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FUNDAMENTAL THEOREMS FOR LINEAR MEASURE DIFFERENTIAL EQUATIONS

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1. Introduction.

The aim of the present paper is to give proofs of some fundamental theorems for linear measure differential equations i.e. equations where some coefficients are allowed to be complex Borel measures. We shall prove theorems for first order linear systems and apply these theorems to scalar linear equations of order $n > 1$ rewritten as systems. In Section 2 we prove an existence and uniqueness theorem, Theorem 2.1, for the two-sided Cauchy problem. We refer to this theorem in the introductory discussion of measure differential equations below.

We look at the simplest model equation. Let δ_1 be the Dirac measure at $x = 1$. Let c and a be constants. We look at the problem

$$Du + a\delta_1 u = 0, \quad x > 0, \quad u(0) = c. \tag{1.1}$$

Formally (1.1) is equivalent to the integral equation

$$u(x) = c - a \int_0^x u(s) d\delta_1(s). \tag{1.2}$$

But (1.2) is ambiguous. If the integration is taken over $[0, x)$, then $u(x) = c$, $x \leqq 1$, and $u(x) = c(1-a)$, $x > 1$.

This corresponds to an Atkinson [2, Sections 11.8, 11.9] interpretation of (1.2). If the integration is taken over $[0, x]$, then $u(x) = c$, $x < 1$, and $u(x) = c(1+a)^{-1}$, $x \geqq 1$. In the last case $u(x)$ does not exist for $x \geqq 1$ if $a = -1$. This is the Sharma interpretation of (1.2), see [29]. In Theorem 2.1 the condition $a \neq -1$ is expressed by condition (2.1). The proof of Theorem 2.1 shows that the forward part of the theorem for first order equations corresponds to the Sharma interpretation and that the backward part corresponds to the Atkinson interpretation of the forward part.

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We have just pointed out the connection between the Sharma interpretation and the Atkinson interpretation. The one is just the reflection of the other. Here we have chosen the interpretation giving right continuous solutions. Anyhow condition (2.1) is very disturbing. It is hard to motivate physically or in other ways that (2.1) must be satisfied. The main result, Theorem 3.1 in Section 3, solves this paradox. If one regularizes the Dirac measures in (1.2) and then takes the limit of the solutions of the regularized problems then this limit does not solve (1.2) in either sense. Theorem 3.1 says that the limit solves a modified version of (1.2) in the Sharma sense. Theorem 3.1 is formulated for a first order linear system. Corollary 3.2 is a reformulation of Theorem 3.1 to the case of a scalar higher order linear equation. A first order version of Corollary 3.2 is found in Persson [23]. Examples for the wave equation corresponding to Corollary 3.2 are found in Persson [25]. See also [21].

In Section 4 we give an existence a uniqueness theorem, Theorem 4.2, for a system of first order stochastic differential equations. In Gihman and Skorohod [9, p. 216] one asks for a satisfactory theory for measure differential equations. We guess that they think of the obstacle set by condition (2.1). In the light of the results of Section 3 we believe that the system of Theorem 4.2 in an approximation situation should be modified according to Theorem 3.1 with the true stochastic part unchanged. But we have no proof of this.

In Section 5 we prove an extension of Libri's theorem on the correspondence between the coefficients of homogeneous linear differential equations and the fundamental set of solutions of the equations, Corollary 5.2. Theorem 5.1 is the corresponding theorem for systems. As to Libri's theorem see Demidov [7], and Coddington and Levinson [5, Theorem 6.2, p. 83]. Condition (2.9) turns out as a nice condition on the Wronskian of the fundamental set.

In connection with an earlier version of this note the referee pointed out two important references which I have missed, Kurzweil [13] and Schwabik, Tvrdý, and Vejvoda [28]. See also Kurzweil [12]. It turns out that Theorem 2.1 is essentially contained in [28, Theorem 3.1.4, p. 106 and Theorem 3.3.1, p. 124]. The book [28] contains a detailed account of what in our terminology is linear measure differential systems. In [12] Kurzweil introduces an integral which turns out to be the Perron-Stieltjes integral, see [28, p. 33]. He uses the integral to define generalized differential equations. In [12], he treats a class of such equations with continuous solutions. In [13] he introduces a problem close to that of Theorem 3.1, see [13, Theorems 5.1 and 5.2]. The difference is that there is only one jump in the limit and the continuous parts of the right hand side does not change in the limiting process whereas the limiting process is more general and the problem is non-linear. Theorems

5.1 and 5.2 are derived from Theorems 4.1 and 4.2. The assumptions of the hypotheses of the latter theorems are complicated and hard to verify.

Section 5 below is also essentially contained in [28, pp. 111–123]. Still we think it deserves its place here given by the definitions of this note. In [28], there seem to be no results of the kind which is expressed by Theorem 3.1.

For earlier special cases and variants of Corollary 2.2 see also Sharma [29], Pandit [15], Deo and Pandit [8], Persson [17], [18]: For results on boundary problems for measure differential equations apart from those contained in the book [28] see Albeverio, Fenstad and Høegh-Krohn [1], Birkeland [4], Kac and Kreïn [11], Mingarelli [14], Persson [15], [22], Bennewitz and Everitt [3]. As another example of Atkinson interpretation see Samoïlenko and Perestyuk [27].

If one disregards the models lying behind measure differential equations one may look at the following mathematical problem. What conditions must the differential equation fulfil in order to have solutions in a certain class. For first order equations we have Peano's existence theorem for continuously differentiable solutions, and Carathéodory's theorem for absolutely continuous solutions. The hypotheses of these theorems can be weakened, Peetre, Persson, see Persson [16]. When the solutions are allowed to be of locally bounded variation then measure differential equations arises in a natural way. Then one may ask if it is possible to have linear equations where some coefficients are distributions and not necessarily measures. In Persson [19] one shows that it is possible to prove an existence and uniqueness theorem for the Cauchy problem for such equations. For equations of order strictly less than three it is just the forward part of Corollary 2.2 with zero initial data. It differs for higher order cases. In Persson [24], these results are extended such that the forward part of Corollary 2.2 is a special case of a general theorem for distribution differential equations.

I like to thank Einar Mjølhus. He challenged me on the question of regularization of first order measure differential equations. His criticism initiated the search which ultimately led to Theorem 3.1. Later the same kind of criticism is found in a book review by O. Hájek [10]. I also want to thank the referee not only for pointing out [13] and [28] to me but also for encouraging me not to leave Theorem 3.1 as a conjecture for $n > 1$.

2. The Cauchy problem.

Here we prove the existence and uniqueness theorem for the two-sided Cauchy problem for measure differential equations. We also prove a theorem of the equiboundedness of the solutions of the corresponding regularized equations for fixed initial data at a fixed initial point. The proof of our main result, Theorem 3.1 of Section 3, depends heavily on the results of this section and also on their proofs.

The space of all complex Borel measures on the real line is called $\mathscr{P}^0$. Let $k > 0$ be an integer and let D denote distribution differentiation. Let f be a complex valued function such that $D^k f \in \mathscr{P}^0$, then f is said to be in $\mathscr{P}^k$. By choice the elements of $\mathscr{P}^k$, $k > 1$, are continuous and those of $\mathscr{P}^1$ are continuous to the right.

The natural dual of $\mathscr{P}^0$ on any compact interval is the space $\mathscr{B}^0$ of all locally bounded Borel measurable complex valued functions. Just as we have chosen the elements of $\mathscr{P}^k$, $k > 1$, continuous and those of $\mathscr{P}^1$ right continuous we go outside distribution theory when we define $\mathscr{B}^k$. Let $k > 0$ be an integer and let f and g be complex valued functions such that $D^k f = g$ in $\mathscr{D}'$ with $g \in \mathscr{B}^0$. Then the pair (f, g) is said to be in $\mathscr{B}^k$. In the same way if $f \in \mathscr{D}'$ and $D^k g = f$ in $\mathscr{D}'$ with $g \in \mathscr{B}^0$, then (f, g) is said to be in $\mathscr{B}^{-k}$. We normally write $f = (f, g)$ and tacitly think of this special g. If $f \in \mathscr{B}^{-1}$, then a primitive function of f is $g + C$ with C a constant. If $f \in \mathscr{P}^0$ then the primitive distribution of f in $\mathscr{P}^1$ is right continuous so we do not have to think of $\mathscr{P}^k$ as a space of pairs.

