COMPARISON OF STATES AND DARBOUX-TYPE PROPERTIES IN VON NEUMANN ALGEBRAS

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0. Introduction and preliminaries.

In connection with his study of quantum comparative probability [3], W. Ochs considered the following problem. Let \( \varphi \) be a normal state on \( B(H) \) and let \( \leq_{\varphi} \) be the relation defined for pairs of projections from \( B(H) \) by \( p \leq_{\varphi} q \) if \( \varphi(p) \leq \varphi(q) \). Does \( \leq_{\varphi} = \leq_{\psi} \) imply \( \varphi = \psi \)? We give the complete solution to the problem for a not necessarily normal state on an arbitrary von Neumann algebra in Section 3. The comparison of states on finite-dimensional von Neumann algebras is described in Section 4.

In [1], H. Choda, M. Enomoto and M. Fujii proved an interesting result: if \( \varphi \) and \( \psi \) are states on a non-atomic von Neumann algebra \( M \), with \( \varphi \) normal, and if, for every projection \( p \) of \( M \), \( \varphi(p) = \frac{1}{2} \) implies \( \psi(p) = \frac{1}{2} \), then \( \varphi = \psi \). One easily notices the strong connection between the theorem and the problem of Ochs (see Section 3). In order, however, that the theorem be applicable to the problem in a nontrivial way, it should be appropriately generalized. We do not require the state \( \varphi \) to be normal, and the algebra \( M \) (although possibly atomic) should not contain a direct summand of type \( I_n, n < \infty \). The above-mentioned theorem was proved in [1] by using a "dyadic" method and a simple Darboux-type property of a normal state \( \varphi \): if \( \varphi(p) = \alpha > 0 \) for some projection \( p \), then \( \varphi \) takes all values less than \( \alpha \) (but \( \geq 0 \)) at some subprojections of \( p \). So to generalize the theorem, one should generalize the property. Our Darboux-type properties (being of interest in themselves) are described in section 1, and the generalized theorem in section 2.

In the sequel, \( M \) denotes a von Neumann algebra, \( Z \) its center, \( \text{Proj} M \) the lattice of all orthogonal projections of \( M \) and \( \varphi, \psi \) (not necessarily normal) positive linear functionals on \( M \). Moreover, for \( r \in \text{Proj} M \) we put

\[
L_r = \{ q \in \text{Proj} M : q \leq r \},
\]

\[
P_r = \{ q \in L_r ; q \sim r - q \sim r \}
\]

(with \( \mathcal{P} = \mathcal{P}_1 \)).
The following theorem is the generalization of the "dyadic" method of [1], suited to our purposes.

**Theorem 0.1.** Fix \( r \in \text{Proj} \, M \). Let \( Q \subseteq \mathcal{L}_r \), \( Q \neq \emptyset \), satisfy the following conditions:

(i) if \( q \in Q \), then \( r - q \in Q \);

(ii) if \( 0 < \gamma < \varphi(q) \) for some \( q \in Q \), then there is a \( q_0 \in Q \), \( q_0 \leq q \) such that \( q - q_0 \in Q \) and \( \varphi(q_0) = \gamma \).

Assume that \( 0 < \alpha < \varphi(r) \) and \( 0 \leq \beta \), and suppose that \( \varphi(p) = \alpha \) implies \( \psi(p) = \beta \) for \( p \in Q \).

Then \( \psi(q) = (\beta/\alpha)\varphi(q) \) for \( q \in Q \).

**Proof.** Note that \( \varphi(q_1) = \alpha/n \) implies \( \psi(q_1) = \beta/n \) for any projection \( q_1 \in Q \) and a positive integer \( n \) satisfying \( \alpha(n+1)/n \leq \varphi(r) \). Indeed, by (i), \( r - q_1 \in Q \), and since \( \varphi(r - q_1) \geq \alpha \), there exist by (ii) mutually orthogonal projections \( q_1 \in Q, q_i \leq r - q_1 \) \( (i = 2, 3, \ldots, n+1) \) such that \( \varphi(q_i) = \alpha/n \) for each \( i \). Denote \( p = \sum_{i=1}^{n+1} q_i \). By supposition,

\[
\varphi(p - q_1) = \varphi(p - q_i) = \alpha
\]

implies

\[
\psi(p - q_1) = \psi(p - q_i) = \beta \quad \text{for} \quad i = 2, 3, \ldots, n+1.
\]

Thus \( \psi(q_1) = \psi(q_i) \) and \( \psi(q_i) = \psi(p - q_1)/n = \beta/n \).

Take now an arbitrary \( q \in Q \). If \( 0 < \varphi(q) < \varphi(r) \), then there are positive integers \( k, n \) (with \( n \) arbitrarily great) such that \( \alpha k/n \leq \varphi(q) \leq \alpha(k+1)/n \) and that \( \alpha(k+1)/n \leq \varphi(r) \). Hence, mutually orthogonal projections \( q_i \in Q \) \( (i = 1, 2, \ldots, k+1) \) can be chosen satisfying \( \sum_{i=1}^{k+1} q_i \leq q \leq \sum_{i=1}^{k+1} q_i \) and \( \varphi(q_i) = \alpha/n \) for each \( i \). Therefore, \( \beta k/n \leq \psi(q) \leq \beta(k+1)/n \) and the conclusion follows. If \( \varphi(q) = \varphi(r) \), find \( q_1, q_2 \in Q \) such that \( 0 < \varphi(q_i) < \varphi(r) \) and that \( q_1 + q_2 = q \), and apply the above result to each of the \( q_i \) \( (i = 1, 2) \) to obtain \( \psi(q) = (\beta/\alpha)\varphi(q) \). Similarly, \( \psi(r) = (\beta/\alpha)\varphi(r) \). If \( \varphi(q) = 0 \), then \( \varphi(r - q) = \varphi(r) \) and what we have got so far shows that \( \psi(q) = 0 \). All the cases having been considered, the proof of the theorem is finished.

