A HOLOMORPHIC REPRODUCING KERNEL FOR Kohn-Nirenberg Domains in $\mathbb{C}^2$

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Introduction.

Let $D$ be a smooth domain in $\mathbb{C}^2$. Any Leray map $\Psi = (\Psi_1, \Psi_2): \Omega \times \partial \Omega \rightarrow \mathbb{C}^2$ gives rise to a Cauchy Fantappié formula which reproduces holomorphic functions that are continuous up to the boundary of $\Omega$. In general, it will be impossible to find a Leray map which is holomorphic in the first variable, therefore the Cauchy-Fantappié form will not be holomorphic in this variable either.

For smooth strictly pseudoconvex domains it was proved among others by Henkin [3] that Leray maps and Cauchy-Fantappié forms that are holomorphic in the first variable exist. Range and Siu [5] obtained a kind of Cauchy-Fantappié formula for intersections of smooth strictly pseudoconvex domains.

In this paper we consider the so-called Kohn-Nirenberg domains in $\mathbb{C}^2$:

$$\Omega = \{ \omega \in \mathbb{C}^2 : \text{Re} \omega_2 + P(\omega_1) < 0 \},$$

where $P$ is a real valued homogeneous polynomial in $\omega_1$ and $\bar{\omega}_1$ with $\Delta P > 0$ when $\omega_1 \neq 0$. To avoid problems stemming from the unboundedness of $\Omega$ we will mainly be concerned with $\Omega_R = \Omega \cap \{ |\omega| < R \}$. In general, it is impossible to find a holomorphic Leray map defined in $\Omega \times \partial \Omega$. However, it was shown by the first author [2] that such a map with fairly good properties exists on $\Omega \times \Sigma$, where $\Sigma = \partial \Omega \setminus \{ \zeta_1 = 0 \}$.

We modify this map slightly and study the related Cauchy-Fantappié formula on $\Omega_R$. Formally this looks exactly like what one would expect in view of the Range-Siu result. Although the kernel we obtain blows up at $\zeta_1 = 0$, we will show that it is integrable over the boundary. It reproduces functions in $A(\Omega_R)$ and maps $C(\partial \Omega_R)$ into $H(\Omega)$.

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1. Preliminaries.

Let \( \Omega = \{(w_1, w_2) \in \mathbb{C}^2 ; \text{Re} \, w_2 + P(w_1) < 0\} \), where \( P \) is a homogeneous polynomial of degree \( 2k \), \( \Delta P > 0 \) when \( w_1 \neq 0 \).

Let \( \Sigma = \{w \in \partial \Omega, w_1 \neq 0\} \). Let \( \Omega_R = \Omega \cap B(0, R), \Sigma_R = \Sigma \cap B(0, R) \).

In this section we give an account of results concerning \( \Omega \). All proofs can be found in [1] or [2].

Let \( \zeta = (\zeta_1, \zeta_2) \in \Sigma, \theta_1 = \text{arg} \, \zeta_1 \).

**Lemma 1.1.** For every \( \zeta_1 \) there exists a unique harmonic polynomial of the form \( \text{Re} \, \alpha w_1^{2k} = P(w_1) + O(|w_1 - \zeta_1|^2) \). The constant \( \alpha = \alpha(\theta_1) \) depends real analytically on \( \theta_1 \).

Write \( P_1(w_1) = P(\theta_1, w_1) = P(w_1) - \text{Re} \, \alpha(\theta_1) w_1^{2k} \).

**Lemma 1.2.** There exist \( \delta > 0, \gamma > 0 \) independent of \( \theta \), such that if

\[
|\text{arg} \, w_1 - \theta_1| \leq \delta, \text{ then } \frac{1}{\gamma} |\text{arg} \, w_1 - \theta_1|^2 |w_1|^{2k} \leq P_1(w) \leq \gamma |\text{arg} \, w_1 - \theta_1|^2.
\]

Introduce

\[
F_1(w_1) = F(\zeta_1, w_1) = w_1^{2k}(w_1 - \zeta_1)^2 e^{-(i(2k+2))\theta_1}.
\]

**Lemma 1.3.** There exist \( \delta > 0, \gamma > 0 \) such that if \( |\text{arg} \, w_1 - \theta_1| \leq \delta \), then

\[
\text{Re} \, F_1(w_1) \geq \frac{1}{2} |w_1|^{2k}|w_1 - \zeta_1|^2 - \gamma |w_1|^{2k+2}(\gamma |w_1 - \theta_1|^2).
\]

Let \( P_\varepsilon(w_1) = P(w_1) - \varepsilon |w_1|^{2k} \); \( \varepsilon \) will be chosen very small below, but at least so small that \( P_{3\varepsilon} \) is strictly subharmonic if \( w_1 \neq 0 \). We change coordinates as follows

\[
\tilde{w}_1 = w_1, \quad \tilde{w}_2 = \tilde{w}_2(\zeta_1, w, M) = w_2 + \alpha(\theta_1) w_1^{2k} - \varepsilon/M \, F_1(w_1), \quad M \gg 0.
\]

Let

\[
Q_1(w_1) = Q(\zeta_1, w_1, M) = P_1(w_1) + \varepsilon/M \, \text{Re} \, F_1(w_1).
\]

Then in these coordinates

\[
\Omega = \{\text{Re} \, \tilde{w}_2 + Q_1(\tilde{w}_1) < 0\}.
\]

**Lemma 1.4.** For every \( R > 0 \) there exists \( M > 0 \) such that if \( \tilde{w} \in \Omega \) and

\[
|\zeta_1|, |\tilde{w}_1| \leq R,
\]

then

\[
\tilde{w} \in \Omega_\varepsilon := \{\text{Re} \, \tilde{w}_2 + P_\varepsilon(\tilde{w}_1) - \text{Re} \, \alpha(\theta_1) w_1^{2k} < 0\}.
\]

By continuity we can find arcs \( I_1, \ldots, I_l \) which cover the unit circle, are centered at \( e^{i\theta^*}, \ldots, e^{i\theta^l} \), respectively, and are shorter than \( \delta \) such that if \( e^{i\theta^j} \in I_j \), then
\[ \Omega_e \subset \Omega_j := \{ \text{Re} \, \hat{w}_2 + P_{2e}(\hat{w}_1) - \text{Re}(\alpha(\theta^j)\hat{w}_1^{2k}) < 0 \}. \]

Each \( \Omega_j \) is contained in the even larger pseudoconvex domain

\[ \Omega'_j := \{ \text{Re} \, \hat{w}_2 + P_{3e}(\hat{w}_1) - \text{Re}(\alpha(\theta^j)\hat{w}_1^{2k}) < 0 \}. \]