THEOREM 2.1. *Let $n \geqq 1$ be an integer and let $a \in \mathrm{R}$. Let A be an $n \times n$ matrix with entries in $\mathscr{P}^0$ and let I be the $n \times n$ identity matrix. Let $f \in (\mathscr{P}^0)^n((\mathscr{B}^{-1})^n)$. We assume that*

$$A(\{x\}) + I \text{ is invertible}, \quad x \in \mathrm{R}. \tag{2.1}$$

Then to each choice of $c = (c_1, \ldots, c_n) \in \mathrm{C}^n$ there is a unique $u \in (\mathscr{P}^1)^n((\mathscr{B}^0)^n)$ such that

$$u(x) = c - \int_{a^+}^{x} dA(t)u(t) + \int_{a^+}^{x} df(t), \quad x \geqq a, \tag{2.2}$$

and

$$u(x) = c + \int_{x^+}^{a} dA(t)u(t) - \int_{x^+}^{a} df(t), \quad x < a. \tag{2.3}$$

PROOF. We solve (2.1) and (2.2) by successive approximations. At first we let $f \in (\mathscr{P}^0)^n$. Now the integrals of (2.2) are taken over the interval $(a, x]$ and those of (2.3) over $(x, a]$ in the ordinary sense. The exact meaning of the last integrals of (2.2) and (2.3), when $f \in (\mathscr{B}^{-1})^n$, is the primitive distribution of f in $(\mathscr{B}^0)^n$ which is zero at $x = a$.

Let

$$v_0(x) = c + \int_{a^+}^{x} df(t),\ x \geqq a, \quad v_0(x) = c - \int_{x^+}^{a} df(t),\ x < a, \tag{2.4}$$

and

$$v_p(x) = \int_{x^+}^{a} dA(t)v_{p-1}(t),\ x < a, \tag{2.5}$$

$$v_p(x) = -\int_{a^+}^{x} dA(t)v_{p-1}(t),\ x \geqq a,\ p = 1, 2, \dots .$$

We chose the norm

$$|d| = \max_{1 \leq j \leq n} |d_j|, \quad d \in \mathbf{C}^n,$$

and

$$\|B\| = \max_{1 \leq k \leq n} \sum_{j=1}^{n} |b_{jk}|,$$

when $B = (b_{jk})$ is an $n \times n$ matrix with entries in C. It follows that $|Bd| \leqq \|B\|\,|d|$. If $h = h' + ih''$, where h' and h'' are signed Borel measures, then we let

$$|f| = (|f_1|, \dots, |f_n|).$$

In the same way we let $|A| = (|a_{jk}|)$.

Let $b' < a < b$ be constants. Then there is always a finite number of points x_k, $k = -j, \ -j+1, \dots, 0, 1, \dots j$ for some j such that $x_0 = a$, $b' = x_{-j} < x_{-j+1} < \dots < x_j = b$, such that

$$\||A|((x_k, x_{k+1}))\| < 2^{-1}, \quad -j \leqq k < j. \tag{2.6}$$

Let the constant C be defined by

$$|c| + \left| \int_{b'^+}^{b} d|f|(t) \right| = C. \tag{2.7}$$

We claim that

$$|v_p(x)| \leqq 2^{-p}C, \quad x_0 \leqq x < x_1,\ p = 0, 1, \dots . \tag{2.8}$$

It follows from (2.7) that (2.8) is true for $p = 0$. Let it be true for $p = l$.

Then (2.4), (2.6), and (2.8) shows that

$$|v_{l+1}(x)| < \sup_{x_0 \leqq x \leqq x_1} |v_l(x)| \left\| \int_{x_0^+}^{x_1^-} d|A|(t)) \right\| \leqq 2^{-l-1}C, \quad x_0 \leqq x < x_1.$$

That shows that (2.8) is true for all p. Let $u = \sum_{p=0}^{\infty} v_p$. It follows that u converges, that $|u(x)| \leqq 2C$, $x_0 \leqq x < x_1$, and that

$$u(x) = c - \int_{a^+}^{x} dA(t)u(t) + \int_{a^+}^{x} df(t)$$

in the same interval. It is now also clear that $u(x_1^-)$ exists and that $|u(x_1^-)| \leqq 2C$. Then we compute $u(x_1)$ from (2.2). We get

$$u(x_1) = u(x_1^-) - A(\{x_1\})u(x_1) + f(\{x_1\}).$$

This is always possible because of condition (2.1). It is obvious from the proof that the solution u in $x_0 \leqq x \leqq x_1$ is unique. It is also obvious how we now can repeat the proof in $x_1 \leqq x < x_2$. Then we get a unique value of $u(x_2)$ and so on. We finally arrive at a unique solution u in $a \leqq x \leqq b$. For the backward part we start by computing $u(a^-)$ from (2.3). We get

$$u(a^-) = c + A(\{a\})c + f(\{a\}).$$

Then we solve

$$u(x) = u(a^-) + \int_{x^+}^{a^-} dA(t)u(t) + \int_{x^+}^{a^-} df(t), \quad x_{-1} \leqq x < a$$

by successive approximations. We do not write down that proof. Then we compute $u(x_{-1}^-)$ from (2.5) and repeat the proof in $x_{-2} \leqq x < x_{-1}$. Finally we arrive at the existence of a unique solution u in $(\mathscr{P}^1)^n$ of (2.4) and (2.6).

If $f = (f, f_0) \in (\mathscr{B}^{-1})^n$, then we let $v(x) = u(x) - f_0(x) + f_0(a)$. Then (2.2) and (2.3) are turned into equations of the same form with u replaced by v and with a new f which this time is in $(\mathscr{P}^0)^n$. It follows that v exists and is unique in $(\mathscr{P}^1)^n$. Then u exisits in $(\mathscr{B}^0)^n$ and is unique. Theorem 2.1 is proved.

COROLLARY 2.2. *Let $n > 0$ be an integer and let $a \in \mathbb{R}$. Let $a_j \in \mathscr{P}^0$, $0 \leqq j < n$, and let $f \in \mathscr{P}^0(\mathscr{B}^{-1})$. Let*

(2.9) $$a_{n-1}(\{x\}) \neq -1, \quad x \in \mathbb{R}.$$

Then to each choice of $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathbf{C}^n$ *there is a unique* $u \in \mathscr{P}^n(\mathscr{B}^{n-1})$ *such that*

$$u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_0 u = f, \quad D^j u(a) = c_j, \; 1 \leqq j < n. \tag{2.10}$$

REMARK. The precise meaning of (2.10) is that if $f \in \mathscr{P}^0$

$$u^{(n-1)}(x) + \int_{a^+}^{x} d(a_{n-1}u^{(n-1)} + \ldots + a_0 u)(t) = c_{n-1} + \int_{a^+}^{x} df(t), \quad x \geqq a, \tag{2.11}$$

and

$$u^{(n-1)}(x) - \int_{x^+}^{a} d(a_{n-1}u^{(n-1)} + \ldots + a_0 u)(t) = c_{n-1} - \int_{x^+}^{a} df(t), \quad x < a, \tag{2.12}$$

with $D^j u(a) = c_j$, $0 \leqq j < n-1$.

PROOF. One rewrites (2.11) and (2.12) as a system just as in the case of usual differential equations. Then (2.9) turns out to be equivalent to (2.1) for this system. Then the conclusion is drawn from Theorem 2.1. As to the case when $f \in \mathscr{B}^{-1}$ we also refer to what is said in the proof of that theorem.