1. **Darboux-type properties.**

We shall need the following simple result (cf. [5; Lemma 1]).

**Proposition 1.1.** Let \( p, q \in \text{Proj} \, M \), \( p \sim q \), \( pq = 0 \) and \( \varphi(p) \leq \gamma \leq \varphi(q) \). Then there is a projection \( r \) such that

\[
\varphi(r) = \gamma, \quad r \leq p + q \quad \text{and} \quad r \sim p \sim p + q - r.
\]
PROOF. Let \( u \in M \) be such that \( u^*u = p, uu^* = q \). Define a norm-continuous function \( \omega : [0, 1] \to \text{Proj} M \) by
\[
\omega(\lambda) = (1 - \lambda^2)p + \lambda^2 q + \lambda(1 - \lambda^2)^\frac{1}{2}(u + u^*)
\]
Then \( \omega(0) = p, \omega(1) = q \), and the sought – for projection \( r \) may be chosen from among the values of the function \( \omega \).

A stronger result will be proved below (Theorem 1.4). We shall use the following Wold-type decomposition [4; Theorem 1.1].

**THEOREM 1.2.** If \( e + r \sim e + s \) for mutually orthogonal \( e, r, s \in \text{Proj} M \), then there are mutually orthogonal projections \( r_1, r_2, s_1, s_2, f, g_n, h_n \) (\( n \geq 1 \)) such that \( r = r_1 + r_2, s = s_1 + s_2 \),
\[
e = f + \sum_{n \geq 1} (g_n + h_n)
\]
and \( r_1 \sim s_1, r_2 \sim g_n, s_2 \sim h_n \) for \( n \geq 1 \).

**LEMMA 1.3.** Any equivalent projections \( p, q \in M \) can be decomposed (in \( \text{Proj} M \)) as follows:
\[
p = r + f + \sum_{n \geq 1} (g_n + h_n),
\]
\[
q = s + f + \sum_{n \geq 1} (g_n + h_n)
\]
so that, for any \( K \subset \mathbb{N} \) with \( \# K = \# \mathbb{N} \setminus K \), the projections
\[
p_K = r + f + \sum_{n \in K} (g_n + h_n)
\]
and
\[
q_K = s + f + \sum_{n \in K} (g_n + h_n)
\]
are unitarily equivalent.

**PROOF.** Choose \( t \in \text{Proj} M \) so that \( pt = tp \) and \( q = vtv \) for some unitary \( v \in M \) (see, for example, [4; 3.9]). Put \( e = pt, r = p - e, s = t - e \) and apply Theorem 1.2. Let further
\[
p_K = f + r + \sum_{n \in K} (g_n + h_n), \quad t_K = p_K - r + s.
\]
Then

\[ p_K = f + r_1 + \left( r_2 + \sum_{n \in K} g_n \right) + \sum_{n \notin K} h_n \]

\[ \sim f + s_1 + \sum_{n \in K} g_n + \left( s_2 + \sum_{n \in K} h_n \right) = t_K, \]

\[ 1 - p_K = 1 - (e + r + s) + s_1 + \sum_{n \notin K} g_n + \left( s_2 + \sum_{n \notin K} h_n \right) \]

\[ \sim 1 - (e + r + s) + r_1 + \left( r_2 + \sum_{n \notin K} g_n \right) + \sum_{n \notin K} h_n = 1 - t_K \]

Put now \( \hat{s} = vsv^*, \hat{f} = vf v^*, \hat{g}_n = vg_n v^*, \hat{h}_n = vh_n v^* \) and conclude that \( p_K \) and \( q_K = vt_K v^* \) are unitarily equivalent.

**Theorem 1.4.** Let \( p \sim q \) for some \( p, q \in \text{Proj } M \), and let \( \varepsilon > 0 \). Then there exists a continuous (in norm) function \( \omega : [0, 1] \to \text{Proj } M \) such that

\[ 1^* \quad \omega(0) \leq p, \quad \omega(1) \leq q; \]

\[ 2^* \quad \varphi(\omega(0)) > \varphi(p) - \varepsilon, \quad \varphi(\omega(1)) > \varphi(q) - \varepsilon. \]

**Proof.** Consider the decomposition from Lemma 1.3 and take a sequence of disjoint subsets \( N_i \subset \mathbb{N} \) with \( \# N_i = \# \mathbb{N} \setminus N_i \). For one of them, the inequalities

\[ \varphi \left( \sum_{n \in N_i} (g_n + h_n) \right) < \varepsilon, \quad \varphi \left( \sum_{n \in N_i} (\hat{g}_n + \hat{h}_n) \right) < \varepsilon \]

hold. The projections

\[ \omega(0) = f + r + \sum_{n \notin N_i} (g_n + h_n), \]

\[ \omega(1) = \hat{s} + \hat{f} + \sum_{n \notin N_i} (\hat{g}_n + \hat{h}_n) \]

satisfy \( 1^* \) and \( 2^* \), and are unitarily equivalent. Thus the required function \( \omega \) exists (see, for example, [2; Theorem 1], or use the connectedness of the unitary group of \( M \)).