**Lemma 1.5.** If \( \varepsilon > 0 \) is small enough, \(|\arg \hat{w}_1 - \theta_1| \leq \delta, |\zeta_1|, |\tilde{w}_1| \leq R\), then

\[ Q_1(\hat{w}_1) \sim (\arg \hat{w}_1 - \theta_1)^2|\tilde{w}_1|^{2k} + |\tilde{w}_1|^{2k}|\tilde{w}_1 - \zeta_1|^2. \]

**Lemma 1.6.** If \( \varepsilon > 0 \) is small enough, \(|\arg w_1 - \theta_1| = \delta\), and \( e^{i\theta_1} \in I_j \), then

\[ P_{3e}(\tilde{w}_1) - \text{Re}(\alpha(\theta^j)\tilde{w}_1^{2k}) > 0. \]

We take \( \varepsilon \) so small that the above requirements are satisfied and such that the sets

\[ \{ \tilde{w}_1 : P_{3e}(\tilde{w}_1) - \text{Re}(\alpha(\theta^j)\tilde{w}_1^{2k}) \leq 0 \} \]

are the closures of their interior for \( j = 1, \ldots, l \).

In the \( \hat{w} \) coordinates one has \( \hat{\zeta}_1 = \zeta_1, \hat{\zeta}_2 = \zeta_2 + \alpha(\theta_1)\zeta_1^{2k} \). Let \( \Phi : \mathbb{C}^2 \to \mathbb{C}^2 \) be given by

\[ \Phi(\hat{w}_1, \hat{w}_2) = (\hat{w}_1, \hat{w}_2^{2k} + \tilde{\zeta}_2) = (\hat{w}_1, \tilde{w}_2). \]

Let \( \hat{\Omega} = \phi^{-1}(\Omega) \), \( \hat{\Omega}_e = \phi^{-1}(\Omega_e) \), \( \hat{\Omega}_j = \phi^{-1}(\Omega_j) \), and \( \hat{\Omega}_j = \phi^{-1}(\Omega_j) \). Note that \( \text{Re} \, \tilde{\zeta}_2 = 0 \). Let \( \hat{\zeta} = \phi^{-1}(\zeta) \). One has

\[ \hat{\Omega}_j = \{ \text{Re} \, \hat{w}_2^{2k} + P_{2e}(\hat{w}_1) - \text{Re}(\alpha(\theta^j)\hat{w}_1^{2k}) < 0 \}, \]

\[ \hat{\Omega}_j = \{ \text{Re} \, \hat{w}_2^{2k} + P_{3e}(\hat{w}_1) - \text{Re}(\alpha(\theta^j)\hat{w}_1^{2k}) < 0 \}. \]

Let \( S_0^l, S_l^j \) be the connected components of \( \hat{\Omega}_j \cap \{ \tilde{w}_2 = 0 \} \). Say \( S_0^l \) is the component of \( \hat{\zeta} \).

Fix any of the \( \hat{\Omega}_j \). To \( \hat{\Omega}_j \) are associated two open Riemann surfaces \( \hat{R}_j \subset R_j \) with the following properties: There is a holomorphic map \( \Pi : R_j \times \mathbb{C} \to \mathbb{C}^2 \) of the form \( \Pi(p, t) = (\alpha(p)t, \beta(p)t) \) where \( \alpha, \beta \) are holomorphic functions on \( R_j \) without common zeros; \( \Pi \) is nonsingular when \( t \neq 0 \); there is an open set \( \hat{\Omega}_j \) in \( R_j \times \mathbb{C} \) such that \( \Pi(\hat{\Omega}_j) \to \hat{\Omega}_j \) is a biholomorphism. Moreover

\[ \hat{\Omega}_j = \bigcup_{p \in \hat{R}_j} \{ p \} \times \hat{S}^p, \]

where \( \hat{S}^p \) is a nonempty connected open sector in \( \mathbb{C} \). For each complex line \( L \subset \mathbb{C}^2 \) through \( 0 \), \( L \cap \hat{\Omega}_j \) is a union of finitely many disjoint, open, connected sectors \( C_1 \ldots C_{m_L} \) and there exist \( q_1 \ldots q_{m_L} \) in \( \hat{R}_j \) such that \( \Pi \) is a linear isomorphism between \( q_i \times \hat{S}^{q_i} \) and \( C_i \).

Let \( p_0 \ldots p_{n_j} \in \hat{R}_j \) be associated to the sectors \( S_0^l \ldots S_{n_j}^l \) in \( \{ \tilde{w}_2 = 0 \} \cap \hat{\Omega}_j \).
There are also $2k$ sectors $C_1 \ldots C_{2k}$ in $\tilde{\Omega}_j \cap \{\tilde{w}_1 = 0\}$ with associated points $q_1 \ldots q_{2k} \in \tilde{R}_j$.

We fix a holomorphic function $\phi : R_j \to \mathbb{C}$, nowhere identically vanishing while $\phi$ vanishes at least to order $2k+1$ at each of the points $p_1, \ldots, p_n$, $q_1, \ldots, q_{2k}$.

2. Construction of the Leray map.

We start with the meromorphic function $1/(\zeta_1 - \tilde{w}_1)$ on $\tilde{\Omega}_j$ and pull it back to $R_j \times \mathbb{C}$ to get the meromorphic function $g = 1/(\zeta_1 - \alpha(p)t)$. Fix a small neighborhood $V$ of $\{p_1, \ldots, p_n\}$ in $R_j$. Observe that $g$ is holomorphic as a function of $\zeta_1$, $p$ and $t$ on $S_0 \times (\tilde{\Omega}_j \cap (V \times \mathbb{C}))$ and that there

$$|g| \leq \inf\{1/|\zeta_1|, 1/|t|\} \quad (\text{i.e. } |g| \leq \text{const.}\{1/|\zeta_1|, 1/|t|\}).$$

Let $\chi \in C^0_\delta(R_j)$, $\chi \equiv 1$ on a neighborhood $V' \subset V$ of $p_1, \ldots, p_n$, $p_0 \notin V$ and $\text{supp } \chi \subset V$. We may assume that $\phi \neq 0$ on $V - \{p_1, \ldots, p_n\}$.