Let A and f be as in the hypothesis of Theorem 2.1 with $f \in (\mathscr{P}^0)^n$. Let $\phi \in C(\mathbf{R})$, $\phi \geqq 0$, with $\int \phi(x)dx = 1$ and $\operatorname{supp} \phi$ in $|x| \leqq 1$. Let $\varepsilon > 0$ and let $\phi(x, \varepsilon) = \phi((x+\varepsilon)/\varepsilon)/\varepsilon$. Let

$$A(x, \varepsilon) = \int \phi(x-t, \varepsilon) dA(t) \quad \text{and} \quad f(x, \varepsilon) = \int \phi(x-t, \varepsilon) df(t).$$

We let $dA(t, \varepsilon) = A(t, \varepsilon)dt$ and $df(t, \varepsilon) = f(t, \varepsilon)dt$. When we regularize in the following we shall always use this regularization. Also the notation will be the same. Thus the reader is asked to notice that $dg(t) = d(g(t)) = g(t)dt$ in the following, if g is a function.

THEOREM 2.3. *Let the hypothesis be as in Theorem* 2.1 *except that* (2.1) *is not necessarily fulfilled. Let* $\varepsilon > 0$ *and let*

$$u(x, \varepsilon) = c - \int_a^x dA(t, \varepsilon) u(t, \varepsilon) + \int_a^x df(t, \varepsilon). \tag{2.13}$$

Let $b' < a < b$ *be constants. Then the family* $(u(x, \varepsilon))_{0 < \varepsilon < 1}$ *is equibounded in* $b' \leqq x \leqq b$.

PROOF. It is no restriction to assume that $a = 0$ and that $b' = -b$. This we do from now on. We solve (2.13) by successive approximations. Let $v_0(x, \varepsilon) = c + \int_0^x df(t, \varepsilon)$ and let

$$v_{p+1}(x, \varepsilon) = - \int_0^x dA(t, \varepsilon) v_p(t, \varepsilon), \quad p = 0, 1, \ldots . \tag{2.14}$$

We assert that there is a $C \geqq 0$ independent of ε, $0 < \varepsilon < 1$, such that

$$|v_p(x, \varepsilon)| \leqq 2^{-p} C \exp\left(2 \left| \int_0^x \|A(t, \varepsilon)\| dt \right|\right), \quad |x| < b. \tag{2.15}$$

We choose $C = |c| + \int_{-b}^{b+1} d|f|(t)$. Then (2.15) is true for $p = 0$. Let (2.15) be true for a certain p. Then (2.14) and (2.15) shows that

$$\begin{aligned}
|v_{p+1}(x, \varepsilon)| &= \left| \int_0^x dA(t, \varepsilon) v_p(t, \varepsilon) \right| \leqq \left| \int_0^x \|A(t, \varepsilon)\| |v_p(t, \varepsilon)| dt \right| \\
&\leqq \left| \int_0^x \|A(t, \varepsilon)\| 2^{-p} C \exp\left(2 \left| \int_0^t \|A(s, \varepsilon)\| ds \right|\right) dt \right| \\
&\leqq 2^{-p-1} C \left(\exp\left(2 \left| \int_0^x \|A(t, \varepsilon)\| dt \right|\right) - 1 \right) \\
&\leqq 2^{-p-1} C \exp\left(2 \left| \int_0^x \|A(t, \varepsilon)\| dt \right|\right).
\end{aligned}$$

By that we have proved that (2.15) is true for all p. Let

$$u(x, \varepsilon) = \sum_{p=0}^{\infty} v_p(x, \varepsilon)$$

Now we notice that

$$2 \left| \int_0^x \|A(t, \varepsilon)\| dt \right| \leqq 2 \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-b}^{b+1} d|a_{jk}|(t) = D.$$

Then we see that $u(x, \varepsilon)$ converges uniformly on $|x| \leqq b$, that $u(x, \varepsilon)$ solves (2.13) and that $|u(x, \varepsilon)| \leqq 2Ce^D$, $|x| \leqq b$. Theorem 2.3 is proved.

3. The limit of solutions of regularized problems.

Theorem 3.1 and Corollary 3.2 below can be seen as a solution of a problem raised by Kurzweil in [13]. See also Hájek [10]. In earlier versions they were only given as conjectures when $n > 1$, Persson [23]. The proofs below are a modification of the proof of the first order scalar result of [23].

THEOREM 3.1. *Let the hypothesis be as in Theorem* 2.1 *with* $f \in (\mathscr{P}^0)^n$ *and without condition* (2.1). *In addition let* $\varepsilon > 0$ *and let*

$$g(D) = \sum_{j=1}^{\infty} (j!)^{-1} D^{j-1},$$

where D is an $n \times n$ matrix with entries in C. *Then when $\varepsilon \to 0$ the solution $u(x, \varepsilon)$ of*

$$u'(x, \varepsilon) + A(x, \varepsilon)u(x, \varepsilon) = f(x, \varepsilon), \quad u(a, \varepsilon) = c, \tag{3.1}$$

converges pointwise to the unique solution $u \in (\mathscr{P}^1)^n$ of

$$u(x) = c - \int_{a^+}^{x} g(A(\{t\}))dA(t)u(t) + \int_{a^+}^{x} g(A(\{t\}))df(t), \quad x \geqq a, \tag{3.2}$$

and

$$u(x) = c + \int_{x^+}^{a} g(A(\{t\}))dA(t)u(t) - \int_{x^+}^{a} g(A(\{t\}))df(t), \quad x < a. \tag{3.3}$$

PROOF. We let $A(\cdot, 0) = A$ and $f(\cdot, 0) = f$. At first we assume that there is only a finite number of points x with $A(\{x\}) \neq 0$. At last we remove this restriction.

We let $f = f_0 + \bar{f}$, where f_0 has no point masses and where $\bar{f} = \sum_{j=1}^{\infty} b_j \delta_{x_j}$ with $b_j \in \mathrm{C}^n$ and δ_{x_j} is the Dirac measure at the point $x = x_j$. If we sum $\sum b_j$ over all j with x_j in a given bounded set, then this sum is absolutely convergent, since f is a complex Borel measure. We shall prove the theorem in an arbitrary fixed interval $b' \leqq x \leqq b$, where $b' < a < b$. We then assume that $\sum_{j=1}^{\infty} b_j$ is absolutely convergent and that $b' \leqq x_j \leqq b$ for all j.

Let $f_N = \sum_{j=1}^{N} b_j \delta_{x_j}$ and let $h_N = \sum_{j=N+1}^{\infty} b_j \delta_{x_j}$. We notice that

$$\delta_{x_j}(x, \varepsilon) = \phi(x - x_j, \varepsilon)$$

in the terminology of Section 2. We solve the regularized problems

$$u_0(x,\varepsilon) = c - \int_a^x dA(s,\varepsilon)u_0(s,\varepsilon) + \int_a^x df_0(s,\varepsilon) \tag{3.4}$$

$$u_j(x,\varepsilon) = -\int_a^x dA(s,\varepsilon)u_j(s,\varepsilon) + \int_a^x \varphi(s-x_j,\varepsilon)ds b_j, \quad 0 < j \leqq N, \tag{3.5}$$

and

$$v_N(x,\varepsilon) = -\int_a^x dA(s,\varepsilon)v_N(s,\varepsilon) + \int_a^x dh_N(s,\varepsilon). \tag{3.6}$$

Let

$$u(x,\varepsilon) = \sum_{j=0}^{N} u_j(x,\varepsilon) + v_N(x,\varepsilon). \tag{3.7}$$

It is then clear that $u(x,\varepsilon)$ solves (3.4) when f_0 is replaced by f and thus also (3.1). Let

$$\begin{aligned} u_0(x,0) &= c - \int_{a^+}^x g(A(\{t\}))dA(t,0)u_0(t,0) + \int_{a^+}^x df_0(s,0), \quad x \geqq a, \\ u_0(x,0) &= c + \int_{x^+}^a g(A(\{t\}))dA(t,0)u_0(t,0) - \int_{x^+}^a df_0(s,0), \quad x < a. \end{aligned} \tag{3.8}$$

It follows from (3.8) and Theorem 2.1 that $u_0(x,0)$ exists.