Two more Darboux-type properties will be used in the sequel.

**Proposition 1.5.** Let \( p \in \text{Proj } M \) be properly infinite and let \( \varphi(p) > \gamma > 0 \). Then \( \varphi(r) = \gamma \) for some \( r \in \text{Proj } M \) such that \( r \leq p \) and \( r \sim p - r \sim p \).

**Proof.** There is a sequence \( \{p_n\} \) of mutually orthogonal projections from \( M \)
such that \( p_n \sim p \) and \( p = \sum p_n \) (see [7; Proposition 4.12]). If \( q = p_n \) with sufficiently large \( n \), then \( \varphi(q) < \gamma, \varphi(p - q) > \gamma \) and, obviously, \( q \sim p - q \sim p \). By Proposition 1.1, there is a projection \( r \) in \( M \) such that \( \varphi(r) = \gamma \) and \( r \sim p - r \sim p \).

**Proposition 1.6.** Let \( p \in \text{Proj} M \) be finite and continuous and let \( \varphi(p) \geq \gamma \geq 0 \). Then \( \varphi(r) = \gamma \) for some \( r \in \text{Proj} M \) with \( r \leq p \).

**Proof.** We may assume that \( M \) is of type \( \text{II}_1 \), \( p = 1 \) and \( \varphi(p) = 1 \). Note also that it suffices to prove the proposition for \( 0 < \gamma \leq 1/2 \). There are two possibilities:

\( 1^* \). \( \varphi(q) = 1/2 \) for each \( q \in \text{Proj} M \) such that \( q \sim 1 - q \).

Let \( T \) denote the canonical center-valued trace on \( M \), \( \mu \) a positive linear functional on \( Z \), and let \( \tau = \mu \circ T \). Moreover, put

\[ Q = \{ p \in \text{Proj} M : T(p) = \beta 1 \text{ for some } \beta, 0 \leq \beta \leq 1 \}. \]

By assumption, \( T(q) = (1/2)1 \) implies \( \varphi(q) = 1/2 \) for \( q \in \text{Proj} M \). In view of the Darboux-type property of \( T \) (see [7; Proposition 7.17]), we may apply Theorem 0.1 with \( \tau \) and \( \varphi \) in place of \( \varphi \) and \( \psi \) to conclude that \( T(q) = \beta 1 \) implies \( \varphi(q) = \beta \) for each \( \beta \in [0,1] \) and \( q \in \text{Proj} M \). Hence, \( \varphi(r) = \gamma \) for a (clearly existing) projection \( r \) such that \( T(r) = \gamma 1 \).

\( 2^* \). \( \varphi(q) = \delta < 1/2 < 1 - \delta = \varphi(1 - q) \) for some \( q \in \text{Proj} M \) satisfying \( q \sim 1 - q \). There are positive integers \( k, n \), \( k \leq 2^n \), such that \( \beta = 2^n \gamma/k \in [\delta, 1 - \delta] \). By Proposition 1.1, \( \varphi(s) = \beta \) for some \( s \in \text{Proj} M \). By repeated use of Proposition 1.1, we get a sequence of mutually orthogonal projections \( r_1, \ldots, r_{2^n} \) from \( M \) such that \( \varphi(r_m) = \beta/2^n \) for each \( m, 1 \leq m \leq 2^n \), and that \( r_1 + \ldots + r_{2^n} = s \). Put \( r = r_1 + \ldots + r_k \) to get \( \varphi(r) = k \beta/2^n = \gamma \).

2. A sufficient condition for the equality of states.

**Lemma 2.1.** Let \( M \) be properly infinite (respectively of type \( \text{II}_1 \)), \( 0 < \alpha < \varphi(1) \) and \( 0 \leq \beta \). If \( \varphi(p) = \alpha \) implies \( \psi(p) = \beta \) for \( p \in \mathcal{P} \) (respectively \( p \in \text{Proj} M \)), then \( \psi(a) = (\beta/\alpha)Q(a) \) for each \( q \in \mathcal{P} \) (respectively \( q \in \text{Proj} M \)). (For the definition of \( \mathcal{P} \) see Introduction.)

**Proof.** Follows at once from Proposition 1.5 (respectively Proposition 1.6) and Theorem 0.1 with \( Q = \mathcal{P} \) (respectively \( Q = \text{Proj} M \)).

**Lemma 2.2.** Let \( M \) be properly infinite. If \( \varphi = \psi \) on \( \mathcal{P} \), then \( \varphi = \psi \) (on \( \text{Proj} M \)).
**Proof.** Will be carried out in two steps.

**Step 1.** \( \varphi = \psi \) on \( \mathcal{P} \) for any \( z \in \mathbb{Z} \). Fix \( p \in \mathcal{P} \) and \( \varepsilon > 0 \). By Proposition 1.5, we may find a projection \( q \in \mathcal{P}_{1-z} \) satisfying \( \varphi(q) < \varepsilon, \psi(q) < \varepsilon \). Note that \( p + q \in \mathcal{P} \). By assumption, \( \varphi(p + q) = \psi(p + q) \) and, consequently, \( |\psi(p) - \varphi(p)| < 2\varepsilon \).