We define a $\bar{\partial}$-closed form $\lambda = \lambda_{\zeta_1}$ on $\tilde{\Omega}_j$ by

$$\lambda = \begin{cases} 
\bar{\partial} \chi g/\phi & \text{if } p \in \text{supp } \bar{\partial} \chi \\
0 & \text{if } p \notin \text{supp } \bar{\partial} \chi.
\end{cases}$$

Then $||\lambda||_{L^2} \leq C||\ln|\zeta_1||^{1/2}$ for a fixed constant $C > 0$. We now apply Hörmanders theory for solving the $\bar{\partial}$-equation, cf. [4]. Because $\lambda_{\zeta_1}$ is an $L^2 - (0, 1)$ form for all $\zeta_1 \in S_0$, we can use the same $L^2$-space for solving the $\bar{\partial}$-equation for all $\zeta_1 \in S_0$. In particular, choosing the solution in the closure of the range of $\bar{\partial}^*$, we obtain a linear solution operator $T$ that satisfies $\bar{\partial}Tf = f$ and $||Tf||_{L^2} \leq C||f||_{L^2}$ for all $\bar{\partial}$-closed $(0, 1)$ forms with coefficients in $L^2$.

We observe that $T\lambda_{\zeta_1}$ is a holomorphic function of $\zeta_1$ on $S_0$, because $T$ is linear.

Next we define $\Psi^1_t(p, t, \zeta_1) = \chi g - \phi T\lambda$. We have

$$||\Psi^1_t||_{L^2} \leq ||\ln|\zeta_1||^{1/2} + 1.$$}

We push $\psi^1_t$ down to $\Omega_j$ to obtain $\psi^1_t(\tilde{w}, \zeta_1) = \psi^1_t(II^{-1}(\hat{w}), \zeta_1)$ and return to the $\tilde{\cdot}$ coordinates as follows. Let $\omega$ be a primitive $2k$-root of unity. Define

$$\psi^2_t(\tilde{w}_1, \tilde{w}_2, \zeta_1) = \frac{1}{2k} \sum_{j=1}^{2k} \psi^1_t(\tilde{w}_1, \omega^j \tilde{w}_2, \zeta_1).$$

This function is holomorphic in $(\tilde{w}_1, \tilde{w}_2, \zeta_1)$, hence it can be pushed down to $\Omega_j$ yielding the holomorphic function

$$\psi^4_t(\tilde{w}_1, \tilde{w}_2, \zeta_1) = \psi^1_t(\tilde{w}_1, (\tilde{w}_2 - \zeta_2)^{1/2k}, \zeta_1).$$

Now $\psi^4_t(\tilde{w}, \zeta)$ is defined implicitly by $1 = (\zeta_1 - \tilde{w}_1)\psi^4_t + (\zeta_2 - \tilde{w}_2)\psi^4_t$. 


Finally this can be written in the original \( w \)-coordinates as follows

\[
1 = \psi_1^4(\tilde{w}, \tilde{\zeta})(\zeta_1 - w_1) + \psi_2^4(\tilde{w}, \tilde{\zeta})[\zeta_2 - w_2 + G(\zeta_1, w_1)(\zeta_1 - w_1)].
\]

where

\[
G(\zeta_1, w_1) = \alpha(\theta_1)(\zeta_1^{2k} - w_1^{2k})/(\zeta_1 - w_1) + (e/M)F(\zeta_1, w_1)/(\zeta_1 - w_1)
\]

and we define the map \( \psi^i \) by

\[
\psi_1^4(w, \zeta) = \psi_1^4(\tilde{w}, \tilde{\zeta}) + \psi_2^4(\tilde{w}, \tilde{\zeta})G(\zeta, \tilde{w})
\]

\[
\psi_2^4(w, \zeta) = \psi_2^4(\tilde{w}, \tilde{\zeta}).
\]

The map \( \psi^i \) satisfies the requirements for a Leray map, but only for \( \zeta \in \Sigma \)

with \( e^{i\theta_1} \in I_j \).

A global map is now easily defined using a partition of unity. Let \( \chi_j \in C^\infty_0(I_j), \chi_j \geq 0, \sum_j \chi_j = 1 \). Define

\[
(2) \quad \psi_i(w, \zeta) = \sum_{j=1}^I \chi_j(\zeta_1/|\zeta_1|)\psi^i_j(w, \zeta) \quad (i = 1, 2) \quad \text{for} \quad \zeta \in \Sigma.
\]

Similarly we can push down each of the functions \( \chi, g, \) and \( \phi T\lambda \). On the \( \tilde{\zeta} \) level this gives functions \( \chi^{4, j}, g^{4, j}, \) and \( (\phi T\lambda)^{4, j} \) living on \( \Omega_j \). We have

\[
\psi_4^4 = \psi_4^{4, j} = \chi^{4, j}g^{4, j} - (\phi T\lambda)^{4, j},
\]

where we used that \( g^{3, j} \) is independent of \( \tilde{w}_2 \).

3. Estimates concerning the Leray map.

**Lemma 3.1.** There exists a constant \( C > 0 \) such that for \( \zeta \in \Sigma_R, \zeta_1/|\zeta_1| \in I_j \)

and \( \tilde{w} \in \tilde{\Omega} \) the following holds:

1. \( \chi^{4, j}(\tilde{w}, \tilde{\zeta}) \equiv 1 \) if \( \tilde{w}_1 \notin S_0^j \) and \( |\tilde{w}_2 - \tilde{\zeta}_2| < 1/C|\tilde{w}_1|^{2k} \).

2. \( \chi^{4, j}(\tilde{w}, \tilde{\zeta}) \equiv 0 \) if \( \tilde{w}_1 \in S_0^j \) or \( |\tilde{w}_2 - \tilde{\zeta}_2| > C|\tilde{w}_1|^{2k} \).

3. \( (\phi T\lambda)^{4, j}(\tilde{w}, \tilde{\zeta}) \) has a zero at \( \tilde{w}_2 = \tilde{\zeta}_2 \).

4. \( g^{4, j}(\tilde{w}, \tilde{\zeta}) = 1/(\tilde{w}_1 - \tilde{\zeta}_1) \).

**Proof.** All these properties are direct consequences of the definition of \( \chi, \phi, \) and \( g \) and their transformation to \( \tilde{\zeta} \) coordinates.

