obtain a new smooth surface by considering $\Sigma' = \Sigma - (J_1 \cup J_2) / \sim$, where \sim is the identification of $\varpi_1(x)$ with $\varpi_2(x)$ for each $x \in (-1, 0] \times [0, 1]$.

Collared surfaces and open embeddings form a category which will be denoted by \mathbf{Surf} . Let $\mathbf{Disc} \subset \mathbf{Surf}$ denote the full subcategory whose objects are disjoint unions of discs each having at least one collared boundary interval. Let $\widetilde{\mathbf{Surf}}$ be the category, enriched over \mathbf{Top} , of finite type surfaces (possibly with boundary and locally modeled on $[0, +\infty) \times [0, +\infty)$) and open embeddings.

2.3. Sheaves of geometric structures

A *smooth sheaf* ζ on $\widetilde{\mathbf{Surf}}$ is an enriched functor $\widetilde{\mathbf{Surf}} \rightarrow \mathbf{Top}$ sending pushout squares to homotopy pullback squares.

Let $\zeta(\Sigma)$ be the space of geometric structures of a given type on an *I-collared* surface Σ . Some examples of such spaces are:

- (1) the set of orientations on Σ ;
- (2) the space of principal G -bundles over Σ .

A smooth sheaf ζ is *connected* if $\zeta((-1, 0) \times I)$ is connected. If ζ is connected, then one can choose a base-point $p \in \zeta((-1, 0) \times I)$. Let $J \subset \partial\Sigma$ be a boundary interval equipped with a collar ϖ . A section $\sigma \in \zeta(\Sigma)$ is trivial at J if the pullback $\sigma \circ \varpi$ restricts to the chosen base-point p on $(-1, 0) \times I$. For a surface Σ endowed with a collection of disjoint collared boundary intervals $\{J_1, \dots, J_n\}$, let $\tilde{\zeta}(\Sigma) \subset \zeta(\Sigma)$ be the subspace consisting of section that are trivial at J_i for each $i \in \{1, \dots, n\}$.

Let A_ζ be the space $\tilde{\zeta}(I \times I)$ of sections that are trivial at each side of the square $I \times I$, which are assumed to be equipped with a collar at each of the intervals $\{0\} \times I$ and $\{1\} \times I$.

PROPOSITION 2.2 ([6, Proposition 4.3.1]). *Gluing squares side-to-side endows A_ζ with an A_∞ -composition making it into a group-like A_∞ -monoid; the homotopy inverse map is induced by rotating the square π . Fixing a collared boundary interval J on a surface Σ , there is a right A_∞ -action of A_ζ on $\tilde{\zeta}(\Sigma)$ given by gluing the right-hand side of a square to J , and a left A_∞ -action given by gluing the left edge of the square to J .*

PROPOSITION 2.3 ([6, Proposition 4.3.2]). *Let J_1 and J_2 be two disjointly collared boundary intervals on a surface Σ , and let Σ' be the result of gluing J_1 to J_2 . There is a homotopy equivalence $\zeta(\Sigma') \simeq \tilde{\zeta}(\Sigma)_{hA_\zeta}$, where the action of A_ζ is given, for a square with ζ -structure $K \in A_\zeta$, by gluing the left edge of one copy of K to J_1 and gluing the left edge of a second copy of K to J_2 .*

2.4. The modular operad of moduli spaces

Cyclic operads were introduced by Getzler, Kapranov and Kontsevich in order to handle algebras endowed with inner products $\langle -, - \rangle$ satisfying $\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle$ in the associative case and $\langle [a, b], c \rangle = \langle a, [b, c] \rangle$ in the Lie case [10, §1.14]. A *cyclic operad* in a symmetric monoidal category \mathbf{C} is given by a functor \mathfrak{P} from the category of non-empty sets and bijections to \mathbf{C} , and for each pair of finite sets I, J , with elements $i \in I$ and $j \in J$, a composition morphism

$$\circ_{i,j}: \mathfrak{P}(I) \otimes \mathfrak{P}(J) \rightarrow \mathfrak{P}(I \sqcup J - \{i, j\})$$

natural in I and J and satisfying the following axioms.

- (1) Given finite sets I, J, K and elements $i \in I, j_1, j_2 \in J$ and $k \in K$, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{P}(I) \otimes \mathfrak{P}(J) \otimes \mathfrak{P}(K) & \xrightarrow{\text{Id} \otimes \circ_{j_2, k}} & \mathfrak{P}(I) \otimes \mathfrak{P}(J \sqcup K - \{j_2, k\}) \\ \circ_{i, j_1} \otimes \text{Id} \downarrow & & \downarrow \circ_{i, j_1} \otimes \text{Id} \\ \mathfrak{P}(I \sqcup J - \{i, j_1\}) \otimes \mathfrak{P}(K) & \xrightarrow{\circ_{j_2, k}} & \mathfrak{P}(I \sqcup J \sqcup K - \{i, j_1, j_2, k\}). \end{array}$$

- (2) Given finite sets I, J and elements $i \in I$ and $j \in J$ the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{K}(I) \otimes \mathfrak{K}(J) & \xrightarrow{\circ_{i,j}} & \mathfrak{K}(I \sqcup J - \{i, j\}) \\ \text{(!)} \downarrow & & \downarrow \mathfrak{K}(\tau) \\ \mathfrak{K}(J) \otimes \mathfrak{K}(I) & \xrightarrow{\circ_{j,i}} & \mathfrak{K}(J \sqcup I - \{j, i\}), \end{array}$$

where τ denotes the bijection $I \sqcup J - \{i, j\} \cong J \sqcup I - \{j, i\}$ and (!) denotes a swapping.