At first we assume that $A(\{x\}) = 0$ for all x in $b' \leqq x \leqq b$. Let $v_0(x,\varepsilon) = u_0(x,\varepsilon) - u_0(x,0)$. We combine (3.8) and (3.4) and get

$$v_0(x,\varepsilon) = -\int_a^x dA(s,\varepsilon)v_0(s,\varepsilon) + g(x,\varepsilon), \tag{3.9}$$

where

$$g(x,\varepsilon) = \int_{a^+}^x d(A(s,\varepsilon) - A(s,0))u_0(s,0) + \int_{a^+}^x d(f_0(s,\varepsilon) - f_0(s,0)), \quad x \geqq a,$$

and

$$g(x, \varepsilon) = -\int_{x^+}^{a} d(A(s, \varepsilon) - A(s, 0))u_0(s, 0) - \int_{x^+}^{a} d(f_0(s, \varepsilon) - f_0(s, 0)), \quad x < a.$$

The measure $(A(\cdot, \varepsilon) - A(\cdot, 0))u_0(\cdot, 0)$ goes setwise to zero as does $f_0(\cdot, \varepsilon) - f_0(\cdot, 0)$. Then [26, Proposition 18, p. 232] shows that $g(x, \varepsilon)$ goes uniformly to zero in $b' \leqq x \leqq b$ when $\varepsilon \to 0$. Then it follows from the proof of Theorem 2.3 that $v_0(x, \varepsilon) \to 0$ when $\varepsilon \to 0$. In other words $u_0(x, \varepsilon) \to u_0(x, 0)$.

Let there be just one x in $b' \leqq x \leqq b$ such that $A(\{x\}) \neq 0$. We call it x'. At first we look at the case $a < x' \leqq b$. The proof above shows that $u_0(x, \varepsilon) \to u_0(x, 0)$ pointwise in $b' \leqq x < x'$. Then we get

$$(3.10) \quad \begin{aligned} u_0(x, \varepsilon) &= u_0(x' - 2\varepsilon, \varepsilon) - \int_{x'-2\varepsilon}^{x} d(A(\{x'\})\phi(t - x', \varepsilon)u_0(t, \varepsilon)) - \\ &- \int_{x'-2\varepsilon}^{x} d(A(t, \varepsilon) - \phi(t - x', \varepsilon)A(\{x'\}))u_0(t, \varepsilon) + \int_{x'-2\varepsilon}^{x} df_0(t, \varepsilon). \end{aligned}$$

We notice that the value of $u_0(x' - 2\varepsilon, \varepsilon)$ is not influenced by the point mass at $x = x'$. Neither is $u_0(x'^-, 0)$. We know that $u_0(x, \varepsilon)$ tends to $u_0(x, 0)$ uniformly on bounded sets when there is no point mass in A. So we see that

$$\begin{aligned} &u_0(x' - 2\varepsilon, \varepsilon) - u_0(x'^-, 0) \\ &\quad = u_0(x' - 2\varepsilon, \varepsilon) - u_0(x' - 2\varepsilon, 0) + u_0(x' - 2\varepsilon, 0) - u_0(x'^- \cdot 0) \end{aligned}$$

goes to zero when $\varepsilon \to 0$. It follows from Theorem 2.3 that $(u_0(x, \varepsilon))_{0 < \varepsilon < 1}$ is equibounded in $x' \leqq x \leqq b$. We rewrite (3.10).

$$(3.11) \quad \begin{aligned} u_0(x, \varepsilon) &= u_0(x' - 2\varepsilon, \varepsilon) - \int_{x'-2\varepsilon}^{x} \phi(t - x', \varepsilon) d(A(\{x'\})u_0(t, \varepsilon)) - \\ &- \int_{x'-2\varepsilon}^{x} dg(t, \varepsilon), \end{aligned}$$

with $g(t, \varepsilon)$ defined by

$$(3.12) \quad g(t, \varepsilon) = (A(t, \varepsilon) - A(\{x'\})\phi(t - x', \varepsilon))u_0(t, \varepsilon) - f_0(t, \varepsilon).$$

The solution of (3.11) is given by

$$(3.13)\quad u_0(x,\varepsilon) = \exp\left(-\int_{x'-2\varepsilon}^{x} \phi(t-x',\varepsilon)A(\{x'\})dt\right) \times$$

$$\times\left(u_0(x'-2\varepsilon,\varepsilon)+\int_{x'-2\varepsilon}^{x} \exp\left(\int_{x'-2\varepsilon}^{t} \phi(s-x',\varepsilon)A(\{x'\})ds\right)g(t,\varepsilon)dt\right).$$

Let

$$h(t,\varepsilon) = \exp\left(\int_{x'-2\varepsilon}^{t} \phi(s-x',\varepsilon)A(\{x'\})ds\right)g(t,\varepsilon).$$

It follows from (3.12) and the equiboundedness of $u_0(x,\varepsilon)$ in $b' \leqq x \leqq b$ that $h(\cdot,\varepsilon)$ tends setwise to a complex Borel measure without point masses in the same set. Then it follows from (3.13) that

$$u_0(x',\varepsilon) \to e^{-A(\{x'\})}u_0(x'^-,0).$$

It follows from (3.8) that

$$u_0(x'^-,0) = (I+g(A(\{x'\}))A(\{x'\})u(x',0) = e^{A(\{x'\})}u(x',0).$$

We now see that $u_0(x',\varepsilon) \to u_0(x,0)$ when $\varepsilon \to 0$. Let

$$v_0(x,\varepsilon) = u_0(x',0) - \int_{x'}^{x} dA(s,\varepsilon)v_0(s,\varepsilon) + \int_{x'}^{x} df_0(s,\varepsilon).$$

It follows from the proof above that $v_0(x,\varepsilon) \to u_0(x,0)$ when $\varepsilon \to 0$ uniformly in $x' \leqq x \leqq b$. From the proof of Theorem 2.3 it follows that $v_0(x,\varepsilon)-$ $-u_0(x,\varepsilon) \to 0$, when $\varepsilon \to 0$ uniformly in $b' \leqq x \leqq b$, since $u_0(x',\varepsilon)-$ $-v_0(x',0) \to 0$. By that we have proved that $u_0(x,\varepsilon) \to u_0(x,0)$ when $\varepsilon \to 0$.

If $b' \leqq x' \leqq a$ then as before we have that $u_0(x,\varepsilon) \to u(x,0)$ uniformly in $x' \leqq x \leqq b$. Let

$$g(t,\varepsilon) = -(A(x,\varepsilon)-\phi(t-x',\varepsilon)A(\{x'\}))u_0(t,\varepsilon)+f_0(t,\varepsilon).$$

We multiply both members of

$$(3.14)\qquad u_0'(x,\varepsilon)+A(\{x'\})\phi(x-x',\varepsilon)u_0(x,\varepsilon) = g(x,\varepsilon)$$

by $\exp(-\int_x^{x'} \phi(t-x', \varepsilon)A(\{x'\})dt)$ and integrate from $x'-2\varepsilon$ to x'. We get

$$u_0(x', \varepsilon) - e^{-A(\{x'\})}u_0(x'-2\varepsilon, \varepsilon) = \int_{x'-2\varepsilon}^{x'} \exp\left(-\int_t^{x'} \phi(s-x', \varepsilon)A(\{x'\})ds\right) g(t, \varepsilon)dt. \tag{3.15}$$

The integrand of the right member is called $h(t, \varepsilon)$. We see from the definition of $g(t, \varepsilon)$ that $h(\cdot, \varepsilon)$ tends setwise to a measure without point masses. Then the right member of (3.15) tends to zero when $\varepsilon \to 0$. Since $u_0(x', \varepsilon) \to u_0(x', 0)$ we see that

$$u_0(x'-2\varepsilon, \varepsilon) \to e^{A(\{x'\})}u_0(x', 0).$$

We notice that (3.8) shows that

$$\begin{aligned} u_0(x'^-, 0) &= (I + g(A(\{x'\}))A(\{x'\})u_0(x', 0) \\ &= e^{A(\{x'\})}u_0(x', 0). \end{aligned}$$

So $u_0(x'-2\varepsilon, \varepsilon) \to u_0(x'^-, 0)$ when $\varepsilon \to 0$.

