DOMINATED AND UNIFORMLY DOMINATED FAMILIES OF LOEB-MEASURES

DIETER LANDERS and LOTHAR ROGGE

Abstract.

It is shown in this paper that each dominated family of Loeb-measures, derived from an internal family of probability contents, is uniformly dominated. As a corollary we obtain some surprising “nonstandard equivalences” for uniform domination in the standard world. An essential tool is an extension of a well-known theorem of Halmos and Savage which is proven by nonstandard methods in a rather direct way.

1. Notations.

Let $P$ and $Q$ be probability contents ($p$-contents) on an algebra $\mathcal{C}$. Then

$Q$ weakly dominates $P$ iff $C \in \mathcal{C}$ and $Q(C) = 0$ imply $P(C) = 0$;

$Q$ dominates $P$ iff for each $\varepsilon > 0$ there exists $\delta > 0$ such that $C \in \mathcal{C}$ and $Q(C) < \delta$ imply $P(C) < \varepsilon$.

If $P, Q$ are $p$-measures on a $\sigma$-algebra then both concepts coincide.

Let $\mathcal{P}$ be a family of $p$-contents on $\mathcal{C}$. Then $\mathcal{P}$ is (weakly) dominated iff there exists a $p$-content $Q$, which (weakly) dominates $\mathcal{P}$; that is $Q$ (weakly) dominates each $P \in \mathcal{P}$.

$\mathcal{P}$ is uniformly dominated iff there exists a $p$-content $Q$, which uniformly dominates $\mathcal{P}$; i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that $C \in \mathcal{C}$ and $Q(C) < \delta$ imply $P(C) < \varepsilon$ for all $P \in \mathcal{P}$.

We assume in this paper that we have a structure containing the real numbers $\mathbb{R}$, and a polysaturated nonstandard model of this structure.

Let $\mathcal{A}$ be an internal algebra, and $Q: \mathcal{A} \rightarrow \ast[0, 1]$ be an internal $p$-content. Then $Q_L(B) := \circ(Q(B)), B \in \mathcal{A}$, defines a $p$-measure on $\mathcal{A}$ and the system $L(Q)$ of all sets $C$ with

$$\sup\{Q_L(B) : C \supset B \in \mathcal{A}\} = \inf\{Q_L(B) : C \subseteq B \in \mathcal{A}\}$$

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is the \( \sigma \)-algebra of all Carathéodory-measurable sets with respect to \( Q_L|\mathcal{B} \). The common value of the above expressions defines the unique extension of \( Q_L|\mathcal{B} \) to a measure on the complete \( \sigma \)-algebra \( L(Q) \). This extension is also denoted by \( Q_L; Q_L \) is the Loeb-measure associated with \( Q \). The described construction was given in [3], [4]. Put

\[
L_n(\mathcal{B}) = \bigcap \{ L(Q) : Q \text{ internal } p\text{-content on } \mathcal{B} \}.
\]

If \( \mathcal{B} \) is a family of internal \( p\)-contents on \( \mathcal{B} \), let \( \mathcal{G}_L|\mathcal{B} = \{ Q_L|\mathcal{B} : Q \in \mathcal{G} \} \).

2. The results.

The following Theorem of Halmos and Savage (see [1]) is an important tool in mathematical statistics and especially in the theory of sufficiency. We give a short and transparent proof using nonstandard techniques.

If \( P \) is a \( p \)-content on an algebra \( \mathcal{A} \), put

\[
N(P) := \bigcup \{ \ast N : N \in \mathcal{A}, P(N) = 0 \}.
\]

As our model is polysaturated, Theorem 1 of [2] implies that \( N(P) \in L_n(\ast \mathcal{A}) \) and \( \ast P_L(N(P)) = 0 \).

The following Lemma will be used several times in the proofs of our results.

1. Lemma. Let \( Q \) be an internal \( p \)-content on an internal algebra \( \mathcal{B} \).

1. If \( Q_L|\mathcal{B} \) is weakly dominated by a \( p \)-content \( v|\mathcal{B} \), then it is dominated by \( v|\mathcal{B} \).

2. If \( P|\mathcal{B} \) is an internal \( p \)-content such that \( P_L \) dominates \( Q_L \) on \( \mathcal{B} \), then it dominates \( Q_L \) on \( L_n(\mathcal{B}) \).

3. Let \( P \) and \( Q \) be \( p \)-contents on an algebra \( \mathcal{A} \). Then \( P|\mathcal{A} \) dominates \( Q|\mathcal{A} \) iff \( \ast P_L|L_n(\ast \mathcal{A}) \) dominates \( \ast Q_L|L_n(\mathcal{A}) \).

Proof. (1) As \( \mathcal{B} \) is an internal algebra, \( v|\mathcal{B} \) is a \( p \)-measure and can be extended to a unique \( p \)-measure on \( \sigma(\mathcal{B}) \). It suffices to show that \( v|\sigma(\mathcal{B}) \) weakly dominates \( Q_L|\sigma(\mathcal{B}) \). Let \( C \in \sigma(\mathcal{B}) \) with \( v(C) = 0 \). Assume indirectly that \( Q_L(C) > 0 \). Then there exists \( B \in \mathcal{B} \) with \( B \subset C \) and \( Q_L(B) > 0 \), contradicting \( v(B) = 0 \).