**Lemma 3.2.** Let \( K \) be a compact subset of \( \Omega_R \). Then there exists a positive constant \( \kappa \) such that for every \( \zeta \in \Sigma_R, \zeta_1/|\zeta_1| \in I_j \) the set \( \tilde{K}_3 \), the pullback of \( K \) to \( \tilde{\Omega}_j \) has distance greater than \( \kappa \) to the boundary of \( \tilde{\Omega}_j \).
Proof. The compactum $K$ is for some positive $\delta$ contained in \(\{\text{Re} \, w_2 - P(w_1) \leq -\delta\}\). In \(\bar{\zeta}\) coordinates this set corresponds to \(\bar{K}_{\zeta} := \{\text{Re} \, \hat{w}_2 - Q_1(\hat{w}_1) \leq -\delta\}\). As the gradient of \(\text{Re} \, \hat{w}_2(\zeta_1) - Q_1(\zeta_1, \hat{w}_1)\) remains uniformly bounded as $\zeta \in \Sigma_R$, it follows that \(\text{dist}(\bar{K}_{\zeta}, \partial \tilde{\Omega}) \geq \delta' > 0\). So for $\zeta_1 / |\zeta_1| \in I_j$:

\[
\text{dist}(\bar{K}, \partial \Omega_j) \geq \delta'
\]

by Lemma 1.4 and the observations following it. Pulling back to $\hat{\Omega}_j$ is done by a translation in the $\text{Im} \, \zeta_2$ direction, which has no influence on the distance to $\partial \Omega_j$, followed by taking the inverse image under a proper map which does not depend on $\zeta$. Hence, there is a compactum in $\hat{\Omega}_j$ which contains the pullbacks of all $\bar{K}_{\zeta}$. Finally \(\Pi : \hat{\Omega}_j \rightarrow \tilde{\Omega}_j\) is a biholomorphism and the lemma follows.

**Lemma 3.3.** For every compact $K \subseteq \Omega_R$ there exists a positive constant $\gamma(K)$ such that $|\zeta_1 - \hat{w}_1| > \gamma(K)$ on \(\text{supp} \, \chi^4, j\), and $|\zeta_2 - \hat{w}_2| > \gamma(K)$ on \(\text{supp} \, 1 - \chi^4, j\), for $w \in K$ and $\zeta \in \Sigma_R$, $\zeta_1 / |\zeta_1| \in I_j$.

**Proof.** Let $\bar{K}$ denote $K$ in the \(\bar{\zeta}\) coordinates. $\bar{K}$ depends on $\zeta$, but by Lemma 3.2 and its proof there exists $d > 0$ such that for $\zeta \in \Sigma_R$ distance $(\bar{K}, \partial \tilde{\Omega}) > d$. Hence

\[
|\hat{w}_1 - \zeta_1| < \frac{1}{2}d \Rightarrow |\hat{w}_2 - \zeta_2| > \frac{1}{2}d.
\]

By Lemma 3.1, $\chi^4, j = 0$ if $|\hat{w}_1|^{2k} < \frac{1}{2}d/\delta'$ or if $\hat{w}_1 \in S_0$. Therefore if $(\hat{w}, \zeta) \in \text{supp} \, \chi^4, j$, then

\[
|\hat{w}_1 - \zeta_1| > \min\{\frac{1}{2}d, d/2\Omega \cdot \min[1, \min \{\arg(\zeta_1 - \arg w_1)\}] := \gamma_1.
\]

$\gamma_1$ is strictly positive because $I_j \subseteq S_{\delta}$ and the sectors $S_{1/k}$ are separated.

Next there exists $\delta' > 0$ such that $\text{Re} \, \hat{w}_2 + Q_1(\hat{w}_1) \leq -\delta'$ on $\bar{K}$ for all $\zeta \in \Sigma_R$. Now $|\hat{w}_2 - \zeta_2| < \frac{1}{2}\delta'$ implies $|\text{Re} \, \hat{w}_2| < \frac{1}{2}\delta'$ because $\text{Re} \, \hat{\zeta}_2 = 0$. Hence $Q_1(\hat{w}_1) < -\frac{1}{2}\delta'$ and by Lemma 1.5, $|\arg \hat{w}_1 - \arg \zeta_1| > \delta$. By Lemma 1.6 and the remark following it, we conclude that $\hat{w}_1 \notin S_0$. From $Q_1(w_1) < -\frac{1}{2}\delta'$ we also infer that $|w_1| > A(\delta')^{1/2k}$, where $A$ is independent of $\zeta$, $|\zeta| < R$. Application of Lemma 3.1 gives $\chi^4, j(\hat{w}, \zeta) = 1$, if

\[
|\hat{w}_2 - \zeta_2| < \min\{\frac{1}{2}\delta', A^{2k}/C \delta'\} =: \gamma_2.
\]

Now take $\gamma(K) = \min\{\gamma_1, \gamma_2\}$. 


Proposition 3.4. For every compact \( K \subseteq \Omega_R \) the functions \( \psi_i, i = 1, 2 \), defined by (2) satisfy

\[
|\psi_i(w, \zeta)| \leq \|\ln|\zeta_1|\|^{1/2} + 1, \quad \zeta \in \Sigma_R, \ w \in K.
\]

Proof. It will be enough to show that \( |\psi_1^i(w, \zeta)| \leq \|\ln|\zeta_1|\|^{1/2} + 1 \), if \( \zeta_1/|\zeta_1| \in I_j, \ z \in \Sigma_R, \ w \in K \). By the definition of \( \psi_1^i \) this reduces to proving that the corresponding \( \psi_1^*, \ \psi_2^* \) are majorized by a constant times \( \|\ln|\zeta_1|\|^{1/2} + 1 \), because \( G(\zeta, \tilde{w}) \) remains bounded. Since \( \psi_1^i \) is the pushdown of \( \psi_1^i \) we have

\[
\|\psi_1^i(\tilde{w}, \tilde{\zeta})\|_K = \|\psi_1^i(p, t, \zeta)\|_{\overline{R}} \leq \|\psi_1^i(p, t, \zeta)\|_{L^2} \leq \|\ln|\zeta_1|\|^{1/2} + 1.
\]

We used Lemma 3.2 for the first inequality, while the last inequality is just (1).