**Step 2.** \( \varphi = \psi \) on \( \text{Proj} \, M \). Fix \( p \in \text{Proj} \, M \). By the comparability theorem (see [6: Theorem V.1.8]), there are \( x, y \in \mathbb{Z} \) such that \( x + y = 1, px \leq (1-p)x, (1-p)y \leq py \). Choose \( q_1, \ldots, q_4 \in \text{Proj} \, M \) so that \( q_1 + q_2 = (1-p)x, q_1 \sim q_2 \sim (1-p)x, q_3 + q_4 = py, q_3 \sim q_4 \sim py \). Note that \( q_1, q_2 \in \mathcal{P}_x, q_3, q_4 \in \mathcal{P}_y \), and \( px = x - q_1 - q_2 \). By Step 1,

\[
\psi(p) = \psi(px) + \psi(py) = \psi(x - q_1) - \psi(q_2) + \psi(q_3) + \psi(q_4) = \varphi(p),
\]

which ends the proof.

**Theorem 2.3.** Let \( M \) be a von Neumann algebra without a direct summand of type \( I_n \) \( (n < \infty) \), \( 0 < \alpha < \varphi(1) \) and \( 0 \leq \beta \). If \( \varphi(p) = \alpha \) implies \( \psi(p) = \beta \) for \( p \in \text{Proj} \, M \), then \( \psi = (\beta/\alpha) \varphi \). In particular, if \( \varphi \) and \( \psi \) are states, then \( \varphi = \psi \) (and \( \alpha = \beta \)).

**Proof.** Let \( z \) be the maximal projection in the center \( Z \) of \( M \), such that \( M_z \) is of type \( II_1 \). Then \( M(1-z) \) is properly infinite. Let us note that, by Propositions 1.5 and 1.6, \( Q = \mathcal{L}_z + \mathcal{P}_{1-z} \) satisfies the assumptions of Theorem 0.1. Thus \( \psi = (\beta/\alpha) \varphi \) on \( \mathcal{L}_z + \mathcal{P}_{1-z} \) and the equality must hold on \( \mathcal{P}_{1-z} \), on \( \mathcal{L}_z \) and, by Lemma 2.2, on \( \mathcal{P}_{1-z} \). The proof is finished.

We have proved, in fact, the following

**Proposition 2.4.** Let \( 0 < \alpha < \varphi(1), 0 \leq \beta, \) and, for a projection \( z \in \mathbb{Z} \), let

(i) \( 0 < \gamma < \varphi(p), p \in \mathcal{L}_z, \) imply \( \varphi(q) = \gamma \) for some \( q \in \mathcal{L}_\beta \);

(ii) \( M(1-z) \) be properly infinite.

If \( \varphi(p) = \alpha \) implies \( \psi(p) = \beta \) for \( p \in \mathcal{L}_z \mathcal{P}_{1-z} \), then \( \psi = (\beta/\alpha) \varphi \).

It follows easily from Theorem 3.3 that if \( M \neq \mathbb{C}I_1 \) and \( M \) has a nonzero direct summand of type \( I_n \) \( (\text{for some } n < \infty) \), then there are two distinct (and equivalent) states \( \varphi, \psi \) on \( M \) such that \( \varphi(p) = 1/2 \) implies \( \psi(p) = 1/2 \) for \( p \in \text{Proj} \, M \). However, we have the following

**Corollary 2.5.** Let \( M \) be a von Neumann algebra with no factor of type \( I_n \), \( n < \infty \), as a direct summand, \( 0 < \alpha < \varphi(1) \) and \( 0 \leq \beta \). Suppose that \( \varphi \) is
normal \( \text{at least on the finite discrete part of } M \), and that \( \varphi(p) = \alpha \) implies \( \psi(p) = \beta \) for \( p \in \text{Proj } M \). Then \( \psi = (\beta/\alpha)\varphi \).

**Proof.** Let \( z \) be the smallest projection in \( Z \) such that \( M(1-z) \) is properly infinite. Then \( Mz = M_1 \oplus M_2 \), where \( M_1 \) is of type \( \Pi_1 \), \( M_2 \) is (finite, discrete and) non-atomic and \( \varphi \) is normal on \( M_2 \). By Proposition 1.6 and \([1; \text{Theorem } 1]\) (see Introduction), the conditions \( 0 < \gamma < \varphi(p), \ p \in \text{Proj } Mz \), imply \( \varphi(q) = \gamma \) for some \( q \in \mathcal{L}_p \). Thus, Proposition 2.4 can be used to end the proof.

### 3. Equivalent and exclusive states.

Each state \( \varphi \) on a von Neumann algebra \( M \) gives rise to the following relation of comparative probability (cf. \([3]\)) on the lattice of projections of \( M \):

\[
p \leq_{\varphi} q \iff \varphi(p) \leq \varphi(q), \ p,q \in \text{Proj } M.
\]

A state \( \psi \) is said to be equivalent to a state \( \varphi \) if \( \leq_{\varphi} = \leq_{\psi} \), i.e. if, for \( p,q \in \text{Proj } M \),

\[
\varphi(p) \leq \varphi(q) \quad \text{is equivalent to } \psi(p) \leq \psi(q).
\]

We shall also say that \( \psi \) is similar to \( \varphi \) if, for \( p,q \in \text{Proj } M \),

\[
\varphi(p) < \varphi(q) \quad \text{implies } \psi(p) \leq \psi(q).
\]

We denote by \( E(\varphi) \) (respectively \( S(\varphi) \)) the set of all states equivalent (respectively similar) to \( \varphi \). If \( E(\varphi) = \{ \varphi \} \) (respectively \( S(\varphi) = \{ \varphi \} \)), then the state \( \varphi \) is called exclusive (respectively strongly exclusive). Note that the equivalence is, in fact, an equivalence relation, and that \( \psi \in S(\varphi) \iff \varphi \in S(\psi) \) i.e., the relation of similarity is symmetric.