Let $h(x, \varepsilon) = 1$, $x \leqq x'-2\varepsilon$, and $h(x, \varepsilon) = 0$, $x > x'-2\varepsilon$. The measures with the densities $h(x, \varepsilon)A(x, \varepsilon)$ and $h(x, \varepsilon)f(x, \varepsilon)$ tend setwise to A and f in $x < x'$. For $x < x'-2\varepsilon$ we get

$$\begin{aligned} u_0(x, \varepsilon) = u_0(x'-2\varepsilon, \varepsilon) &+ \int_x^{x'^-} dA(s, 0)u_0(s, \varepsilon) - \int_x^{x'} df_0(s, 0) + \\ &+ \int_x^{x'^-} d(h(s, \varepsilon)A(s, \varepsilon) - A(s, 0))u_0(s, \varepsilon) - \int_x^{x'} d(h(s, \varepsilon)f_0(s, \varepsilon) - f_0(s, 0)). \end{aligned}$$

Let $v_0(x, \varepsilon) = u_0(x, \varepsilon) - u_0(x, 0)$ and let

$$\begin{aligned} g(x, \varepsilon) = \int_{x^+}^{x'^-} & d((h(t, \varepsilon)A(t, \varepsilon) - A(t, 0))u_0(t, \varepsilon) - h(t, \varepsilon)f_0(t, \varepsilon) + \\ & + f_0(t, 0)) + u_0(x'-2\varepsilon, \varepsilon) - u_0(x'^-, 0). \end{aligned}$$

One sees from above that $g(x, \varepsilon) \to 0$ uniformly in $b' \leqq x \leqq x'$. We see that

$$v_0(x, \varepsilon) = g(x, \varepsilon) + \int_{x^+}^{x'^-} dA(s, 0)v_0(s, \varepsilon).$$

It then follows from the proof of Theorem 2.1 that $v_0(x, \varepsilon)$ tends to zero uniformly on compact subsets of $b' \leqq x < x'$. By that we have proved that $u_0(x, \varepsilon) \to u_0(x, 0)$, $\varepsilon \to 0$, when there is precisely one x with $A(\{x\}) \neq 0$. The extension to the case with finitely many points with that property is immediate from the proof above. We do not write it down.

Let $j > 0$ and let us assume that $x_j > a$. We solve (3.5). We see that $u_i(x, \varepsilon) = 0$ for $x < x_j - 2\varepsilon$. We rewrite (3.5) as

$$u_j'(x, \varepsilon) + \phi(x - x_j, \varepsilon)A(\{x_j\})u_j(x, \varepsilon) - \phi(x - x_j, \varepsilon)b_j = g(x, \varepsilon), \tag{3.16}$$

where

$$g(x, \varepsilon) = -(A(x, \varepsilon) - \phi(x - x_j, \varepsilon)A(\{x_j\}))u_j(x, \varepsilon).$$

By Theorem 2.3 we know that $(u_j(x, \varepsilon))_{0 < \varepsilon < 1}$ is equibounded in $b' \leqq x \leqq b$. We multiply both members of (3.16) by

$$\exp\left(\int_{x_j - 2\varepsilon}^{x} \phi(t - x_j, \varepsilon)A(\{x_j\})dt \right)$$

and integrate from $x_j - 2\varepsilon$ to x_j. In the new equation the right member tends to zero as $\varepsilon \to 0$, just as in (3.15). We get

$$\begin{aligned} &\exp(A(\{x_j\}))u_j(x, \varepsilon) - \\ &- \int_{x_j - 2\varepsilon}^{x_j} \phi(t - x_j, \varepsilon)\left(\sum_{k=0}^{\infty} \left(\int_{x_j - 2\varepsilon}^{t} \phi(s - x_j, \varepsilon)A(\{x_j\})ds \right)^k / k! \right) b_j dt \\ &= \exp(A(\{x_j\}))u_j(x_j, \varepsilon) - g(A(\{x_j\}))b_j \to 0, \quad \varepsilon \to 0. \end{aligned}$$

In other words

$$u_j(x_j, \varepsilon) \to \exp(-A(\{x_j\}))g(A(\{x_j\}))b_j, \quad \varepsilon \to 0.$$

We define $u_j(x, 0)$ by

$$\begin{aligned} u_j(x, 0) &= -\int_{a^+}^{x} g(A(\{t\}))dA(t, 0)u_j(t, 0) + \int_{a^+}^{x} g(A(\{t\}))b_j d\delta_{x_j}(t), \quad x \geqq a, \\ u_j(x, 0) &= \int_{x^+}^{a} g(A(\{t\}))dA(t, 0)u_j(t, 0) - \int_{x^+}^{a} g(A(\{t\}))b_j d\delta_{x_j}(t), \quad x < a. \end{aligned} \tag{3.17}$$

Now (3.17) gives

$$u_j(x_j, 0) = \exp(-A(\{x_j\}))g(A(\{x_j\})b_j.$$

This shows that $u_j(x_j, \varepsilon) \to u_j(x_j, 0)$, $\varepsilon \to 0$. Then the proof for $j = 0$ applies in $x_j \leqq x \leqq b$. So $u_j(x, \varepsilon) \to u_j(x, 0)$ pointwise in $b' < x \leqq b$ if $a \leqq x_j \leqq b$.

Let $x_j \leqq a$. Then $u_j(x, \varepsilon) = 0$, $x_j \leqq x \leqq b$. Now we multiply (3.16) by

$$\exp\left(-\int_x^{x_j} \phi(t-x_j, \varepsilon)A(\{x_j\})dt\right)$$

and integrate from $x_j - 2\varepsilon$ to x_j. This time we get

$$-\exp(-A(\{x_j\}))u_j(x_j-2\varepsilon, \varepsilon) - g(-A(\{x_j\})b_j \to 0, \quad \varepsilon \to 0.$$

An easy computation shows that $e^B g(-B) = g(B)$ for any matrix B. It follows from (3.17) that $u_j(x_j^-, 0) = -g(A(\{x_j\}))b_j$. By that we have proved that

$$u_j(x_j - 2\varepsilon, \varepsilon) \to u_j(x_j^-, 0),\ \varepsilon \to 0.$$

The proof for $j = 0$ then shows that $u_j(x, \varepsilon) \to u_j(x, 0)$, $\varepsilon \to 0$, pointwise in $b' \leqq x < x_j$. By that we have now proved that $u_j(x, \varepsilon) \to u_j(x, 0)$, $\varepsilon \to 0$, $j = 0, 1, \ldots$, pointwise in $b' \leqq x \leqq b$. Let

$$\begin{aligned} v_N(x,0) &= -\int_{a^+}^{x} g(A(\{t\})dA(t,0)v_N(t,0) + \int_{a^+}^{x} g(A(\{t\}))dh_N(t), \quad x \geqq a, \\ v_N(x,0) &= \int_{x^+}^{a} g(A(\{t\})dA(t,0)v_N(t,0) - \int_{x^+}^{a} g(A(\{t\}))dh_N(t), \quad x < a. \end{aligned} \tag{3.18}$$

It follows from the proof of Theorem 2.1 that $v_N(x, 0)$ goes to zero uniformly in $b' \leqq x \leqq b$. It follows from the proof of Theorem 2.3 that $v_N(x, \varepsilon)$ tends to zero uniformly in $b' \leqq x \leqq b$, $0 < \varepsilon < 1$, when $N \to \infty$. Let

$$w_N(x, 0) = \sum_{j=0}^{N} u_j(x, 0).$$

Then we let $u(x, 0) = \sum_{j=0}^{\infty} u_j(x, 0)$. We see that $u(x, 0) = w_N(x, 0) + v_N(x, 0)$ and that $u(x, \varepsilon) \to u(x, 0)$, $\varepsilon \to 0$. It is now also obvious that $u(x, \varepsilon)$ solves (3.1). We see that

$$u(x,0) = c - \int_{a^+}^{x} g(A(\{t\}))dA(t)w_N(t,0) + \int_{a^+}^{x} g(A(\{t\}))df_N(t) +$$

$$+ v_N(x,0) \to c - \int_{a^+}^{x} g(A(\{t\}))dA(t)u(t,0) + \int_{a^+}^{x} g(A(\{t\}))df(t),$$

when $N \to \infty$, since $w_N(t,0) \to u(t,0)$ uniformly when $N \to \infty$ and since $f_N \to f$ setwise, [26, Proposition 18, p. 232]). We do not repeat the argument for $x < a$. It follows that $u(x,0) = u(x)$ with u from (3.2) and (3.3). By that we have proved the theorem for the case where there are only a finite number of points x with $A(\{x\}) \neq 0$.