Next, we deal with

\[
\psi_2^*(\tilde{w}, \zeta) = \frac{1 - (\zeta_1 - \tilde{w}_1)\psi_1^*(\tilde{w}, \zeta)}{\overline{\zeta}_2 - \tilde{w}_2}.
\]

We have for \( \zeta \in \Sigma_R, \ w \in K \), if \( |\zeta_2 - \tilde{w}_2| \geq \gamma(K)/2 \)

\[
|\psi_2^*| \leq \frac{2}{\gamma(K)} (\|\ln|\zeta_1|\|^{1/2} + 1),
\]

while if \( |\zeta_2 - \tilde{w}_2| \leq \gamma(K)/2 \), by Lemmas 3.1, 3.3, and the fact that \((\phi T \lambda)^{a,j}\) is holomorphic if \( |\zeta_2 - \tilde{w}_2| \leq \gamma(K) \)

\[
|\psi_2^*(w, \zeta)| = \left| \frac{1 - \chi^{a,j}(\tilde{w}, \zeta) + (\zeta_1 - \tilde{w}_1)(\phi T \lambda)^{a,j}}{\overline{\zeta}_2 - \tilde{w}_2} \right| \\
= \left| \frac{(\zeta_1 - \tilde{w}_1)(\phi T \lambda)^{a,j}}{\overline{\zeta}_2 - \tilde{w}_2} \right| \leq \|\cdot\|_{L^2} \leq \|\ln|\zeta_1|\|^{1/2} + 1.
\]

4. Reproducing kernels on \( \Omega_R \).

Let \( \psi \) be the Leray map for \( \Sigma \) as constructed in section 2, let \( \psi^2 \) be the Leray map for \( \partial B(0, R) \), that is

\[
\psi^2_i(w, \zeta) = -\frac{\overline{\zeta}_i}{(w \cdot \zeta - R^2)}, \quad \text{where} \ w \cdot \zeta = \sum w_i \overline{\zeta}_i, \ (\zeta, w) \in \partial B(0, R) \times B(0, R).
\]

We also need the map

\[
\psi^3_i(w, \zeta) = \frac{\zeta_i - w_i}{\|w - \zeta\|^2},
\]

associated with the Bochner Martinelli formula.
We introduce smooth cut-off functions as follows. Let \( \alpha \in C^\infty (R^+) \), \( 0 \leq \alpha \leq 1 \), \( \alpha(t) = 1 \) for \( t \leq 0 \), \( \alpha(t) = 0 \) for \( t \geq 1 \). Put
\[
\tau^1_\varepsilon (\zeta) = \alpha \left( \frac{|\zeta|_1 - \varepsilon_1}{\varepsilon_2} \right), \quad \varepsilon = (\varepsilon_1, \varepsilon_2), \varepsilon_i > 0,
\]
\[
\tau^2_\varepsilon (\zeta) = \alpha \left( \frac{|\zeta|-R - \varepsilon_1}{\varepsilon_2} \right), \quad \varepsilon = (\varepsilon_1, \varepsilon_2), \varepsilon_i > 0.
\]

We distinguish the following parts in \( \partial \Omega_R \)
\[
F_1 = \partial \Omega \cap B(0, R), \quad F_2 = \partial B(0, R) \cap \Omega, \quad F_3 = F_1 \cap F_2.
\]

We form for \((w, \zeta) \in \Omega_R \times \partial \Omega_R\)
\[
\psi^{\varepsilon, \eta}_i(w, \zeta) = [\tau^2_\varepsilon (\zeta) \psi^{\eta}_i(w, \zeta) + (1 - \tau^2_\varepsilon (\zeta)) \psi_i(w, \zeta)] (1 - \tau^1_\eta (\zeta)) + \tau^1_\eta (\zeta) \psi^{\eta}_i(w_1, \zeta)
\]
\(\varepsilon = (\varepsilon_1, \varepsilon_2), \eta = (\eta_1, \eta_2).\)

Observe that we have for \((w, \zeta) \in \Omega_R \times \partial \Omega_R\)
\[
1 = \sum_{i=1}^{2} \psi^{\varepsilon, \eta}_i(w, \zeta)(\zeta_i - w_i)
\]
and we can extend \(\psi^{\varepsilon, \eta}_i\) smoothly to a neighborhood of \(\Omega^\delta_R \times \partial \Omega_R\), such that the above identity remains valid.

**Proposition 4.1.** Let \( f \in A(\Omega_R) \). Then for \( w \in \Omega_R \)
\[
4\pi^2 f(w) = \int_{\partial \Omega_R} f(\zeta) K^{\varepsilon, \eta}(w, \zeta),
\]
where
\[
K^{\varepsilon, \eta}(w, \zeta) = (\psi^{\varepsilon, \eta}(w, \zeta) \overline{\psi}_2^{\varepsilon, \eta}(w, \zeta) - \psi_2^{\varepsilon, \eta}(w, \zeta) \overline{\psi}_1^{\varepsilon, \eta}(w, \zeta)) \wedge d\zeta_1 \wedge d\zeta_2.
\]

The proof is a copy of the proof for the case of smooth domains: Fix \( w \in \Omega_R \), after changing \( \psi \) in a small neighborhood of \( w \), we can assume that on this neighborhood \( \psi \equiv \psi^3 \). Then by using Stokes' Theorem, we see that
\[
4\pi^2 f(w) = \int_{\partial \Omega_R^\delta} f(\zeta) K^{\varepsilon, \eta}(w, \zeta),
\]
where \( \Omega_R^\delta \) form an increasing family of smooth domains which contain \( w \) and exhaust \( \Omega_R \) when \( \delta \to 0 \). If we let \( \delta \) go to 0, we obtain the required formula.

We will let \( \varepsilon \) and \( \eta \) tend to 0. Then \( K^{\varepsilon, \eta} \) will tend to a form \( K^0 \) which is holomorphic in \( w \). This already would yield an integral representation, but
perhaps only in some principal value sense. We proceed by proving that $K^0$ is a form with integrable coefficients.

Define

\begin{align*}
K_1(w, \zeta) &= (\psi_1 \overline{\partial}_\zeta \psi_2 - \psi_2 \overline{\partial}_\zeta \psi_1) \wedge d\zeta_1 \wedge d\zeta_2 \\
K_2(w, \zeta) &= (\psi_1 \overline{\partial}_2 \psi_2 - \psi_2 \overline{\partial}_2 \psi_1) \wedge d\zeta_1 \wedge d\zeta_2 \\
K_3(w, \zeta) &= (\psi_1 \psi_2 - \psi_2 \psi_1) d\zeta_1 \wedge d\zeta_2.
\end{align*}

**Theorem 4.2.** Let $K \subseteq \Omega_R$. There exists a constant $A$ such that

\[ \int_{F_i} |K_i(w, \zeta)| < A \quad \text{for } w \in K \quad (i = 1, 2, 3). \]