The notions of quantum comparative probability, equivalence and exclusiveness of states were introduced by Ochs \([3]\). He showed that (with the equivalence relation restricted to the set of normal states) each normal state on a factor of type \( \Gamma_\infty \) is exclusive. He also stated a necessary and sufficient condition for the exclusiveness of a state on a factor of type \( \Gamma_n, n < \infty \), and proved that each nonfaithful state on such a factor is exclusive.

In this section we shall describe those von Neumann algebras which admit only exclusive states. The subsequent section contains a thorough description of the sets \( E(\varphi) \) and \( S(\varphi) \) for factors of type \( \Gamma_n, n < \infty \).

We shall start with two simple lemmas.

**Lemma 3.1.** Let \( K \) be a commutative von Neumann algebra. There is a state \( \varphi \) on \( K \) such that \( \varphi(\text{Proj } K) = \{0, 1\} \).
Proof. We may assume that $K = L^\infty(\Omega, \mathcal{F}, v)$ where $(\Omega, \mathcal{F}, v)$ is a finite measure space. Let $U$ be an ultrafilter in $\Omega$ containing the complements of all $v$-negligible subsets of $\Omega$. For $A \in \mathcal{F}$, put $\mu(A) = 0$ when $A \notin U$ and $\mu(A) = 1$ when $A \in U$. Since $\mu$ is a finitely additive measure on $\mathcal{F}$, absolutely continuous with respect to $v$, it yields a state $\varphi$ with the desired property.

Lemma 3.2. Let $M(1 - z)$ be of type $\text{II}_1$ and let $Mz$ be properly infinite for some $z \in \mathbb{Z}$. If $\psi$ is similar to $\varphi$, then $\varphi(p) = 1/2$ implies $\psi(p) = 1/2$ for any $p \in \mathcal{L}_{1 - z} + \mathcal{P}_z$.

Proof. Let $\varphi(p) = 1/2$, $p \in \mathcal{L}_{1 - z} + \mathcal{P}_z$, and $\varepsilon > 0$. By Propositions 1.5 and 1.6, we can always find mutually orthogonal projections $r_n \in \mathcal{L}_{1 - z} + \mathcal{P}_z$, $r_n \leq p$, satisfying $\varphi(r_n) > 0$ for every $n$. For sufficiently large $n_0$, $\varphi(r_{n_0}) < \varepsilon$.

Hence, $\varphi(p) < \varphi(1 - p + r_{n_0})$ implies

$$\psi(p) \leq \psi(1 - p + r_{n_0}) < \psi(1 - p) + \varepsilon.$$

Thus, $\psi(p) \leq \psi(1 - p)$ and, replacing $p$ by $1 - p$, $\psi(p) \geq \psi(1 - p)$, which gives the assertion of the lemma.

Theorem 3.3. For a von Neumann algebra $M$, the following conditions are equivalent:

(i) $M = C1$ or $M$ has no direct summand of type $I_n$, $n < \infty$;

(ii) each state on $M$ is strongly exclusive;

(iii) each state on $M$ is exclusive.

Proof. (i) $\Rightarrow$ (ii). We may assume that $M$ has no direct summand of type $I_n$, $n < \infty$. By virtue of Lemma 3.2 and Proposition 1.6, we can use Proposition 2.4 with $\alpha = \beta = 1/2$ to obtain (ii).

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Assume the contrary. There are two cases:

1°. $M$ has a nonzero commutative direct summand $K$ and $M \neq C1$. We may assume that $M = K \oplus N$ for some nonzero von Neumann subalgebra $N$. Let $\varphi$ be state on $K$ such that $\varphi(\text{Proj} K) = \{0, 1\}$, which exists by Lemma 3.1, and let $\psi$ be any state on $N$. Put

$$\varphi_1 = (2/3)\varphi \oplus (1/3)\psi \quad \text{and} \quad \varphi_2 = (3/4)\varphi \oplus (1/4)\psi.$$

Then the inequality

$$\varphi_i(p \oplus r) \leq \varphi_i(q + s)$$

is equivalent to the alternative $\varphi(p) < \varphi(q)$ or $\varphi(p) = \varphi(q)$, $\psi(p) \leq \psi(q)$.

Hence, $\varphi_1$ and $\varphi_2$ are distinct and equivalent.