Let there be a sequence $(x_j)_{j=1}^{\infty}$ such that $x \neq x_j$, all j, implies that $A(\{x\}) = 0$. Let $M > 1$ be an integer and let

$$A_M = A - \sum_{j=M+1}^{\infty} A(\{x_j\})\delta_{x_j}.$$

Let

$$u^M(x,\varepsilon) = c - \int_a^x dA_M(t,\varepsilon)u^M(t,\varepsilon) + \int_a^x df(t,\varepsilon)$$

and let $u(x,\varepsilon)$ be defined by (3.1). Let $v^M(x,\varepsilon) = u(x,\varepsilon) - u^M(x,\varepsilon)$. Then one has

$$v^M(x,\varepsilon) = -\int_a^x dA(t,\varepsilon)v^M(t,\varepsilon) + \int_a^x d(A-A_M)(t,\varepsilon)u^M(t,\varepsilon). \tag{3.19}$$

It follows from the proof of Theorem 2.3 that $(u^M(x,\varepsilon))_{0<\varepsilon<1}$ is equibounded in $b' \leqq x \leqq b$, with a bound independent of M. Then we see from the definition of A_M that the last integral of the right member of (3.19) tends uniformly to zero in $b' \leqq x \leqq b$, $0 < \varepsilon < 1$, when $M \to \infty$.

Let $u(x)$ solve (3.2) and (3.3) and let $u^M(x)$ solve the same equations when A is replaced by A_M. Let $v^M(x) = u(x) - u^M(x)$. It follows from the proof of Theorem 2.1 that $v^M(x)$ tends to zero uniformly in $b' \leqq x \leqq b$, when $M \to \infty$. We also see that for fixed M $u^M(x,\varepsilon) \to u^M(x)$, $\varepsilon \to 0$. Let $\varepsilon' > 0$. Then we have

$$|u(x,\varepsilon) - u(x)| \leqq |u(x,\varepsilon) - u^M(x,\varepsilon)| + |u^M(x,\varepsilon) - u^M(x)| + |u^M(x) - u(x)|.$$

Now we choose M so big that the first and the last term of the right member both are less than ε'. Then for this M the middle term is less than ε' for all small ε according to the first part of the proof. Theorem 3.1 is proved.

COROLLARY 3.2. *Let the hypothesis be as in Corollary 2.2 with the restriction $f \in \mathscr{P}^0$ and without condition (2.9). Let $\varepsilon > 0$ and let $g(s) = (e^s - 1)/s$, $s \neq 0$, and $g(0) = 1$. Then the solution $u(x, \varepsilon)$ of*

$$u^{(n)}(x,\varepsilon)+a_{n-1}(x,\varepsilon)u^{(n-1)}(x,\varepsilon)+\ldots+a_0(x,\varepsilon)u(x,\varepsilon) = f(x,\varepsilon), \tag{3.20}$$

$$u^{(j)}(a,\varepsilon) = c_j,\ 0 \leqq j < n,$$

tends to the solution $u \in \mathscr{P}^n$ of

$$(3.21)\quad \begin{aligned} &u^{(n-1)}(x)+\int_{a^+}^{x} g(a_{n-1}(\{t\}))d(a_{n-1}u^{(n-1)}+\cdots+a_0u-f)(t) = c_{n-1}, \quad x \geqq a,\\ &u^{(n-1)}(x)-\int_{x^+}^{x} g(a_{n-1}(\{t\}))d(a_{n-1}u^{(n-1)}+\cdots+a_0u-f)(t) = c_{n-1}, \quad x < a,\end{aligned}$$

$$u^{(j)}(a) = c_j,\ 0 \leqq j < n-1. \tag{3.22}$$

PROOF. We rewrite (2.10) as a system $U'+AU = F$. Let

$$G(D) = \sum_{j=0}^{\infty} D^j((j+1)!)^{-1}.$$

If $D = A(\{t\})$, then $D_{ij} = 0$, $1 \leqq i < n$, $1 \leqq j \leqq n$. Of course we have let $U_j = u^{(j-1)}$ and we let $F_j = 0$, $1 \leqq j < n$, and $F_n = f$. We also notice that $D_{nj} = a_{j-1}(\{t\})$, $1 \leqq j \leqq n$. We let $c = (c_0, c_1, \ldots, c_{n-1})$. According to Theorem 3.1 the limit U of the regularized solutions fulfils (3.2) and (3.3) with $U = u, f$ replaced by F and $g = G$ as in the hypothesis of Theorem 3.1. $G(A(\{t\})) \neq I$ only at a point with $A(\{t\}) \neq 0$. At such a point

$$(D^jA(\{t\}))_{nk} = (D_{nn})^j D_{nk}, \quad 1 \leqq k \leqq n,$$

and

$$(D^jF)_k = 0, \quad 1 \leqq k < n,$$

$$(D^jF)_n = (D_{nn})^j f \quad \text{for all } j.$$

Then we let $U_1 = u$ and one now realizes that u fulfils (3.21) and (3.22) where strictly speaking each coefficient should be modified by addition of a bounded function which is zero outside a denumerable set. But that does not alter the modified coefficients as members of $\mathscr{P}^0$. Corollary 3.2 is proved.

REMARK. We have chosen a regularization in Theorem 3.1 leading to a right continuous limit of the solutions of the regularized problems. We now choose ϕ as before but let $\phi(x, \varepsilon) = \varepsilon^{-1}\phi(x/\varepsilon)$. We call the solution of the problem regularized by this $\phi(x, \varepsilon)$ $u(x, \varepsilon)$. Let

$$\alpha = \int_{-1}^{0} \phi(t)dt$$

and let $u(x)$ be defined by Theorem 3.1. A slight modification of the proof of that theorem shows that

$$\lim_{\varepsilon \to 0} u(x, \varepsilon) = e^{-\alpha A(\{x\})}u(x^-) + g(-\alpha A(\{x\}))\alpha f(\{x\}). \tag{3.23}$$

We do not write down that proof.

If $\alpha = 1$, then we get the solution of Theorem 3.1. When $\alpha = 0$, the limit is left continuous corresponding to a reflection of the problem of Theorem 3.1 at the point $x = a$. But the most important thing is that outside the points with point masses in A or f the limit is independent of the choice of the regularizing functions.

I owe most of the remark above to the referee.

4. A stochastic measure differential equation.

In this section we combine measure differential equations with stochastic measure differential equations. Theorem 4.2 below can be seen as a generalization of Theorem 2.1. If one sees the Dirac measure as an idealization of something physical concentrated around a point but with no point mass at that point I guess that one should modify γ and μ of (4.7) below as is done in Theorem 3.1 for the limit of solutions of regularized problems and leave the true stochastic part unaltered. However the regularization which we use in order to get right continuous solutions causes problems since it uses values at t, $t > t'$, to define the value of the regularized solution at t'. If one instead chooses regularizations leading to a left continuous limit then the regularizations are compatible with the stochastic process. That would lead to a modification of γ and μ corresponding to the left continuous version of Theorem 3.1 which we have not written down here. It is just reflexion at the origin of Theorem 3.1. At the end, this left continuous solution would differ from the solution of (4.7) modified as suggested by Theorem 3.1 only at the points of discountinuity. So we stick to the right continuous version here.

For the probability theory the reader is referred to Gihman and Skorohod [9]. However the author's source has been Da Prato [6].