- (3) For each set $A = \{a, b\}$ of cardinality 2 there is a morphism $u_A: 1_{\mathbf{C}} \rightarrow \mathfrak{K}(A)$, where $1_{\mathbf{C}}$ denotes the monoidal unit, that is natural in A and such that for any finite set and an element $i \in I$ the following diagram commutes:

$$\begin{array}{ccc} 1_{\mathbf{C}} \otimes \mathfrak{K}(I) & \xrightarrow{u_A \otimes \text{Id}} & \mathfrak{K}(A) \otimes \mathfrak{K}(I) \\ & \searrow \mathfrak{K}(\beta) & \downarrow \circ_{a,i} \\ & & \mathfrak{K}(A \sqcup I - \{a, i\}), \end{array}$$

where β is the canonical bijection $I \cong A \sqcup I - \{a, i\}$ sending i to b .

A *modular operad* in \mathbf{C} is a cyclic operad \mathfrak{K} together with natural self-composition maps $\circ_{i,j}: \mathfrak{K}(I) \rightarrow \mathfrak{K}(I - \{i, j\})$ that commute with the cyclic operad composition maps and with each other.

EXAMPLE 2.4. An important modular operad is the Deligne-Mumford moduli spaces of stable curves of genus g with $n + 1$ marked points, where the composition maps are defined by intersecting along with their marked points.

Given a cyclic operad \mathfrak{D} , its *modular envelope* is the modular operad freely generated by \mathfrak{D} .

A (finite) graph γ is given by a (finite) set of vertices $V(\gamma)$, a (finite) set of half-edges $H(\gamma)$, a map $\lambda: H(\gamma) \rightarrow V(\gamma)$ and an involution $\iota: H(\gamma) \rightarrow H(\gamma)$. For a graph γ , one says that two half-edges a, b form an *edge* if $\iota(a) = b$; a half-edge a is connected to a vertex v if $\lambda(a) = v$. A *leg* in γ is a univalent vertex. A *corolla* is a graph consisting of a single vertex with several legs connected to it.

The topological realization of a graph γ is a 1-dimensional CW complex. A tree is a graph whose topological realization is contractible; a forest is a union of trees.

Let **Graph** be the symmetric monoidal category whose objects are disjoint unions of corollas and where the arrows are given by graphs; one may think

of such a morphism as assembling a set of corollas to create a graph, followed by contracting all the edges so the result is again a set of corollas. There is a subcategory **Forest** \subset **Graph** whose objects are those of **Graph** with no isolated vertices and whose arrows are given by forests.

2Disc and **2Surf** are the strict 2-categories with the same objects as **Disc** and **Surf**, respectively, and with space of diffeomorphisms given by the groupoid of diffeomorphisms and isotopy classes of isotopies. Let $S: \mathbf{Graph} \rightarrow (\mathbf{Cat} \downarrow \mathbf{2Surf})$ be the strict 2-functor which sends a corolla τ to the strict 2-category of collared surfaces with components identified with τ and collared boundary intervals compatibly identified with the legs of τ . The functor to **2Surf** is given by forgetting the extra identification data. One writes $D: \mathbf{Forest} \rightarrow (\mathbf{Cat} \downarrow \mathbf{2Disc})$. These two functors define cyclic and modular operads (respectively). For a corolla τ , $S(\tau)$ consists of a 2-category \mathbf{C} and a functor $\mathfrak{F}: \mathbf{C} \rightarrow \mathbf{Surf}$.

Let \mathfrak{M}_ζ be the moduli space modular operad associated to a connected smooth sheaf ζ by sending an object $\tau \in \text{Obj}(\mathbf{Graph})$ to the homotopy colimit (see [6] for a definition of the concept)

$$\mathfrak{M}_\zeta(\tau) := \text{hocolim}_{S(\tau)} \tilde{\zeta}.$$

The cyclic operad \mathfrak{D}_ζ of the moduli space of discs is defined as

$$\mathfrak{D}_\zeta(\tau) := \text{hocolim}_{D(\tau)} \tilde{\zeta}.$$

The following result is the analogue to Proposition 3.3.4 in [2]:

THEOREM 2.5 ([6, Theorem 4.5.2]). *The derived modular envelope of the cyclic operad \mathfrak{D}_ζ is weakly homotopy equivalent as a modular operad to \mathfrak{M}_ζ .*

3. Open equivariant TCFTs

This section contains the equivalence between open equivariant TCFTs and Calabi-Yau A_∞ -categories endowed with group action. The reasoning relies on the description of the categories governing these theories and follow the steps taken in [4] and [5]. Henceforth \mathbb{K} will be a field of characteristic 0.