**Proof.** The major part is the case $i = 1$. On $\Omega \times \partial \Omega$ we have the following equality

\[
\psi_1 \overline{\partial}_\zeta \psi_2 - \psi_2 \overline{\partial}_\zeta \psi_1 = \frac{\overline{\partial}_\zeta \psi_2(w, \zeta)}{w_1 - \zeta_1} \sum_{j=1}^l \chi_j(\zeta_1/|\zeta_1|) \psi_2^j(w, \zeta) = \frac{\sum_j (\overline{\partial}_\zeta \chi_j) \psi_2^j(w, \zeta)}{w_1 - \zeta_1} + \sum \frac{\chi_j \overline{\partial}_\zeta \psi_2^j(w, \zeta)}{w_1 - \zeta_1}.
\]

As $\chi_j$ depends only on $\arg \zeta_1$, we have $||\overline{\partial}_\zeta \chi_j|| \leq 1/|\zeta_1|$. If $|w_1 - \zeta_1| > \gamma(K)$ we conclude that (3) is majorized by a **constant** times

\[
\frac{1}{\gamma(K)} \frac{1}{|\zeta_1|} \sup_{z \in K} |\psi_2^j(w, \zeta)| + \frac{1}{\gamma(K)} \sup_{w \in K} |\overline{\partial}_\zeta \psi_2^j(w, \zeta)|.
\]

Now

\[
\psi_2^j(w, \zeta) = \psi_2^j(\tilde{w}, \zeta) = \tilde{\psi}(\tilde{w}_1, (\tilde{w}_2 - \zeta_1), \zeta_1)
\]

which is a holomorphic function of three variables. Hence

\[
\frac{\partial \psi_2^j}{\partial \zeta_1} = \frac{\partial \tilde{\psi}}{\partial \zeta_1} \frac{\partial \zeta_1}{\partial \zeta_1} + \frac{\partial \tilde{\psi}}{\partial (\tilde{w}_2 - \zeta_2)} \frac{\partial (\tilde{w}_2 - \zeta_2)}{\partial \zeta_1} + \frac{\partial \tilde{\psi}}{\partial \tilde{w}_1} \frac{\partial \tilde{w}_1}{\partial \zeta_1}.
\]

In view of the form of the \(~\) coordinates we obtain

\[
\overline{\partial}_\zeta \psi_2^j = \frac{\partial \tilde{\psi}}{\partial (\tilde{w}_2 - \zeta_2)} \frac{\partial (\tilde{w}_2 - \zeta_2)}{\partial \zeta_1} d\zeta_1.
\]
Fix an open $\Omega_k$ such that $K \subseteq \Omega_k \subseteq \Omega_R$. Because $\tilde{\psi}$ is holomorphic we have, using Proposition 3.4

$$\sup_{w \in K} \left| \frac{\partial \tilde{\psi}}{\partial (\tilde{w}_2 - \zeta_2)} \right| \leq \sup_{w \in \Omega_k} |\tilde{\psi}| = \sup_{w \in \Omega_k} |\psi_{\frac{1}{2}}| \leq |\ln|\zeta_1||^{1/2} + 1$$

and also

$$\left| \frac{\partial (\tilde{w}_2 - \zeta_2)}{\partial \zeta_1} \right| \leq \frac{|w_1 - \zeta_1|}{|\zeta_1|}.$$

We infer that (4) is bounded by a constant times

$$\frac{|\ln|\zeta_1||^{1/2} + 1}{|\zeta_1|} \text{ for } |w_1 - \zeta_1| > \gamma(K).$$

Now for $|w_1 - \zeta_1| < \gamma(K)$ we proceed as follows. We have $|\tilde{w}_2 - \tilde{\zeta}_2| > \gamma(K)$ and by section 2

$$\psi_{\frac{1}{2}}(w, \zeta) = \psi_{\frac{1}{2}}(w, \tilde{\zeta}) = \frac{1 - \psi_{\frac{1}{2}}(w, \tilde{\zeta})}{(\tilde{w}_2 - \tilde{\zeta}_2)}.$$

Hence

$$\sup_{w \in K} ||\sum \tilde{\zeta}_j \psi_{\frac{1}{2}}/(w_1 - \zeta_1)|| \leq \frac{1}{\gamma(K)} \frac{1}{|\zeta_1|} \sup_{w \in K} ||\psi_{\frac{1}{2}}(w, \zeta)||$$

where we used that $\sum \tilde{\zeta}_j \tilde{\zeta}_j = 0$. Similarly

$$\sup_{w \in K} ||\sum \tilde{\zeta}_j \psi_{\frac{1}{2}}/(w_1 - \zeta_1)|| \leq \sup_{w \in \Omega_k} \left( \left\| \frac{1}{w_1 - \zeta_1} \frac{1}{\tilde{w}_2 - \tilde{\zeta}_2} \right\| + \frac{1}{\gamma(K)} \|\sum \tilde{\zeta}_j \psi_{\frac{1}{2}}(w, \zeta)||\right).$$

As before, we use that $\psi_{\frac{1}{2}}$ is holomorphic as a function of $\tilde{w}_1, \tilde{w}_2 - \tilde{\zeta}_2$ and $\zeta_1$ as well as the estimate for $\partial (\tilde{w}_2 - \tilde{\zeta}_2)/\partial \zeta_1$, to majorize (6) by a constant times

$$\frac{1}{|\zeta_1|} \left( 1 + \sup_{w \in \Omega_k} |\psi_{\frac{1}{2}}(w, \zeta)| \right).$$

Proposition 3.4 combined with (5) the estimate for (6) gives that (3) is bounded by a constant times $|\ln|\zeta_1||^{1/2} + 1)/|\zeta_1|$ for $w \in K$. On $F_1$ we can take as local coordinates $\Re \zeta_1$, $\Im \zeta_1$, and $\Im \zeta_2$. The Jacobian determinants remain bounded and we obtain

$$\int_{F_1} |K_1| \leq C \int_{\left| \zeta_1 \right| < R} \int_{\ln|\zeta_2| < R} \frac{|\ln|\zeta_1||^{1/2} + 1}{|\zeta_1|} d\Re \zeta_1 d\Im \zeta_1 d\Im \zeta_2 \leq A$$

for some constant $A$. 
(i = 2): Just note that \( \psi_i^2 \) and hence \( K_2 \) are smooth and bounded for \( w \in K \subset B(0, r) \), \( \zeta \in \hat{B}(0, R) \).