Let w be an n-dimensional Brownian motion on the probability space $(\Omega, \mathscr{E}, P)$. Let $\mathscr{F}_t = \sigma\{w(s); s \leqq t\}$ be the σ-algebra generated by the random variables $w(s)$, $s \leqq t$. Let u_0 be an n-dimensional random variable which is independent of $\cup_{t>0}\mathscr{F}_t$. Let $\mathscr{G}_t = \sigma\{\mathscr{F}_t, u_0\}$. Let $L^2_w(0, T; \mathsf{R}^n)$ be the set of n-dimensional stochastic processes adopted to $\mathscr{G} = (\mathscr{G}_t)_{t \geqq 0}$ such that

$$P\left(\int_0^T |u(s)|^2 ds < \infty\right) = 1. \tag{4.1}$$

Let $M^2_w(0, T; \mathsf{R}^n)$ be those u in $L^2_w(0, T; \mathsf{R}^n)$ such that

$$E\left(\int_0^T |u(s)|^2 ds\right) < \infty. \tag{4.2}$$

Let $C^+_w(0, T; \mathsf{R}^n)$ be those u in $M^2_w(0, T; \mathsf{R}^n)$ for which $u(t^-)$ and $u(t^+)$ exists with $u(t) = u(t^+)$, $0 \leqq t \leqq T$, with probability one and which also fulfil

$$\|u\|^2 = E\left(\sup_{0 \leqq t \leqq T} |u(t)|^2\right) < \infty.$$

Then $C^+_w(0, T; \mathsf{R}^n)$ is a Banach space with norm $\|\cdot\|$.

Definition 4.1. Let $\mu(\cdot, \omega)$ be an $n \times n$ matrix of signed Borel measures on $[0, T]$ *for each* $\omega \in \Omega$. Let

$$|\mu|(\cdot, \omega) = \sum_{i,j} |\mu_{ij}|(\cdot, \omega).$$

It is assumed that $|\mu|([0, T], \omega)$ is bounded on Ω and that for some integer N there are numbers t_k, $0 = t_0 < \ldots < t_N = T$ with

$$|\mu|((t_{k-1}, t_k), \omega) < 1/8, \quad 1 \leqq k \leqq N, \omega \in \Omega. \tag{4.3}$$

The process $(t, \omega) \to \int_0^t d\mu(s, \omega)u(s, \omega)$ is assumed to be adopted to $\mathscr{G}$ for all $u \in M^2_w(0, T; \mathsf{R}^n)$. Then μ is called a bounded properly adopted $n \times n$ matrix of signed Borel measures.

Let $\gamma(\cdot, \omega) = (\gamma_1(\cdot, \omega), \ldots, \gamma_n(\cdot, \omega))$ where $\gamma_j(\cdot, \omega)$, $1 \leqq j \leqq n$, is a signed Borel measure on $[0, T]$ such that $|\gamma_j|([0, T], \omega)$ is bounded on Ω. It is assumed that $(t, \omega) \to \int_0^t d\gamma(s, \omega)$ is adopted to $\mathscr{G}$ with $\omega \to \gamma(\{0\}, \omega) = u_0(\omega)$, $\omega \in \Omega$. Then γ

is called a bounded properly adopted n-dimensional vector of signed Borel measures. Here and in the rest of the section an integral sign with lower limit a and upper limit b denotes integration over $[a, b]$.

THEOREM 4.2. *Let T, $(\Omega, \mathcal{E}, P)$, $(w(t))_{t \geqq 0}$, u_0, and $C_w(0, T; \mathbb{R}^n)$ be as in Definition 4.1. Let γ be a bounded properly adopted n-dimensional vector and let μ be a bounded properly adopted $n \times n$ matrix of signed Borel measures. Let u_0 be square integrable. Let $G(t, x, \omega)$, $0 \leqq t \leqq s \leqq T$ be measurable in $\mathcal{B}([0, T] \times \mathbb{R}^n) \times \mathcal{G}_s$ with values in the space of real $n \times n$ matrices. Here $\mathcal{B}$ stands for the σ-algebra of Borel sets. It is assumed that $G(t, u(t, \omega), \omega)$ is an adopted stochastic process for each u in $M_w^2(0, T, \mathbb{R}^n)$ and that there exists a constant M such that with the Euclidean norm on $\mathbb{R}^{n^2}$*

$$(4.4) \qquad |G(t, x, \omega)| \leqq M(1 + |x|), \quad (t, x, \omega) \in [0, T] \times \mathbb{R}^n \times \Omega,$$

and

$$(4.5) \quad |G(t, x, \omega) - G(t, y, \omega)| \leqq M|x - y|, \quad 0 \leqq t \leqq T, \ x, y \in \mathbb{R}^n, \ \omega \in \Omega,$$

are true with probability one. Let I be the $n \times n$ identity matrix. It is assumed that

$$(4.6) \qquad \mu(\{t\}, \omega) - I \text{ is invertible}, \ 0 \leqq t \leqq T, \ \omega \in \Omega.$$

Then there is a unique $u \in C_w^+(0, T; \mathbb{R}^n)$ solving

$$(4.7) \quad u(t, \omega) = u_0(\omega) + \int_{0^+}^{t} d\gamma(s, \omega) + \int_{0^+}^{t} d\mu(s, \omega) u(s, \omega) + \int_{0^+}^{t} G(s, u(s, \omega), \omega) dw(s).$$

with probability one.

PROOF. Let $u \in C_w^+(0, T; \mathbb{R}^n)$. Let $\phi(u)(t, \omega)$ be equal to the right member of (4.7). Then the first two integrals in (4.7) represent a function in $C_w^+(0, T; \mathbb{R}^n)$. From (4.4) one sees that $G_{ij}(s, u(s, \omega), \omega)$ is in $M_w^2(0, T; \mathbb{R}^n)$ for $1 \leqq i, j \leqq n$. This shows that

$$t \to \int_{0}^{t} G(s, u(s, \omega), \omega) dw(s)$$

is continuous with probability one. Further one sees that

$$E\left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \int_{0}^{t} G_{ij}(s, u(s, \cdot), \cdot) dw(s)\right)^2\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} E\left(\int_{0}^{t} (G_{ij}(s, u(s, \cdot), \cdot))^2 ds\right), \quad 0 \leqq t \leqq T.$$

This combined with (4.4) gives

$$E\left(\sup_{0 \leqq t \leqq T}\left|\int_0^t G(s, u(s, \cdot), \cdot)dw(s)\right|^2\right) \leqq 2M^2T(1+\|u\|^2) < \infty.$$

That means that ϕ is a map from $C_w^+(0, T; \mathrm{R}^n)$ into itself.

We now refine the partition of the interval $[0, T]$ used in (4.3) such that for a new N both

$$t_k - t_{k-1} < (8M^2)^{-1}, \quad 1 \leqq k \leqq N, \tag{4.8}$$

and (4.3) are true. We shall use this N from now on.

We first assume that $N = 1$. We let f and g be in $C_w^+(0, T; \mathrm{R}^n)$ both with the value $u_0(\omega)$ at $(0, \omega)$. It follows from the definition of ϕ that

$$E(|(\phi(f)-\phi(g))(t,\cdot)|^2) \leqq 2E\left(\left|\int_{0^+}^t d\mu(s,\cdot)(f-g)(s,\cdot)\right|^2\right)+$$

$$+2E\left(\left|\int_0^t (G(s,f(s,\cdot),\cdot)-G(s,g(s,\cdot),\cdot))dw(s)\right|^2\right)$$

$$\leqq 2E\left(\left(\sup_{0 \leqq s < T}|f-g|(s,\cdot)\int_{0^+}^{T^-} d|\mu|(s,\cdot)\right)^2\right)+$$

$$+2E\left(\int_0^T |G(s,f(s,\cdot),\cdot)-G(s,g(s,\cdot),\cdot)|^2 ds\right).$$

We use (4.3), (4.5) and (4.8) and get

$$E\left(\sup_{0 \leqq s < T}|\phi(f)-\phi(g)|^2(s,\cdot)\right) \leqq 2^{-1}E\left(\sup_{0 \leqq s < T}|f-g|^2(s,\cdot)\right). \tag{4.9}$$

The restriction to $[0, T)$ of $C_w^+(0, T, \mathrm{R}^n)$ with the norm modified in the natural way gives a new Banach space. Then (4.9) shows that ϕ is a contraction mapping giving a fixed point u fulfilling (4.7) with probability one. Then $u(T, \omega)$ is uniquely defined from (4.6) and

$$u(T,\omega) = u(T^-,\omega)+\mu(\{T\},\omega)u(T,\omega)+\gamma(\{T\},\omega)$$

for those ω for which $u(T^-, \omega)$ is defined. This u is unique in the sense of the theorem.