3.1. Moduli space of surfaces with principal G -bundle

Let Λ be a set of D -branes and G a discrete group. Let \mathbf{M}_Λ^G be the topological category where:

- (1) The objects are triples of the form $\alpha := ([O], s, t)$ for $O \in \mathbb{N}$ and maps $s, t: [O] \rightarrow \Lambda$. One writes $[O]$ for $\{0, \dots, O-1\}$.
- (2) The space of morphisms $\mathbf{M}_\Lambda^G(\alpha, \beta)$ is given by moduli spaces \mathbf{M}_Λ^G (observe the abuse of notation) of collared Riemann surfaces Σ equipped

with a principal G -bundle with a trivialization of the G -bundle over each boundary component. The surfaces Σ have open boundary components given by disjoint parameterized intervals, embedded in the boundary and labeled by $[O]$, and free boundary components, which are intervals in $\partial\Sigma$ labeled by D -branes in a way compatible with the labeling $\{s(i), t(i)\}$.

Let T be the total space of the trivial bundle over an interval and consider $\alpha := ([O_\alpha], s, t)$ and $\beta := ([O_\beta], s', t')$; a morphism in $\mathbf{M}_\Lambda^G(\alpha, \beta)$ is given by the moduli space \mathbf{M}_Λ^G of Riemann surfaces Σ connecting intervals, together with a principal G -bundle $\pi: P \rightarrow \Sigma$, such that $P|_{\text{incoming boundary}} = \bigsqcup_i T_i$ and $P|_{\text{outgoing boundary}} = \bigsqcup_j T_j$, for $i \in \{0, \dots, O_\alpha - 1\}$ and $j \in \{0, \dots, O_\beta - 1\}$. Open intervals have associated an ordered pair $\{s(i), t(i)\}$ (for $0 \leq i \leq O - 1$), indicating where the interval begins and where it ends.

The composition of morphisms is given by gluing surfaces and the associated bundles; the trivializations are required when gluing two G -bundles. One glues incoming open (resp. closed) boundary components with outgoing open boundary components. Open boundary components can only be glued together if their D -brane labeling agree. Disjoint union makes \mathbf{M}_Λ^G into a symmetric monoidal category.

The positive boundary condition is required: Riemann surfaces are required to have at least one incoming closed boundary component on each connected component.

REMARK 3.1. Although surfaces in $\mathbf{M}_\Lambda^G(\alpha, \beta)$ are required to be stable, the following exceptional surfaces are allowed: the disc and the annulus with no open or closed boundary components and only free boundary components. These surfaces are unstable therefore their associated moduli space is defined to be a point.

Let $\mathfrak{C}: \mathbf{Top} \rightarrow \mathbf{Comp}_\mathbb{K}$ be the functor of singular chains. Applying the functor \mathfrak{C} to \mathbf{M}_Λ^G yields a differential graded symmetric monoidal category $\mathbf{O}_\Lambda^G = \mathfrak{C}(\mathbf{M}_\Lambda^G)$ whose objects are triples $\alpha = ([O], s, t)$ and where $\text{Hom}_{\mathbf{O}_\Lambda^G}(\alpha, \beta) := \mathfrak{C}(\text{Hom}_{\mathbf{M}_\Lambda^G}(\alpha, \beta))$.

REMARK 3.2. Henceforth, all the categories will be differential graded symmetric monoidal categories (DGSM for short), and all the functors will be assumed to be differential graded functors.

Given DGSM categories (\mathbf{A}, \sqcup) and (\mathbf{B}, \otimes) , a symmetric monoidal functor $\mathfrak{F}: (\mathbf{A}, \sqcup) \rightarrow (\mathbf{B}, \otimes)$ satisfying $\mathfrak{F}(a \sqcup b) \cong \mathfrak{F}(a) \otimes \mathfrak{F}(b)$ for objects $a, b \in \text{Obj}(\mathbf{A})$ is called *split*. If one has quasi-isomorphisms instead (see [5, §1.7] for a definition), then \mathfrak{F} is called an *h-split* functor.

DEFINITION 3.3. An open equivariant topological conformal field theory (henceforth an open ETCFT) is a pair (Λ, \mathfrak{F}) where Λ is finite set of D -branes and \mathfrak{F} is an h-split symmetric monoidal functor of the form $\mathfrak{F}: \mathbf{O}_\Lambda^G \rightarrow \mathbf{Comp}_{\mathbb{K}}$.

3.2. Homotopical approach to moduli spaces of surfaces with principal G -bundle

Vector bundles over contractible spaces are always trivial. This fact leads, if one considers $\zeta(\Sigma)$ as the space of principal G -bundles over a Riemann surface Σ , to the following weak homotopy equivalence of cyclic operads: $D_{g,n,I} \simeq \mathfrak{D}_\zeta$, where $D_{g,n,I}$ is the cyclic operad defined in [2] of Riemann surfaces with boundary, with marked points and possibly nodes on the boundary whose irreducible components are discs. The homotopy equivalence follows from [2, Theorem 1.0.1]. By considering their modular envelopes one obtains

$$\mathrm{Mod}(D_{g,n,I}) \simeq \mathrm{Mod}(\mathfrak{D}_\zeta).$$

Following [9], one can think of an operad \mathfrak{A} as a category with objects given by \mathbb{N} and space of morphisms of the form $\mathfrak{A}(n, 1)$, for $n \in \mathbb{N}$. Therefore, \mathfrak{M}_ζ is equivalent to \mathbf{M}_Λ^G , when one thinks of the space $\zeta(\Sigma)$ as the space of principal G -bundles over a Riemann surface Σ . Theorem 2.5 applies now in order to conclude

$$\mathfrak{M}_\zeta \simeq \mathrm{Mod}(\mathfrak{D}_\zeta) \simeq \mathrm{Mod}(D_{g,n,I}).$$