(2) Let \( C \in L_n(\mathcal{B}) \) with \( P_L(C) = 0 \). If \( Q_L(C) > 0 \), we obtain a contradiction as in (1).

(3) By transfer it can be seen that \( P|\mathcal{A} \) dominates \( Q|\mathcal{A} \) iff \( \ast P_L|\ast \mathcal{A} \) dominates \( \ast Q_L|\ast \mathcal{A} \). Now by (2), applied to \( P|B = \ast P_L|\ast \mathcal{A} \) and \( Q|B = \ast Q_L|\ast \mathcal{A} \), we obtain (3).

2. Theorem. Let \( \mathcal{P} \) be a dominated family of \( p \)-measures on a \( \sigma \)-algebra \( \mathcal{A} \). Then there exists \( P_n \in \mathcal{P}, n \in \mathbb{N} \), such that \( \sum_{n \in \mathbb{N}} 2^{-n} P_n \) dominates \( \mathcal{P} \).
\textbf{Proof.} Let $\mathcal{P}$ be dominated by a $p$-content $\mu$. Let $\{P_n : n \in \mathbb{N}\} \subset \mathcal{P}$ be such that
\begin{equation}
*\mu_L \left( \bigcap_{n \in \mathbb{N}} N(P_n) \right) = \inf \left\{ *\mu_L \left( \bigcap_{P \in \mathcal{P}_0} N(P) \right) : \mathcal{P}_0 \subset \mathcal{P} \text{ countable} \right\}.
\end{equation}

By (1) we have for each $Q \in \mathcal{P}$ that
\[ N := \bigcap_{n \in \mathbb{N}} N(P_n) \subset N(Q) \quad *\mu_L|_{L^*(\mathcal{A})}\text{-a.e.} \]

As $*Q_L|_{L^*(\mathcal{A})}$ is dominated by $*\mu_L|_{L^*(\mathcal{A})}$ according to Lemma 1, we obtain $N \subset N(Q) \ *Q_L$-a.e. Put
\[ P_0 := \sum_{n \in \mathbb{N}} \frac{1}{2^n} P_n \]
and let $A \in \mathcal{A}$ with $P_0(A) = 0$. Then $*A \subset N(P_0) \subset \bigcap_{n \in \mathbb{N}} N(P_n) = N$. Hence $Q(A) = *Q_L(*A) = 0$ for all $Q \in \mathcal{P}$.

Using once more nonstandard techniques, we obtain the following generalization of the Theorem of Halmos-Savage.

3. Corollary. Let $\mathcal{P}$ be a dominated family of $p$-contents on an algebra $\mathcal{A}$. Then there exist $P_n \in \mathcal{P}$, $n \in \mathbb{N}$, such that $\sum_{n \in \mathbb{N}} 2^{-n} P_n$ dominates $\mathcal{P}$.

\textbf{Proof.} Let $\mathcal{P}$ be dominated by a $p$-content $\mu$. By Lemma 1 we have that $\{ *P_L : P \in \mathcal{P} \}$ is dominated by $*\mu_L$ on $\sigma(*\mathcal{A})$, the $\sigma$-algebra generated by $*\mathcal{A}$. According to Theorem 2 there exist $P_n \in \mathcal{P}$, $n \in \mathbb{N}$, such that $\sum_{n \in \mathbb{N}} 2^{-n} (P_n)_L$ dominates $\{ *P_L : P \in \mathcal{P} \}$ on $\sigma(*\mathcal{A})$. Hence $\sum_{n \in \mathbb{N}} 2^{-n} P_n$ dominates $\mathcal{P}$.

Now we prove a result for certain families of Loeb-measures which is obviously false for general families of measures.

4. Theorem. Let $\mathcal{B}$ be an internal algebra and let $\mathcal{G}$ be an internal family of $p$-contents on $\mathcal{B}$. If $\mathcal{G}_L|_\mathcal{B}$ is weakly dominated, then there exists an internal $p$-content $v|_\mathcal{B}$ such that $v_L|_\mathcal{B}$ uniformly dominates $\mathcal{G}_L|_\mathcal{B}$.

\textbf{Proof.} According to Lemma 1, $\mathcal{G}_L|_\mathcal{B}$ is dominated. Hence there exist $Q_n \in \mathcal{G}$, $n \in \mathbb{N}$, such that $\sum_{n \in \mathbb{N}} 2^{-n} (Q_n)_L$ dominates $\mathcal{G}_L|_\mathcal{B}$ (use Corollary 3). Since $\mathcal{G}$ is an internal set, and since our model is polysaturated, there exists an internal extension $(Q_H)_{H \in \ast \mathbb{N}} \subset \mathcal{G}$ of $(Q_n)_{n \in \mathbb{N}}$. Put
\[ v = \sum_{H \in \ast \mathbb{N}} \frac{1}{2^H} Q_H. \]

Then $v|_\mathcal{B}$ is an internal $p$-content. Furthermore, $v_L|_\mathcal{B}$ dominates $\mathcal{G}_L|_\mathcal{B}$.
If $B \in \mathcal{B}$ and $v_L(B) = 0$, then $Q_n(B) \approx 0$ for all $n \in \mathbb{N}$ and hence $Q_L(B) = 0$ for all $Q \in \mathcal{G}$; therefore $\mathcal{G}_L$ is dominated by $v_L$ according to Lemma 1.