(i = 3): By the boundedness of \( \psi_i^2 \) and Proposition 3.4

\[
|\psi_1 \psi_2^2 - \psi_1^2 \psi_2| \leq |\ln|\zeta_1||^{1/2} + 1.
\]

Also

\[
|dz_1 \wedge d\zeta_2| \leq |dx_1^1 \wedge dx_2^1| + |dx_1^1 \wedge dx_3^2| + |dx_2^1 \wedge dx_3^2| + |dx_1^2 \wedge dx_2^2|,
\]

where \( \zeta_j = x_j^1 + ix_j^2 \), \( j = 1, 2 \). Using \( x_1^1 = P(x_1^1, x_2^1) \) with \( \partial P/\partial x_1^1 \), \( \partial P/\partial x_2^1 \) bounded for \( |X| < R \) we can estimate

\[
\int_{F_j} |K_3(w, \zeta)| \leq \int_{-R}^R \int_{-R}^R (|\ln|\tau||^{1/2} + 1) dtds
\]

and the latter integral is bounded.

**Lemma 4.3.** If \( \psi^j(w, \zeta), i, j = 1, 2 \) satisfy

\[
1 = (w_1 - \zeta_1)\psi_1^1(w, \zeta) + (w_2 - \zeta_2)\psi_2^1(w, \zeta),
\]

for \( (w, \zeta) \) in a neighborhood of \( \Omega \times \partial \Omega \), then

\[
\psi_1^1 \overline{\psi}_1^1 - \psi_1^2 \overline{\psi}_1^2 - \psi_1^1 \overline{\psi}_1^2 - \psi_1^2 \overline{\psi}_1^1 = \psi_1^2 \overline{\psi}_1^2 - \psi_1^2 \overline{\psi}_1^1.
\]

**Proof.**

\[
(w_1 - \zeta_1)[(\psi_1^1 - \psi_1^2)\overline{\psi}_1^1 - (\psi_1^2 - \psi_2^2)\overline{\psi}_1^2]
\]

\[
= (\psi_1^2 - \psi_2^2)[-(w_2 - \zeta_2)\overline{\psi}_1^1 - (w_1 - \zeta_1)\overline{\psi}_1^1]
\]

\[
= -(\psi_1^2 - \psi_2^2)\overline{\psi}_1^2[(w_1 - \zeta_1)\psi_1^1 + (w_2 - \zeta_2)\psi_1^2] = 0.
\]

Similarly

\[
(w_2 - \zeta_2)[(\psi_1^1 - \psi_1^2)\overline{\psi}_1^1 - (\psi_1^2 - \psi_2^2)\overline{\psi}_1^2] = 0
\]

and the Lemma follows.

**Theorem 4.4.** Let \( f \) be a continuous function on \( \partial \Omega_R \), then

\[
C[f](w) := \sum_{j=1}^3 \int_{F_j} f(\zeta)K_j(w, \zeta)
\]

is a holomorphic function on \( \Omega_R \). Moreover, if

\[
f \in A(\Omega_R) \ (= C(\Omega_R) \cap H(\Omega_R)),
\]

then \( C[f] = f \).
Proof. The first statement follows easily by differentiating under the integral sign, the Cauchy formula and dominated convergence. Next, if \( f \in A(\Omega_R), w \in \Omega_R \) we have by Proposition 4.1

\[
f(w) = \int_{\partial \Omega_R} f(\zeta)K^{<\eta}(w, \zeta) .
\]

We put

\[
\tilde{\psi}_i = \tilde{\psi}_i(w, \zeta) = \tau_1^n(\zeta)\psi_i^2(w, \zeta) + (1 - \tau_1^n(\zeta))\psi_i(w, \zeta), \quad i = 1, 2.
\]

Evaluation of \( K^{<\eta} \) yields, using Lemma 4.3

\[
K^{<\eta}(w, \zeta) = \left[ (1 - \tau_1^n)^2(\psi_1^1\psi_2^1 - \psi_2^1\psi_1^1) + \right.
\]

\[
+ (\tau_1^n)^2(\psi_1^2\psi_2^3 - \psi_2^3\psi_1^3) +
\]

\[
+ \tau_1^n(1 - \tau_1^n)[\psi_1^3\psi_2^1 + \psi_1^1\psi_2^3 - \psi_2^1\psi_1^3 - \psi_2^3\psi_1^1] +
\]

\[
+ \tau_1^n\psi_1^1(1 - \tau_1^n)[\psi_1^3\psi_2 - \psi_2^3\psi_1] +
\]

\[
+ (1 - \tau_1^n)\psi_1^1\tau_1^n[\psi_2^1\psi_1^3 - \psi_2^3\psi_1^1] \wedge d\zeta_1 \wedge d\zeta_2
\]

\[
= \left[ (1 - \tau_1^n)(\psi_1^1\psi_2^1 - \psi_2^1\psi_1^1) + \tau_1^n(\psi_1^2\psi_2^3 - \psi_2^3\psi_1^3) +
\right.
\]

\[
+ \psi_1^1\tau_1^n(\psi_2^3\psi_1^1 - \psi_2^1\psi_1^3) \right] d\zeta_1 \wedge d\zeta_2.
\]

We plug this in (4.1) and let \( \eta_2 \to 0 \). Then \( \tau_1^n \) will tend in measure to the characteristic function of the disc \( |\zeta_1| < \eta_1 \), while \( (\partial \tau_1^n/\partial \zeta_1) d\zeta_1 \wedge d\zeta_2 \) will tend in measure to arc length on \( |\zeta_1| = \eta_1 \), compare the proof of a slightly more involved but similar assertion in the sequel. Now we let \( \eta_1 \to 0 \), then

\[
\int_{\partial \Omega_R} f \tau_1^n(\psi_1^1\psi_2^1 - \psi_2^1\psi_1^1) d\zeta_1 \wedge d\zeta_2
\]

will vanish, because the integrand is a continuous function and integration is over \( \partial \Omega_R \cap \{ |\zeta_1| < \eta_1 \} \). Also if \( \eta_1 \to 0 \)

\[
\int_{\zeta \in \partial \Omega_R, |\zeta_1| = \eta_1} f(\psi_2^1\psi_1^1 - \psi_2^1\psi_1^1) d\zeta_2 d\sigma_1 \to 0 \quad (\sigma_1 = \text{arc length on } |\zeta_1| = \eta)
\]

because the integrand is bounded by a constant times \( ||\ln|\zeta_1||^{1/2} + 1 \), in view of Proposition 3.4. Therefore

\[
f(w) = \int_{\partial \Omega_R} (\psi_1^1\psi_2^1 - \psi_2^1\psi_1^1) d\zeta_1 \wedge d\zeta_2.
\]
We perform the same manipulation to obtain
\[
 f(w) = \int_{F_1} f(\xi)(1 - \tau_2^2(\xi))(\psi_1(w, \xi)\overline{\psi}_2(w, \xi) - \psi_2(w, \xi)\overline{\psi}_1(w, \xi)) d\xi_1 \wedge d\xi_2 + \\
 + \int_{\partial \Omega_R} f(\xi)\tau_2^2(\xi)(\psi_1(w, \xi)\overline{\psi}_2(w, \xi) - \psi_2(w, \xi)\overline{\psi}_1(w, \xi)) \wedge d\xi_1 \wedge d\xi_2 + \\
 + \int_{F_1} f(\xi)\overline{\eta}_1(\tau_2^2(\xi))(\psi_1(w, \xi)\overline{\psi}_2(w, \xi) - \psi_2(w, \xi)\overline{\psi}_1(w, \xi)) \wedge d\xi_1 \wedge d\xi_2.
\]

We used that \(\text{supp}(1 - \tau_2^2) \cap \partial \Omega_R \subset F_1\) for all \(\varepsilon_1, \varepsilon_2 > 0\).