Let $\overline{\mathbf{N}}_\Lambda^G(\alpha, \beta)$ be the moduli space of stable Riemann surfaces with principal G -bundle: its elements are stable collared Riemann surfaces with O_α incoming boundary components equipped with a principal G -bundle and a trivialization of the G -bundle over each component. Surfaces have O_β outgoing boundary components. There are open marked points labeled by $[O_\alpha]$ and $[O_\beta]$ which are distributed all along with the boundary components of the surfaces and replace the intervals with the trivialization. Surfaces in $\overline{\mathbf{N}}_\Lambda^G(\alpha, \beta)$ have free boundary components, which are the intervals between open marked points and those components with no marked points on them; free boundary components must be labeled by D -branes (in Λ) in a way compatible with the maps $s, t: [O] \rightarrow \Lambda$. Surfaces in $\overline{\mathbf{N}}_\Lambda^G(\alpha, \beta)$ may have nodes and marked points, but only boundary nodes and marked points on the boundary are allowed.

Although surfaces in $\overline{\mathbf{N}}_\Lambda^G(\alpha, \beta)$ are asked to be stable, the following exceptional surfaces are allowed: the disc with zero, one or two open marked points, and the annulus with no open or closed points. Let $\mathbf{N}_\Lambda^G(\alpha, \beta) \subset \overline{\mathbf{N}}_\Lambda^G(\alpha, \beta)$ be the subspace of non-singular Riemann surfaces with principal G -bundle.

These surfaces are unstable therefore their associated moduli space is defined to be a point.

Since homotopy equivalences of topological spaces induce isomorphisms between the sets of isomorphism classes of principal G -bundles over those spaces, the arguments used in [5, §5] apply in the equivariant setting to conclude:

PROPOSITION 3.4 (cf. [5, Proposition 5.3]). *The DGSM category $\mathfrak{C}(\overline{\mathbf{N}}_\Lambda^G)$ is quasi-isomorphic to \mathbf{O}_Λ^G .*

Let $\mathbf{D}_\Lambda^G(\alpha, \beta) \subset \overline{\mathbf{N}}_\Lambda^G(\alpha, \beta)$ be the subspace consisting of Riemann surfaces whose irreducible components are all discs.

As one deals with not-necessarily connected graphs, it is required to take the modular envelope of $D_{g,n,I}$ and to remark that it is equivalent to \mathbf{D}_Λ^G . Furthermore, the work done in [2], [3] applies in this setting to provide $\overline{\mathbf{N}}_\Lambda^G$ with an orbispace structure (in the sense of Costello, see [4]), whereas \mathbf{D}_Λ^G is endowed with such structure using the same arguments used in [5, §5]. All together this leads to:

PROPOSITION 3.5. *There is a homotopy equivalence of orbispace $\overline{\mathbf{N}}_\Lambda^G(\alpha, \beta) \cong \mathbf{D}_\Lambda^G(\alpha, \beta)$.*

PROOF. This result follows from a concatenation of homotopy equivalences:

$$\mathbf{D}_\Lambda^G \simeq \text{Mod}(D_{g,n,I}) \simeq \mathfrak{M}_\zeta \simeq \mathbf{M}_\Lambda^G \simeq \overline{\mathbf{N}}_\Lambda^G.$$

Let $\mathfrak{C}^{\text{cell}}$ be a functor taking finite cell complexes $\mathbf{Comp}_\mathbb{K}$ [4, Appendix A]. Applying $\mathfrak{C}^{\text{cell}}$ to $\mathbf{D}_\Lambda^G(\alpha, \beta)$ one gets the following category and abuse of notation

$$\mathbf{D}_\Lambda^G(\alpha, \beta) := \mathfrak{C}^{\text{cell}}(\mathbf{D}_\Lambda^G(\alpha, \beta)),$$

which, due to the quasi-isomorphism $\mathfrak{C}^{\text{cell}}(X) \rightarrow \mathfrak{C}(X)$ (where X is an orbicell complex, see [4, Appendix A]), leads to the following result, the equivariant analogue of [4, Lemma 6.1.7]:

PROPOSITION 3.6. *There is a quasi-isomorphism of DGSM categories $\mathbf{D}_\Lambda^G \cong \mathbf{O}_\Lambda^G$.*

3.3. Calabi-Yau equivariant A_∞ -categories

REMARK 3.7. Henceforth it will be assumed that the field \mathbb{K} has characteristic zero and that, for a discrete group G , the field is equipped with the identity map as group action: $G \times \mathbb{K} \xrightarrow{\text{Id}} \mathbb{K}$ with $(g, k) \mapsto k$.

Let \mathbf{C} be a monoidal \mathbb{K} -category. An automorphism of \mathbf{C} is an invertible on the morphisms \mathbb{K} -linear functor $\mathfrak{F}: \mathbf{C} \rightarrow \mathbf{C}$ such that

- (1) $\mathfrak{F}(1) = 1$,
- (2) $\mathfrak{F}(c_1 \otimes c_2) = \mathfrak{F}(c_1) \otimes \mathfrak{F}(c_2)$,
- (3) $\mathfrak{F}(f \otimes g) = \mathfrak{F}(f) \otimes \mathfrak{F}(g)$ for arrows f, g in \mathbf{C} ,
- (4) there is compatibility with the morphisms defining the monoidal structure of \mathbf{C} .