Let $\varepsilon \in \mathbb{R}_+$ be fixed and put for each $\delta \in \mathbb{R}_+$

$$\mathcal{G}_{\delta} := \{Q \in \mathcal{G} \mid (\forall B \in \mathcal{B})(v(B) < \delta \Rightarrow Q(B) < \varepsilon)\}.$$

As $v_L|\mathcal{B}$ dominates $\mathcal{G}_L|\mathcal{B}$, we obtain $\mathcal{G} = \bigcup_{\delta \in \mathbb{R}_+} \mathcal{G}_{\delta}$. As $\mathcal{G}$ and $\mathcal{G}_{\delta}$, $\delta \in \mathbb{R}_+$, are internal sets and since our model is polysaturated, there exists $\delta \in \mathbb{R}_+$ such that $\mathcal{G} = \mathcal{G}_{\delta}$. Consequently $\mathcal{G}_L|\mathcal{B}$ is uniformly dominated by $v_L|\mathcal{B}$.

5. Theorem. Let $\mathcal{A}$ be an algebra, and let $\mathcal{G}$ be an internal family of $p$-contents on $\mathcal{A}$. If for each $Q \in \mathcal{G}$ there exists a $p$-content $P|\mathcal{A}$ such that $*P_L|\mathcal{A}$ dominates $Q_L|\mathcal{A}$, then there exists a $p$-content $\mu|\mathcal{A}$ such that $*\mu_L|\mathcal{A}$ uniformly dominates $\mathcal{G}_L|\mathcal{A}$.

Proof. Let $\varepsilon, \delta \in \mathbb{R}_+$, $P|\mathcal{A}$ be a $p$-content and put

$$\mathcal{G}_{P,\delta,\varepsilon} := \{Q \in \mathcal{G} \mid (\forall A \in \mathcal{A})(*P(A) < \delta \Rightarrow Q(A) < \varepsilon)\}.$$

By assumption we obtain for each $\varepsilon \in \mathbb{R}_+$ that $\mathcal{G} = \bigcup \{\mathcal{G}_{P,\delta,\varepsilon} : \delta \in \mathbb{R}_+, P|\mathcal{A}$ p-content $\}$. Since $\mathcal{G}$ and $\mathcal{G}_{P,\delta,\varepsilon}$ are internal and since our model is polysaturated there exists $p$-contents $P_{\varepsilon}^1, \ldots, P_{n(\varepsilon)}^\varepsilon$ on $\mathcal{A}$ and $\delta(\varepsilon) \in \mathbb{R}_+$ such that for all $A \in \mathcal{A}$ and all $Q \in \mathcal{G}$:

$$*P_i^\varepsilon(A) < \delta(\varepsilon) \quad \text{for } i = 1, \ldots, n(\varepsilon) \Rightarrow Q(A) < \varepsilon.$$

Let $P_n$, $n \in \mathbb{N}$, be a denumeration of $\{P_n^\varepsilon : \nu \leq n(\varepsilon), \varepsilon = 1/m, m \in \mathbb{N}\}$ and put $\mu = \sum_{n \in \mathbb{N}} 2^{-n}P_n$. Then $*\mu_L|\mathcal{A}$ uniformly dominates $\mathcal{G}_L|\mathcal{A}$.

6. Corollary. Let $\mathcal{P}$ be a family of $p$-contents on an algebra $\mathcal{A}$. Then the following four conditions are equivalent:

(i) $\mathcal{P}|\mathcal{A}$ is uniformly dominated;
(ii) $*\mathcal{P}_L|\mathcal{A}$ is weakly dominated;
(iii) $*\mathcal{P}_L|\mathcal{A}$ is uniformly dominated;
(iv) for each $Q \in *\mathcal{P}$ there exists a $p$-content $P|\mathcal{A}$ such that $*P_L|\mathcal{A}$ dominates $Q_L|\mathcal{A}$.

Proof. (i) $\Rightarrow$ (iv) by transfer; (iv) $\Rightarrow$ (iii) by Theorem 5; (iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i): If (ii) holds, then by Theorem 4 there exists an internal $p$-content $v|\mathcal{A}$ such that $v_L|\mathcal{A}$ uniformly dominates $*\mathcal{P}_L|\mathcal{A}$. Hence $\mathcal{P}|\mathcal{A}$ is uniformly dominated by the $p$-content $P|\mathcal{A}$, given by $P(A) := v_L(A)$ for $A \in \mathcal{A}$. 


REFERENCES