If \(\varepsilon_1, \varepsilon_2 \to 0\), then the first integral tends to \(\int_{F_1} f(\xi)K_1(w, \xi)\) by dominated convergence, in view of Theorem 4.2. Similarly the second integral tends to \(\int_{F_1} f(\xi)K_2(w, \xi)\). For the third one, observe that
\[
||\overline{\eta}_1(\tau_2^2(\xi))|| \leq C/\varepsilon_2 \text{ and supp } \overline{\eta}_1(\tau_2^2(\xi)) \subset \partial \Omega \cap \{R - \varepsilon_1 - \varepsilon_2 \leq |\xi| \leq R - \varepsilon_1\}.
\]

We let first \(\varepsilon_1\) go to 0. Again, by dominated convergence
\[
\lim_{\varepsilon_1 \to 0} \int_{F_1} f\overline{\eta}_1(\tau_2^2(\xi))d\xi_1 \wedge d\xi_2 = \int_{\partial \Omega} f\overline{\eta}_1(\tau_2^2(\xi))d\xi_1 \wedge d\xi_2.
\]

We claim that
\[
\lim_{\varepsilon_2 \to 0} \int_{\partial \Omega} f\overline{\eta}_1(\tau_2^2(\xi))(\psi_1(w, \xi)\overline{\psi}_2 - \psi_2(w, \xi)\overline{\psi}_1) \wedge d\xi_1 \wedge d\xi_2 = \int_{\partial \Omega} f d\xi_1 \wedge d\xi_2.
\]

This is seen as follows: Put
\[
g(w, \xi) = (\psi_1(w, \xi)\overline{\psi}_2 - \psi_2(w, \xi)\overline{\psi}_1),
\]

let \(f(\xi)\) be a \(C_0\) function on \(\partial \Omega\) such that
\[
\sup_{\xi \in F_1} |f - \overline{f}| < \delta
\]
and let \(g(\xi)\) be a \(C_0\) function on \(\partial \Omega\) such that \(|g| \leq g\) on \(F_1\) and \(g = g\) on \(F_1 \cap \{|\xi_1| \geq \delta\}\). Then
\[
(7) \quad \left| \int_{\partial \Omega} fg\overline{\eta}_1(\tau_2^2(\xi)) \wedge d\xi_1 \wedge d\xi_2 - \int_{F_3} fg d\xi_1 \wedge d\xi_2 \right|
\]
\begin{align*}
\leq & \left| \int_{\partial \Omega} (fg - \tilde{f}\tilde{g}) \bar{\partial} \tau_{0,1,2} \wedge d\zeta_1 \wedge d\zeta_2 \right| + \left| \int_{\partial \Omega} \tilde{f}\tilde{g} \bar{\partial} \tau_{0,1,2} d\zeta_1 \wedge d\zeta_2 \right| \\
& - \left| \int_{F_3} \tilde{f}\tilde{g} d\zeta_1 \wedge d\zeta_2 \right| + \left| \int_{F_3} (\tilde{f}\tilde{g} - fg) d\zeta_1 \wedge d\zeta_2 \right|.
\end{align*}

Taking \( \text{Re}\zeta_1, \text{Im}\zeta_1, \text{Im}\zeta \) as coordinates we have, because the involved Jacobian determinants are bounded

\begin{align*}
(8) \quad \left| \int_{\partial \Omega} (fg - \tilde{f}\tilde{g}) \bar{\partial} \tau_{0,1,2} \wedge d\zeta_1 \wedge d\zeta_2 \right| & \leq C\delta \int_{\Omega} g/\varepsilon_2 \left| d\text{Re}\zeta_1 d\text{Im}\zeta_1 d\text{Im}\zeta_2 + \\
& + C \int_{\Omega} \left| d\text{Im}\zeta_1 d\text{Im}\zeta_1 d\text{Im}\zeta_2. \right|
\end{align*}

Now we integrate first with respect to \( \text{Im}\zeta_2 \), and observe that for fixed \( \zeta_1 \), \( \text{Im}\zeta_2 \) runs over an interval of length \( C' \varepsilon_2 \) and that \( g \) is bounded by \( C'' (|\ln| \zeta_1 |^1/2 + 1) \), which is integrable. We infer that (8) tends to 0 with \( \delta \).

For the second term in the righthand side of (7) we have by integrating by parts

\begin{align*}
\int_{\partial \Omega} \tilde{f}\tilde{g} \bar{\partial} \tau_{0,1,2} \wedge d\zeta_1 \wedge d\zeta_2 &= \int_{\partial \Omega} \bar{\partial} (\tilde{f}\tilde{g}) \tau_{0,1,2} \wedge d\zeta_1 \wedge d\zeta_2,
\end{align*}

which leads to

\begin{align*}
\int_{\partial \Omega \setminus F_1} \bar{\partial} \zeta (\tilde{f}\tilde{g}) \wedge d\zeta_1 \wedge d\zeta_2 \quad \text{as} \quad \varepsilon_2 \to 0.
\end{align*}

By Stokes' Theorem

\begin{align*}
\int_{\partial \Omega \setminus F_1} \bar{\partial} \zeta (\tilde{f}\tilde{g}) \wedge d\zeta_1 \wedge d\zeta_2 &= \int_{\partial \Omega \setminus F_1} d\zeta (\tilde{f}\tilde{g} d\zeta_1 \wedge d\zeta_2) = \int_{F_3} \tilde{f}\tilde{g} d\zeta_1 \wedge d\zeta_2.
\end{align*}

Finally the third term in (7) is \( O(\delta \log \delta) \) as \( \delta \to 0 \).
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