The group of automorphisms of \mathbf{C} will be denoted by $\text{Aut}(\mathbf{C})$. The elements of $\text{Aut}(\mathbf{C})$ are required to be the identity on the objects.

Let \mathbf{C} be a differential graded monoidal \mathbb{K} -category equipped with a morphism $G \xrightarrow{\varphi} \text{Aut}(\mathbf{C})$ pairing an element $\alpha \in G$ with an automorphism of \mathbf{C} , $\varphi_\alpha: \mathbf{C} \rightarrow \mathbf{C}$. For a given morphism $f: c_1 \rightarrow c_2$ in $\text{Hom}_{\mathbf{C}}$, one defines the map $\varphi_\alpha(f): \varphi_\alpha(c_1) \rightarrow \varphi_\alpha(c_2)$. Such a category will be called *equivariant category*. A functor $\mathfrak{E}: (\mathbf{C}_1, \varphi) \rightarrow (\mathbf{C}_2, \varphi')$ between equivariant categories is a functor of the underlying DG categories such that $\varphi'_\alpha \circ \mathfrak{E} = \mathfrak{E} \circ \varphi_\alpha$ for each $\alpha \in G$.

LEMMA 3.8 ([13, p. 7]). *For maps $f: c_1 \rightarrow c_2$, $g: c_2 \rightarrow c_3$ in $\text{Hom}_{\mathbf{C}}$ and $\alpha, \beta \in G$, the following identities hold:*

- (1) $\varphi_\alpha(f \circ g) = \varphi_\alpha(f) \circ \varphi_\alpha(g)$;
- (2) $\varphi_\alpha(f \otimes g) = \varphi_\alpha(f) \otimes \varphi_\alpha(g)$;
- (3) $\varphi_\alpha(\text{Id}_{c_1}) = \text{Id}_{\varphi_\alpha(c_1)}$;
- (4) $\varphi_{\alpha\beta}(f) = \varphi_\alpha(\varphi_\beta(f))$;
- (5) $\varphi_1(f) = \varphi_\alpha(\varphi_{\alpha^{-1}}(f)) = \varphi_{\alpha^{-1}}(\varphi_\alpha(f)) = f$.

An *equivariant A_∞ -category* is an equivariant category \mathbf{C} where, for $c_1, c_2 \in \text{Obj}(\mathbf{C})$, the space $\text{Hom}_{\mathbf{C}}(c_1, c_2)$ is a finite-dimensional complex of \mathbb{K} -modules. Further, for each sequence of objects $c_0, \dots, c_n \in \text{Obj}(\mathbf{C})$ with $n \geq 2$, there are cyclically symmetric maps

$$m_n: \text{Hom}_{\mathbf{C}}(c_0, c_1) \otimes \cdots \otimes \text{Hom}_{\mathbf{C}}(c_{n-1}, c_n) \rightarrow \text{Hom}_{\mathbf{C}}(c_0, c_n)$$

of degree $n - 2$ satisfying the usual conditions one finds in [4, §7.1] and the following equality:

$$\varphi_\alpha(m_n(f_0 \otimes \cdots \otimes f_{n-1})) = m_n(\varphi_\alpha(f_0) \otimes \cdots \otimes \varphi_\alpha(f_{n-1})).$$

If for each $c \in \text{Obj}(\mathbf{C})$ there exists an element $1_c \in \text{Hom}_{\mathbf{C}}(c, c)$ of degree zero such that

- (1) $b_2(f \otimes 1_c) = f$ and $b_2(1_c \otimes g) = g$ for $f \in \text{Hom}_{\mathbf{C}}(c', c)$ and $g \in \text{Hom}_{\mathbf{C}}(c, c')$,

(2) for $0 \leq i \leq n$, if $f_i \in \text{Hom}_{\mathbf{C}}(c_i, c_{i+1})$ and $j = j + 1$, then

$$b_n(f_0 \otimes f_1 \otimes \cdots \otimes 1_{c_j} \otimes \cdots \otimes f_{n-1}) = 0,$$

one says that the A_∞ -category \mathbf{C} is *unital*.

A functor $\mathfrak{C}: (\mathbf{C}_1, \varphi) \rightarrow (\mathbf{C}_2, \varphi')$ between equivariant A_∞ -categories is a functor of the underlying A_∞ -categories (see [8, §5.1.2]) such that $\varphi'_\alpha \circ \mathfrak{C} = \mathfrak{C} \circ \varphi_\alpha$ for each $\alpha \in G$. Equivariant A_∞ -categories and functors between them form a category.

A *Calabi-Yau equivariant A_∞ -category* is an equivariant A_∞ -category \mathbf{C} endowed with a trace map $\text{Tr}: \text{Hom}_{\mathbf{C}}(c_1, c_1) \rightarrow \mathbb{K}$ compatible with the group action and a symmetric and non-degenerate pairing $\langle -, - \rangle_{c_1, c_2}$ for objects $c_1, c_2 \in \text{Obj}(\mathbf{C})$:

$$\begin{aligned} \langle -, - \rangle_{c_1, c_2}: \text{Hom}_{\mathbf{C}}(c_1, c_2) \otimes \text{Hom}_{\mathbf{C}}(c_2, c_1) &\longrightarrow \mathbb{K}, \\ f \otimes g &\longmapsto \text{Tr}(g \circ f), \end{aligned}$$

which is required to be symmetric, non-degenerate and must satisfy

- (1) $\langle m_{n-1}(c_0 \otimes \cdots \otimes c_{n-2}), c_{n-1} \rangle = (-1)^{(n+1)+|c_0| \sum_{i=1}^{n-1} |c_i|} \langle m_{n-1}(c_1 \otimes \cdots \otimes c_{n-1}), c_0 \rangle$, and
- (2) $\langle \varphi_\alpha(f), \varphi_\alpha(g) \rangle_{\varphi_\alpha(c_1), \varphi_\alpha(c_2)} = \langle f, g \rangle_{c_1, c_2}$.