Let $N > 1$. It follows from the proof with $N = 1$ that we can find a $u \in C_w^+(0, T; \mathsf{R}^n)$ such that (4.7) is true with probability one for $0 \leqq t \leqq t_1$. The restriction of u to this interval is unique with probability one. Let it be true that there is a $u \in C_w^+(0, T; \mathsf{R}^n)$ such that u solves (4.7) in $[0, t_k]$. Then we look at those functions in $C_w^+(0, T; \mathsf{R}^n)$ which restricted to $[0, t_k]$ are equal to u on this interval. Then we replace 0 by t_k and T by t_{k+1} in the argument used for $N = 1$. Then this argument shows that there is another $u \in C_w^+(0, T; \mathsf{R}^n)$ solving (4.7) for $0 \leqq t \leqq t_{k+1}$. Here we uses the fact the subspace of $C_w^+(0, T; \mathsf{R}^n)$ where each element is equal to the unique solution of (4.7) on $0 \leqq t \leqq t_k$, is itself a Banach space. This induction proves the theorem in the general case.

The result of this section is essentially taken from Persson [20].

5. An extension of Libri's theorem.

Here we refer the reader to [5] and especially to [5, Theorem 6.2, p. 83].

THEOREM 5.1. *Let A be an $n \times n$ matrix with entries in $\mathscr{P}^0$ and let I be the $n \times n$ identity matrix such that*

$$A(\{x\}) + I \text{ is invertible}, \quad x \in \mathsf{R}. \tag{5.1}$$

Let $\phi_1, \ldots, \phi_n$ be n linearily independent solutions of $u' + Au = 0$ in the sense of Theorem 2.1 that is $u = \phi_j$ solves (2.2) and (2.3) with $f = 0$ and $c = c^{(j)}$, where the vectors $c^{(j)}$ are linearly independent. Let Φ be the $n \times n$ matrix having ϕ_j as its jth column. Then to each compact interval $K \subset \mathsf{R}$ there is a constant $b > 0$ such that

$$|\det \Phi(x)| \geqq b, \quad x \in K. \tag{5.2}$$

On the other hand let Φ be an $n \times n$ matrix with entries in $\mathscr{P}^1$. If to each compact interval K there is a constant $b > 0$ such that (5.2) is true, then there is a unique $n \times n$ matrix A with entries in $\mathscr{P}^0$ such that (5.1) is true and such that $\Phi' + A\Phi = 0$ in the sense of Theorem 2.1.

PROOF. Let Φ fulfil the hypothesis of the first part of the theorem. If for some compact interval K, (5.2) is not true for any constant $b > 0$, then there exists points $x_j \in K$, $j = 1, 2, \ldots$, such that $\det \Phi(x_j) \to 0$, $j \to \infty$. Since K is compact we may assume that x_j tends to a $x' \in K$, $j \to \infty$. Then either $\det \Phi(x') = 0$ or $\det \Phi(x'^-) = 0$. In the first case $\Phi(x')$ is not invertible so it must be the second case. If $\Phi(x'^-)$ is not invertible, then (5.1) and $u = \Phi$ in (2.2) gives

$$\Phi(x') = (A(\{x'\})+I)^{-1}\Phi(x'^{-}).$$

So also this leads to the contradiction of the first case.

Let the hypothesis of the second part of the theorem be fulfilled. It follows that $\Phi(x)^{-1}$ exists for all x. Let $A = -\Phi'\Phi^{-1}(x)$. We choose a compact interval K and then a $b > 0$ such that (5.2) is true. Let $\phi_j'(\{x'\}) = b_j$, $1 \leqq j \leqq n$, for a fixed $x' \in K$. Let $H_{x'}(x) = 0$, $x < x'$, and $H_{x'}(x) = 1$, $x \geqq x'$. Let B be an $n \times n$ matrix with b_j as its jth column. Let $\bar{\Phi} = \Phi - H_{x'}(x)B$. Then $\bar{\Phi}$ is continuous at $x = x'$. We notice that $A(\{x'\}) = B\Phi(x')^{-1}$ and $\Phi(x')\Phi(x')^{-1} = I$. We get

$$I + A(\{x'\}) = \Phi(x')\Phi(x')^{-1} - B\Phi(x')^{-1} = \bar{\Phi}(x')\Phi(x')^{-1} = \Phi(x'^{-})\Phi(x')^{-1}.$$

Then (5.2) shows that (5.1) is true for this A. That A is unique follows as in the classical proof. Theorem 5.1 is proved.

COROLLARY 5.2. *Let $n > 0$ be an integer and let*

$$(5.3) \qquad Lu = u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_0 u, \quad u \in \mathscr{P}^n,$$

with $a_j \in \mathscr{P}^0$, $0 \leqq j < n$. Let

$$(5.4) \qquad a_{n-1}(\{x\}) \neq -1, \ x \in \mathrm{R}.$$

Let $\phi_1, \ldots, \phi_n$ be n linearly independent solutions of $Lu = 0$ in $\mathscr{P}^n$ in the sense of Corollary 2.2. Let W denote the Wronskian. Then to each compact interval $K \in \mathrm{R}$ there is a constant $b > 0$ such that

$$(5.5) \qquad |W(\phi_1, \ldots, \phi_n)(x)| \geqq b, \quad x \in K.$$

On the other hand if $\phi_j \in \mathscr{P}^n$, $1 \leqq j \leqq n$, and if to each compact K, there is a $b > 0$ such that (5.5) *is fulfilled, then*

$$u \to (-1)^n (W(\phi_1, \ldots, \phi_n)(x))^{-1} W(u, \phi_1, \ldots, \phi_n) = Lu$$

fulfils the hypothesis of the first with the given ϕ_j as a fundamental set of solutions. There is no other operator of (5.3) *type fulfilling the same conditions.*

PROOF. For $n = 1$, it is just Theorem 5.1 for $n = 1$. Let $n > 1$.

Let the hypothesis of the first part of the corollary be fulfilled. Define a new matrix $A = (a_{jk})$ with entries in $\mathscr{P}^0$ by letting $a_{jj+1} = 1$, $1 \leqq j < n$, let $a_{nk} = a_{k-1}$, $1 \leqq k \leqq n$, and let the other $a_{jk} = 0$. Let $U = (u, u', \ldots, u^{(n-1)})$. Then

$$Lu = 0 \Leftrightarrow U' + AU = 0$$

and (5.1) is true if and only if (5.4) is fulfilled. We see that

$$\psi_j = (\phi_j, \phi_j', \ldots, \phi_j^{(n-1)}), \quad 1 \leqq j \leqq n,$$

is a fundamental set of solutions of $U' + AU = 0$. Let Ψ be the $n \times n$ matrix whose jth column is ψ_j. We see that $\det \Psi = W(\phi_1, \ldots, \phi_n)$. Thus (5.5) follows from the first part of Theorem 5.1.

Let the hypothesis of the second part of the corollary be fulfilled and let Ψ be defined as in the proof of the first part. Then $A_0 = -\Psi'\Psi^{-1}$ is the unique matrix with entries in $\mathscr{P}^0$ such that $\Psi' + A_0\Psi = 0$ and such that $A_0(\{x\}) + I$ is invertible for all x according to the second part of Theorem 5.1. Define A as in the first part of the proof. We see that $\Psi' + A\Psi = 0$. Theorem 5.1 then says that $A = A_0$. But $I + A(\{x\})$ is triangular with entries equal to one on the diagonal except in the lower right corner where it is $1 + a_{n-1}(\{x\})$. Then (5.1) is equivalent to (5.4). Corollary 5.2 is proved.

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