2°. $M$ has a nonzero direct summand $M_n$ of type $I_n$, $1 < n < \infty$. Write
$M_n$ as $F_n \otimes K$ where $F_n$ is a factor of type $I_n$ and $K$ is commutative. By Theorem 4.5, there are two distinct and equivalent states $\psi_1$, $\psi_2$ on $F_n$. Let $\varphi$ be a state on $K$ such that $\varphi(\text{Proj } K) = \{0, 1\}$ (Lemma 3.1). Define a Fubini mapping $h$ on $F_n \otimes K$ by $h(a \otimes b) = a \varphi(b)$ (cf. [6; Section 9.8]). Since $F_n \otimes K$ is norm-dense in $M_n$, we may extend $h$ to the whole algebra $M_n$. It is easy to see that $h$ is a homomorphism of $M_n$ onto $F_n$. Hence, $\psi_1 \circ h$ and $\psi_2 \circ h$ are distinct and equivalent states on $M_n$ which can be extended to distinct and equivalent states on $M$ in an obvious way. Thus, the proof of the theorem is finished.

4. Equivalence and similarity of states in factors of type $I_n$, $n < \infty$.

Throughout this section, $M$ is a factor of type $I_n$, $n < \infty$, $\tau$ is the normalized trace on $M$, $\zeta = \{e \in \text{Proj } M : \tau(e) = 1/n\}$ and $\zeta_p = \zeta \cap \mathcal{L}_p$ for any $p \in \text{Proj } M$. The following lemma generalizes Lemma 2 from [3].

**Lemma 4.1.** Let $\varphi = \tau(v \cdot)$ and $\psi = \tau(w \cdot)$ be two arbitrary hermitian functionals on $M$ with density operators

$$v = \sum_{i=1}^{k} \alpha_i p_i, \quad w = \sum_{i=1}^{m} \beta_i q_i,$$

where $p_1, \ldots, p_k$ (respectively $q_1, \ldots, q_m$) are nonzero mutually orthogonal projections from $M$,

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{m} q_i = 1,$$

and $\alpha_1 < \ldots < \alpha_k$, $\beta_1 < \ldots < \beta_m$. If $\varphi$ is not a multiple of $\tau$ and

$$\varphi(e) < \varphi(f) \quad \text{implies} \quad \psi(e) < \psi(f) \quad \text{for} \quad e, f \in \zeta,$$

then $k = m$, $p_i = q_i$ for $i = 1, \ldots, k$ and $\beta_i = \gamma \alpha_i + \delta$ for some $\gamma, \delta \in \mathbb{R}, i = 1, \ldots, k$.

**Proof.** Take $j < k$ and assume that $m \geq j$ and $p_i = q_i$ for $i < j$. Then

1. $p_j + \ldots + p_k = q_j + \ldots + q_m$;
2. $e \in \zeta_{p_j}, f \in \zeta_{p_j} + \ldots + \zeta_{p_k} \setminus \zeta_{p_j}$ implies $\varphi(e) < \varphi(f)$.

Take $e, f \in \zeta_{p_j}$ and $\varepsilon > 0$. We can choose $g \in \zeta_{p_j} + \ldots + \zeta_{p_k} \setminus \zeta_{p_j}$ so that $\|f - g\| < \varepsilon$. By (2), $\varphi(e) < \varphi(g)$, which implies $\psi(e) < \psi(g) < \psi(f) + \varepsilon \|\psi\|$. Hence

3. $e, f \in \zeta_{p_j}$ implies $\psi(e) = \psi(f)$.
By (1) and (2), \( f \in \zeta_{p_1 + \ldots + p_k} \setminus \zeta_{p_i} \) implies \( f \in \zeta_{q_1 + \ldots + q_m} \setminus \zeta_{q_i} \), so that 
\( \zeta_{q_i} \subset \zeta_{q_j} \). Since \( \zeta_{q_i} \neq \emptyset \) and \( \zeta_{p_i} \subset \zeta_{q_1 + \ldots + q_m} \), therefore (3) implies \( \zeta_{q_i} = \zeta_{p_i} \), thus, \( p_j = q_i \) and \( m \geq j + 1 \).

We have proved so far that \( m \geq k \) and that \( p_i = q_i \) for \( i < k \). Since \( \varphi \) is not a multiple of \( \tau \), we must have \( k \geq 2 \), which implies \( p_1 = q_1 \). Replacing \( \varphi \) by \( -\varphi \) and \( \psi \) by \( -\psi \), we get \( p_k = q_m \), which gives \( k = m \) and \( p_i = q_i \) for \( i = 1, \ldots, k \).

Choose now, for each \( i = 1, \ldots, k \), a projection \( e_i \in \zeta_{p_i} \). Let
\[
f_{\lambda} = \lambda e_1 + (1 - \lambda)e_k + \lambda^{1/2}(1 - \lambda)^{1/2}(u + u^*)
\]
where \( u^*u = e_1 \) and \( uu^* = e_k \) (\( 0 \leq \lambda \leq 1 \)). For \( i = 2, \ldots, k - 1 \), we choose \( \lambda_i \) so that \( \beta_i = \lambda_i \beta_1 + (1 - \lambda_i)\beta_k \). Then \( \psi(e_i) = \psi(f_{\lambda_i}) \), which implies \( \varphi(e_i) = \varphi(f_{\lambda_i}) \). The last equality yields \( \alpha_i = \lambda_i\alpha_1 + (1 - \lambda_i)\alpha_k \), and the existence of \( \gamma \) and \( \delta \) such that \( \beta_i = \gamma\alpha_i + \delta \) follows. This ends the proof of the lemma.