3.4. Generators and relations

The techniques in [4] apply in the equivariant setting as it is not required to include any further generator, as required in [5].

REMARK 3.9. The disc with one incoming and one outgoing boundary component, which is equipped with a principal G -bundle and a trivialization being the identity on one of the boundary components and the action by $g \in G$ on the other, will play the role of the group action when describing Calabi-Yau equivariant A_∞ -categories.

Let $\mathbf{D}_{\Lambda, G}^+ \subset \mathbf{D}_{\Lambda}^G$ be the subcategory with the same objects, but where a morphism is given by a disjoint union of discs endowed with a G -bundle and a trivialization on the boundaries, with each connected component having exactly one outgoing boundary marked point. For an ordered set of D -branes, with $n \geq 1$, let $[\lambda_n] := \{\lambda_0, \dots, \lambda_{n-1}\}$ be the object in $\text{Obj}(\mathbf{O}_{\Lambda}^G)$ with $O = n$, $s(i) = \lambda_i$, $t(i) = \lambda_{i+1}$ for $0 \leq i \leq O - 1$; one writes $[\lambda_n]^c := \{\lambda_1, \dots, \lambda_{n-1}, \lambda_0\}$. Let $(D^+(\lambda_0, \dots, \lambda_{n-1}), \varphi)$ be a disc in $\mathbf{D}_{\Lambda, G}^+$ with n marked points, with D -brane labeling given by the different λ_i and a principal G -bundle with a trivialization $\varphi = \{\varphi_i\}_{i \in \{0, \dots, n-1\}}$ on each marked point. All the boundary marked points are incoming except for that between

λ_{n-1} and λ_0 , which is outgoing. The boundary components of the discs are compatibly oriented.

Let $\mathbf{C} \subset \mathbf{D}_\Lambda^G$ be the subcategory with $\text{Obj}(\mathbf{C}) = \text{Obj}(\mathbf{D}_\Lambda^G)$, but whose arrows are not allowed to have connected components which are the disc with at most one open marked point, or the disc with two open marked incoming points, or the annulus with neither open nor closed marked points. The morphisms in \mathbf{C} are assumed to be not complexes but graded vector spaces.

Let $\xi = (D \times G, \pi, D)$ be a principal G -bundle over a disc D . It is known that the automorphisms $\text{Aut}(D \times G)$ are defined by an element $g \in G$:

THEOREM 3.10 ([7, Chapter 5, Theorem 1.1]). *Let us consider a principal G -bundle of the form $\xi = (B \times G, \pi, B)$. Then the automorphisms $\xi \rightarrow \xi$ over B are in bijection with maps of the form $B \rightarrow G$. Particularly, an automorphism $\xi \rightarrow \xi$ has the form $h_g(b, s) = (b, g(b)s)$ for a certain map $g: B \rightarrow G$.*

Given two trivializations of ξ , namely φ_1 and φ_2 , we have

$$\varphi_2(b, s) = h_g(\pi(b, s))\varphi_1(b, s),$$

therefore any two trivializations over D differ from an element of the group G . A trivialization ψ is defined by an element of the group G .

In the equivariant setting one is interested in pairs of the form (ξ, φ) where ξ is a principal G -bundle over a disc and φ is a trivialization over the boundary, what means that φ is given by a collection $\{\varphi_1, \dots, \varphi_n\}$ with one trivialization φ_i for each boundary component of the given disc. Observe that, as each trivialization φ is given by an element of the group G , such a trivialization is determined by an n -tuple $\{g_1, \dots, g_n\}$ with $g_i \in G$ for $i \in \{1, \dots, n\}$. Two pairs (ξ, φ) and (ξ', φ') are isomorphic if there is an element $h \in G$ such that $(g'_1, \dots, g'_n) = (hg_1, \dots, hg_n)$, where $\{g'_1, \dots, g'_n\}$ is the n -tuple associated to φ' . One denotes by $\overline{(\xi, \varphi)}$ the isomorphism class of (ξ, φ) .

A G -twisted disc is a pair $(D^g(\lambda_0, \lambda_1), \varphi)$ given by a disc $D(\lambda_0, \lambda_1)$ with two marked points endowed with a principal G -bundle ξ and trivializations $\varphi = \{\varphi_1, g\varphi_1\}$ for $g \neq e$, the neutral element of G . A disc $(D(\lambda_0, \dots, \lambda_n), \varphi)$ is *untwisted* if all the trivializations φ_i agree (for $i \in \{0, \dots, n\}$), that is, if they differ by the identity $e \in G$.

PROPOSITION 3.11 (cf. [4, Proposition 6.2.1]). *Let $\overline{(D(\lambda_0, \dots, \lambda_{n-1}), \varphi)}$ be the isomorphism class of an untwisted disc with the marked points all incoming. The subcategory \mathbf{C} is freely generated, as a symmetric monoidal category over $\text{Obj}(\mathbf{D}_\Lambda^G)$, by $\overline{(D(\lambda_0, \dots, \lambda_{n-1}), \varphi)}$, for $n \geq 3$, and the discs $\overline{(D_{\text{out}}^g(\lambda_i, \lambda_j), \varphi)}$ with two outgoing marked points, subject to the relation that*

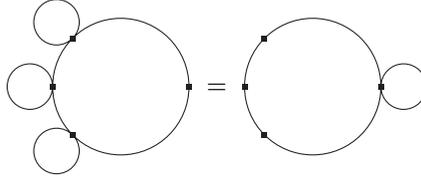
$\overline{(D(\lambda_0, \dots, \lambda_{n-1}), \varphi)}$ is cyclically symmetric:

$$\overline{(D(\lambda_0, \dots, \lambda_{n-1}), \varphi)} = \pm \overline{(D(\lambda_1, \dots, \lambda_{n-1}, \lambda_0), \varphi')}.$$

REMARK 3.12. It is required to permute the components of φ also in the previous proposition to get the trivialization φ' .

COROLLARY 3.13 (cf. [4, Lemma 6.2.2]). *The category $\mathbf{D}_{\Lambda, G}^+$ is freely generated as a DGSM category over $\text{Obj}(\mathbf{D}_{\Lambda, G})$ by the untwisted discs $\overline{(D^+(\lambda_0, \dots, \lambda_{n-1}), \varphi)}$ and the discs $\overline{(D^{\natural}(\lambda_i, \lambda_{i+1}), \varphi)}$ modulo the following relations:*

- (1) for $n \geq 2$ and $0 \leq i \leq n-2$, gluing pairs $\overline{(D^{\natural}(\lambda_i, \lambda_{i+1}), \{\psi_i, \psi_{i+1}\})}$ to the incoming boundary components of $\overline{(D^+(\lambda_0, \dots, \lambda_{n-1}), \varphi)}$ is equivalent to gluing discs $\overline{(D^+(\lambda_0, \lambda_{n-1}), \{\psi_0, \psi_{n-1}\})}$ to the only outgoing boundary component of the disc $\overline{(D^+(\lambda_0, \dots, \lambda_{n-1}), \varphi)}$;



- (2) for $n = 3$ the following relation holds:

$$\begin{aligned} \overline{(D^+(\lambda_0, \lambda_0, \lambda_1), \varphi)} \circ \overline{(D^+(\lambda_0), \psi)} \\ = \text{Id}_{\{\lambda_0, \lambda_1\}} = \overline{(D^+(\lambda_0, \lambda_1, \lambda_1), \xi)} \circ \overline{(D^+(\lambda_1), \zeta)}; \end{aligned}$$

- (3) for $n \geq 4$: $\overline{(D^+(\lambda_0, \dots, \lambda_i, \lambda_i, \dots, \lambda_{n-1}), \varphi)} \circ \overline{(D^+(\lambda_i), \psi)} = 0$.

THEOREM 3.14 (cf. [4, Theorem 6.2.3]). *The category \mathbf{D}_{Λ}^G is freely generated, as a DGSM category over $\text{Obj}(\mathbf{D}_{\Lambda}^G)$, by $\mathbf{D}_{\Lambda, G}^+$ and the discs $\overline{(D_{in}^{\natural}(\lambda_0, \lambda_1), \varphi)}$ with two incoming or $\overline{(D_{out}^{\natural}(\lambda_0, \lambda_1), \psi)}$ with two outgoing boundary components modulo the following relations:*

- (1) gluing a disc with two outgoing boundary components to a disc with two incoming boundary components yields the identity;
- (2) the untwisted disc $\overline{(D(\lambda_0, \dots, \lambda_{n-1}), \varphi)}$ is cyclically symmetric under the existing permutation isomorphism $[\lambda_n] \cong [\lambda_n]^c$.

Let $\mathbf{D}_{\Lambda, G}^+$ be the $\text{Obj}(\mathbf{O}_{\Lambda}^G) - \mathbf{D}_{\Lambda, G}^+$ -bimodule with the generators and relations stated above.

The complexes $\mathbf{D}_{\Lambda, G}^+$ admit a differential d which is defined as follows: if $*$ denotes the gluing of the open marked points between λ_i and λ_j , then

$$\begin{aligned} d((D(\lambda_0, \dots, \lambda_{n-1}), \varphi)) \\ = \sum_{\substack{0 \leq i \leq j \leq n-1 \\ 2 \leq j-i}} \pm (D(\lambda_i, \dots, \lambda_j), \varphi) * (D(\lambda_j, \dots, \lambda_i), \varphi). \end{aligned}$$

3.5. Calabi-Yau equivariant categories and TCFTs

The main result states that the category of open equivariant TCFTs is quasi-isomorphic to the category of Calabi-Yau equivariant A_∞ -categories. Observe that for such a category \mathbf{C} , the elements of $\text{Aut}(\mathbf{C})$ will be given, for each $\alpha \in G$, by the disc with one incoming and one outgoing boundary component. Products m_n are obtained from the generators of the categories defined in the previous sections.