Now, let \( \varphi \) be a state on \( M \). We shall examine the following sets of states:
\[
X_\varphi = \{ \psi : \varphi(p) < \varphi(q), \tau(p) = \tau(q) \text{ imply } \psi(p) < \psi(q) \text{ for } p, q \in \text{Proj } M \};
\]
\[
Y_\varphi = \{ \psi : \varphi(p) < \varphi(q), \tau(p) \neq \tau(q) \text{ imply } \psi(p) < \psi(q) \text{ for } p, q \in \text{Proj } M \};
\]
\[
Z_\varphi = X_\varphi \cap Y_\varphi.
\]

Observe that
\[
(4) \quad E(\varphi) = \{ \psi \in Z_\varphi : \varphi \in Z_\varphi \} \subset Z_\varphi;
\]
\[
(5) \quad \psi \in S(\varphi) \text{ iff } (1 - \varepsilon)\psi + \varepsilon\varphi \in Z_\varphi \quad \text{for each } 0 < \varepsilon \leq 1.
\]

The following characteristic \( \delta_\varphi \) of \( \varphi \) will play an important role in the sequel:

\[
\delta_\varphi = \gamma_1 + \ldots + \gamma_s - \gamma_{n-s+2} - \ldots - \gamma_n \quad \text{with } n = 2s \text{ or } n = 2s - 1
\]

\[
(\delta_\varphi = \gamma_1 \quad \text{for } n = 1, 2),
\]

where \( \varphi = \text{tr}(t \cdot) \) and
\[
(6) \quad r = \sum_{i=1}^{n} \gamma_i e_i
\]

with \( \gamma_1 \leq \ldots \leq \gamma_n \) and mutually orthogonal \( e_i \in \zeta \) (\( \sum_{i=1}^{n} e_i = 1 \)). Obviously, \( \delta_\varphi \leq \gamma_1 \leq 1/n \), and \( \delta_\varphi = 1/n \text{ iff } \varphi = \tau \). Moreover, if \( \varphi \neq \tau \), \( \gamma > 0 \) and \( \psi = \gamma\varphi + (1 - \gamma)\tau \), then
\[
(7) \quad \psi \geq 0 \text{ and then } \delta_\psi > 0 \text{ iff } \gamma < (1 - n\delta_\varphi)^{-1}.
\]
Lemma 4.2. $X_\varphi$ is the set of all states on $M$, and

$$X_\varphi = \{ \psi = \gamma \varphi + (1-\gamma)\tau; \gamma > 0, \psi \geq 0 \} \text{ for } \varphi \neq \tau.$$ 

Proof. The assertion is, of course, valid for $\varphi = \tau$. If $\varphi \neq \tau$, the inclusion \(\supseteq\) is obvious and the inclusion \(\subseteq\) is an immediate consequence of Lemma 4.1.

Lemma 4.3. If $\delta_\varphi > 0$, then $Y_\varphi = \{ \psi; \delta_\psi > 0 \}$.

Proof. Step 1. Let $\varphi = \text{tr}(v \cdot \cdot)$ with $v$ given by (6). Then the condition $\delta_\varphi > 0$ is equivalent to:

$$\tau(p) < \tau(q) \text{ implies } \varphi(p) < \varphi(q) \text{ for } p, q \in \text{Proj } M.$$ 

In fact, it is not difficult to check that the following conditions are equivalent:

$$\gamma_1 + \ldots + \gamma_s > \gamma_{n-s+2} + \ldots + \gamma_n, \text{ where } n = 2s \text{ or } n = 2s-1;$$

$$\gamma_1 + \ldots + \gamma_j > \gamma_{n-j+2} + \ldots + \gamma_n, \text{ for each } j = 1, \ldots, n;$$

$$\varphi(e_1 + \ldots + e_j) = \gamma_1 + \ldots + \gamma_j > \gamma_{n-1+1} + \ldots + \gamma_n = \varphi(e_{n-i+1} + \ldots + e_n) \text{ for each } 0 \leq i < j \leq n;$$

$$\varphi(q) > \varphi(p) \text{ for each } p, q \in \text{Proj } M, \tau(p) = i/n < j/n = \tau(q).$$

Step 2. Assume that $\delta_\varphi > 0$. By Step 1, the condition $\varphi(p) < \varphi(q)$, $\tau(p) \neq \tau(q)$ is equivalent to $\tau(p) < \tau(q)$. Using Step 1 once more (with $\varphi$ replaced by $\psi$), we get $\psi \in Y_\varphi$, iff $\delta_\psi > 0$.

Lemma 4.4. If $\delta_\varphi < 0$, then $Z_\varphi = \{ \varphi \}$.

Proof. Let $\delta_\varphi < 0$. By Lemma 4.2, it is enough to prove that $\psi = \gamma \varphi + (1-\gamma)\tau \in Y_\varphi$ implies $\gamma = 1$. There are two cases to be considered:

1. $\gamma_s > 0 \text{ (n = 2s or n = 2s-1, } \gamma_i \text{ as in (6)).}$ Put

$$p = e_1 + \ldots + e_{s-1}, \quad q = e_{n-s+2} + \ldots + e_n.$$ 

Then $pq = 0$, $p \sim q$ and $\varphi(p) < \varphi(p+e_s) < \varphi(q)$. For any sufficiently small $\varepsilon > 0$, there are, by Proposition 1.1, projections $r_1, r_2 \sim p$ satisfying

$$\varphi(r_1) = \varphi(p+e_s) - \varepsilon, \quad \varphi(r_2) = \varphi(p+e_s) + \varepsilon.$$ 

Thus

$$\tau(r_1) = \tau(r_2) = \tau(p+e_s) - 1/n, \psi(r_1) < \psi(p+e_s) < \psi(r_2)$$ 

and, consequently, $-\varepsilon \gamma/n < 1 - \gamma < \varepsilon \gamma/n$. Hence $\gamma = 1$. 