Let $\mathfrak{F}: \mathbf{D}_{\Lambda, G}^+ \rightarrow \mathbf{Comp}_{\mathbb{K}}$ be a split symmetric monoidal functor. For each $O \in \mathbb{N}$ and D -brane labeling given by $\{s(i), t(i)\}$, with $0 \leq i \leq O-1$, the following isomorphism holds:

$$\mathfrak{F}([O], s, t) \cong \bigotimes_{i=0}^{O-1} \mathfrak{F}(\{s(i), t(i)\}). \quad (3.1)$$

Let the pair $\{s(i), t(i)\}$ corresponds to the pair of D -branes $\{\lambda_i, \lambda_{i+1}\}$. By associating open boundaries to the space $\text{Hom}_{\mathbf{B}}(\lambda_i, \lambda_{i+1})$ one can define a category \mathbf{B} with $\text{Obj}(\mathbf{B}) := \Lambda$ and $\text{Hom}_{\mathbf{B}}(\lambda_i, \lambda_{i+1}) := \mathfrak{F}(\{s(i), t(i)\})$. Composition of morphisms in \mathbf{B} is well defined since \mathfrak{F} is split.

LEMMA 3.15. *A split symmetric monoidal functor $\mathfrak{F}: \mathbf{D}_{\Lambda, G}^+ \rightarrow \mathbf{Comp}_{\mathbb{K}}$ is the same as a unital equivariant A_∞ -category \mathbf{B} with set of objects Λ .*

PROOF. The proof follows from the isomorphism (3.1) above. One observes that

- (1) the homomorphism of groups $\varphi: G \rightarrow \text{Aut}(\mathbf{B})$ is given by $(D^{\mathfrak{g}}(\lambda_0, \lambda_1), \psi)$,
- (2) the discs $(D^+(\lambda_0, \dots, \lambda_{n-1}), \psi)$ yield the products

$$m_{n-1}: \text{Hom}_{\mathbf{B}}(\lambda_0, \lambda_1) \otimes \cdots \otimes \text{Hom}_{\mathbf{B}}(\lambda_{n-2}, \lambda_{n-1}) \rightarrow \text{Hom}_{\mathbf{B}}(\lambda_0, \lambda_{n-1}),$$

- (3) the differential d gives the A_∞ -relations between the m_n ,
- (4) for $n = 2$, $(D^+(\lambda_0, \lambda_1), \psi)$ yields the identity $\text{Hom}_{\mathbf{B}}(\lambda_0, \lambda_1) \rightarrow \text{Hom}_{\mathbf{B}}(\lambda_0, \lambda_1)$, and

(5) for $n = 1$, $(D^+(\lambda), \psi)$ yields the unit $\mathbb{K} \rightarrow \text{Hom}_{\mathbf{B}}(\lambda, \lambda)$.

Relation (1) in Corollary 3.13 shows that the products m_n preserve the group action.

LEMMA 3.16. *A split symmetric monoidal functor $\mathfrak{F}: \mathbf{D}_{\Lambda}^G \rightarrow \mathbf{Comp}_{\mathbb{K}}$ is the same as a Calabi-Yau unital equivariant A_{∞} -category \mathbf{B} with set of objects Λ .*

PROOF. The proof follows the same arguments of Lemma 3.15. There are two more generators (cf. Theorem 3.14): the discs with two incoming and two outgoing marked points, which yield the map

$$\text{Hom}_{\mathbf{B}}(\lambda_0, \lambda_1) \otimes \text{Hom}_{\mathbf{B}}(\lambda_1, \lambda_0) \rightarrow \mathbb{K}$$

and its inverse. The extra relations on \mathbf{D}_{Λ}^G correspond to the cyclic symmetry condition. Relation (1) of Corollary 3.13 yields the identity $\langle \phi_{\alpha}(f), \phi_{\alpha}(g) \rangle = \langle f, g \rangle$.

A Calabi-Yau unital extended equivariant A_{∞} -category with objects in Λ is an h-split symmetric monoidal functor $\mathfrak{F}: \mathbf{D}_{\Lambda}^G \rightarrow \mathbf{Comp}_{\mathbb{K}}$. By considering $\mathbf{D}_{\Lambda, G}^+$ instead of \mathbf{D}_{Λ}^G one obtains the concept unital extended equivariant A_{∞} -category. By considering split functors instead of h-split functors, one gets the concept of unital Calabi-Yau equivariant A_{∞} -category and the concept of a unital equivariant A_{∞} -category respectively.

PROPOSITION 3.17. *The category of Calabi-Yau unital extended equivariant A_{∞} -categories with a set of objects Λ is quasi-equivalent to the category of open equivariant TCFTs.*

PROOF. An open equivariant TCFT is an h-split symmetric monoidal functor

$$\mathfrak{R}: \mathbf{O}_{\Lambda}^G \rightarrow \mathbf{Comp}_{\mathbb{K}}.$$

The result follows from Lemma 3.16 and the quasi-isomorphism between \mathbf{O}_{Λ}^G and \mathbf{D}_{Λ}^G , in Proposition 3.6.

Having in mind the fact that the *twisted disk* in [5] plays an analogous role concerning group actions in this paper, one can follow the arguments in [5, Proposition 6.8] to prove the following result:

THEOREM 3.18 (cf. [5, Proposition 6.8]). *The following categories are homotopy equivalent:*

- (1) *unital extended equivariant A_{∞} -categories;*
- (2) *unital equivariant A_{∞} -categories; and*
- (3) *unital equivariant DG categories.*

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