2'. \( \gamma_s = 0 \). Then \( \psi \geq 0 \) implies \( \gamma \leq 1 \) and, for \( p,q \) and \( r_2 \) as in 1', we have
\[ \psi(p + e_s) < \psi(r_2) \] and \( 1 - \gamma < \varepsilon \gamma / n \). Hence \( \gamma = 1 \).

The proof is finished.

We sum up our results in the following

**Theorem 4.5.** For a state \( \varphi \) on \( M \):

(i) \( \delta_\varphi < 0 \) implies \( E(\varphi) = S(\varphi) = \{ \varphi \} \);
(ii) \( \delta_\varphi = 0 \) implies \( E(\varphi) = \{ \varphi \} \); \( S(\varphi) = \{ \gamma \varphi + (1 - \gamma) \tau : 0 \leq \gamma \leq 1 \} \);
(iii) \( 0 < \delta_\varphi < 1/n \) implies \( E(\varphi) = \{ \psi = \varphi \psi + (1 - \gamma) \tau : 0 < \gamma < (1 - n \delta_\varphi)^{-1} \} \);
\[ S(\varphi) = \{ \psi = \gamma \varphi + (1 - \gamma) \tau : 0 \leq \gamma \leq (1 - n \delta_\varphi)^{-1} \} \];
(iv) \( \delta_\varphi = 1/n \) is equivalent to \( \varphi = \tau \), and then, \( E(\tau) = \{ \tau \} \); \( S(\tau) = \{ \psi : \delta_\psi \geq 0 \} \).

**Proof.** (i). Follows from (4), (5), and Lemma 4.4.

(iii). Let \( 0 < \delta_\varphi < 1/n \). By Lemmas 4.2 and 4.3,
\[ Z_\varphi = \{ \psi = \gamma \varphi + (1 - \gamma) \tau : \gamma > 0, \psi \geq 0, \delta_\psi > 0 \} \].
Thus \( 0 < \delta_\psi < 1/n \) and \( \varphi = \gamma^{-1} \psi + (1 - \gamma^{-1}) \tau \in Z_\varphi \) for \( \psi = \gamma \varphi + (1 - \gamma) \tau \in Z_\varphi \).
By (4) and (7),
\[ E(\varphi) = Z_\varphi = \{ \psi = \gamma \varphi + (1 - \gamma) \tau : 0 < \gamma < (1 - n \delta_\varphi)^{-1} \} \].

Now, the form of \( S(\varphi) \) is easily obtained from (5).

(iv). By Lemmas 4.2 and 4.3, \( Z_\tau = \{ \varphi : \delta_\varphi > 0 \} \), and \( \tau \notin Z_\varphi \) if \( \varphi \neq \tau \). Hence, \( E(\tau) = \{ \tau \} \) by (4), and \( S(\tau) = \{ \varphi \} : \delta_\varphi \geq 0 \} \) by (5).

(ii). Suppose \( \delta_\varphi = 0 \). By (i), (iv), and (iii) with (7), \( \tau \notin E(\varphi) \) if \( \delta_\varphi \neq 0 \). Hence
\[ E(\varphi) = \{ \varphi ; \psi \in E(\varphi) \} \subset \{ \varphi ; \delta_\varphi = 0 \} \].

Since \( E(\varphi) \subset X_\varphi \), the equality \( E(\varphi) = \{ \psi \} \) follows from Lemma 4.2. Similarly, by (i),

(8)
\[ S(\varphi) = \{ \varphi ; \psi \in S(\varphi) \} \subset \{ \varphi ; \delta_\varphi \geq 0 \}. \]

By (5) and Lemma 4.2,

(9)
\[ S(\varphi) \subset X_\varphi = \{ \varphi = \lambda \psi + (1 - \lambda) \tau : \lambda \geq 0, \varphi \geq 0 \}. \]

If \( \varphi = \lambda \psi + (1 - \lambda) \tau \geq 0 \) and \( 0 < \delta_\varphi < 1/n \), then, by (iii),

(10)
\[ \psi \in S(\varphi) = \{ \gamma \varphi + (1 - \gamma) \tau : 0 \leq \gamma \leq (1 - n \delta_\varphi)^{-1} = \lambda^{-1} \}. \]
By (9), (10) and (iv),

\[ S(\psi) \subset \{ \varphi = \lambda \psi + (1 - \lambda)\tau : \varphi \geq 0, 0 < \delta_\varphi < 1/n \} \cup \{ \tau, \psi \} \]

\[ \subset \{ \varphi : \psi \in S(\varphi) \} = S(\psi) : \]

Now, if \( \varphi = \lambda \psi + (1 - \lambda)\tau \in S(\psi) \), then, by (9), \( \lambda \geq 0 \). Also, \( \delta_\varphi = (1 - \lambda)/n \geq 0 \)
by (8). Hence, \( 0 \leq \lambda \leq 1 \), and the proof of the theorem is finished.

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