ALGEBRAIC AND GEOMETRIC APPROACH TO THE CLASSIFICATION OF SEMISPACES

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Let $L$ be a linear space over an ordered field $K$ and let $M$ be a subspace of $L$. Maximal convex subsets of $L \setminus M$ are called semispaces of $L$ at $M$ (comp. Hammer [2]). Klee [4] gave a classification of semispaces in the case of $K = \mathbb{R}$ and $M = 0$. Those semispaces correspond to linear orderings of $L$ (see [3] and [6]). What is more, the conditions

(1) $S$ is a semispace at $M$ in $L$,
(2) $S$ is a (strong) preordering on $L$ with $S_0 = M$,

(see Section 1 for definitions) are equivalent ([6, Proposition 2.2]).

Consideration of those equivalent algebraic objects gives some advantages. Firstly, instead of semispaces at $M$ in $L$, one can consider semispaces at $0$ in $L/M$. Secondly, the classical results about ordered groups (see [1]) can be adopted in a natural way to linear orderings. Since the field of reals is the only complete ordered field, indecomposable real orderings are one-dimensional. This fact enables one to obtain Klee’s classification purely algebraically.

Thanks to Theorem 1.5 or more general Theorem 2.5, a classification of semispaces of a linear space over an Archimedean field (and a classification of ordered abelian groups) can be reduced to the case of the field of reals, that is, to the classification of Klee.

It seems that the algebraic method of the proof of Theorem 1.5 cannot be applied to the proof of the more general Theorem 2.5 and to the proof of Theorem 2.4. The last theorem (announced in the introduction of [6]) and the description [5] of convex half-spaces (i.e. convex sets with convex complements) in $\mathbb{R}^n$ give a description of convex half-spaces in any finite-dimensional space over an Archimedean field.

The first part of the paper is written in terms of orderings and the second independent one in terms of convexity.

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1. Description of linear preorderings.

Let $K$ be an ordered field with the set $K^+$ of positive elements, and let $L$ be a linear space over $K$. A set $S \subseteq L$ is called a (strong) $K$-preordering on $L$, if it satisfies the following conditions

(i) $S + S \subseteq S$, $K^+S \subseteq S$,
(ii) $L = S \cup S_0 \cup (-S)$ for some linear subspace $S_0$ of $L$,

where the symbol $\cup$ stands for the disjoint union. Then $L$ becomes a partially ordered linear space with the relation $<$ defined as follows: $a < b \iff b - a \in S$. If $a < b$ for every $a \in A, b \in B$, we write $A < B$. The space $L$ is fully ordered by the relation $<$ if $S_0 = 0$. In this case $S$ is called a (strong) $K$-ordering on $L$. It is easy to see that the above definitions coincide with those of [6]. We leave to the reader the proof of the following result.

**Lemma 1.1.** Let $M$ be a subspace of $L$ and let $v : L \to L/M$ denote the natural homomorphism. There is a natural one-to-one correspondence between $K$-preorderings on $L$ satisfying $S_0 = M$ and $K$-orderings $\bar{S}$ on $L/M$, which is given by $\bar{S} = v(S)$ and $S = v^{-1}(\bar{S})$.

The above Lemma reduces the investigation of $K$-preorderings to the study of more familiar $K$-orderings. Many properties of them are analogous to those of [1], stated for ordered groups (see Propositions 1.2–1.4 below). Therefore some proofs are sketched only.

Let $\mathcal{V}$ be a chain of subspaces of $L$. For $V \in \mathcal{V}$ denote by $\bar{V}$ the union of all $W \in \mathcal{V}$ properly contained in $V$. We say that the chain $\mathcal{V}$ is admissible, if for every $x \in L$ there exists a smallest subspace $V_x \in \mathcal{V}$ containing $x$. In other words, $x \in V_x \setminus \bar{V}_x$ and, as a consequence, $V = \bar{V}$ if and only if $V \neq V_x$ for every $x \in L$. For an admissible chain $\mathcal{V}$ on $L$ we have

$$L = \bigcup_{x \in L} (V_x \setminus \bar{V}_x) = \bigcup_{V \in \mathcal{V}} (V \setminus \bar{V}),$$

and it is clear that we can reject every $V \in \mathcal{V}$ equal to $\bar{V}$, obtaining another admissible chain with the same operator $\bar{L}$. Any chain closed under intersections is admissible. Each admissible chain $\mathcal{V}$ contains a smallest subspace $V_0 \in \mathcal{V}$. Evidently $\bar{V}_0 = \emptyset$, and except for this case, every $\bar{V}$ is a subspace of $V$.

**Proposition 1.2.** Let $\mathcal{V}$ be an admissible chain of subspaces of $L$ with the smallest subspace 0, and let $\mathcal{W} = \mathcal{V} \setminus \{0\}$. There is a natural one-to-one correspondence between the set of $K$-orderings $S$ on $L$ satisfying $S \cap V < S \setminus V$ for all $V \in \mathcal{W}$ and the set of systems $(\bar{S}_V; V \in \mathcal{W})$ of $K$-orderings $S_V$ on $V/\bar{V}$. 
PROOF. Let $S$ be a $K$-ordering on $L$. Then $L \setminus 0 = \cup_{V \in \mathcal{W}} (V \setminus \bar{V})$ and $S = \cup_{V \in \mathcal{W}} S_V$ where $S_V = S \cap (V \setminus \bar{V})$. The assumption $S \cap V < S \setminus V$ means that $S_V < S_W$ for $V \notin W$. Obviously, $S_V$ is a $K$-preordering on $V$ with $(S_V)_0 = \bar{V}$, and this gives us a $K$-ordering $S_V$ on $V/\bar{V}$, for all $V \in \mathcal{W}$. Conversely, the inverse images $S_V$ of $S_V$ are $K$-preorderings on $V$ (for all $V \in \mathcal{W}$), satisfying $S_V \pm S_W \subset S_V$ for $W \notin V$. Thus we get a $K$-ordering $S = \cup_{V \in \mathcal{W}} S_V$ on $L$ satisfying the desired condition.

We say that $S$ in question (Proposition 1.2) is the ordinal sum of $K$-orderings $S_V$. We call a $K$-ordering $S$ decomposable if it is a proper ordinal sum, that is, if there exists a proper non-zero subspace $V$ of $L$ satisfying $S \cap V < S \setminus V$. Otherwise $S$ is called indecomposable. The next proposition gives us the unique decomposition of a $K$-ordering $S$ into an ordinal sum of indecomposable $K$-orderings. The family of these indecomposable $K$-orderings is called the skeleton of $S$.

**Proposition 1.3.** If $S$ is a $K$-ordering on $L$, then the family $\mathcal{V}$ of all subspaces $V \subset L$ satisfying $S \cap V < S \setminus V$ is an admissible chain with the smallest subspace $0$ and it is closed under intersections and unions. The corresponding $K$-orderings $S_V$ are indecomposable.

**Proof.** Let $V, W \in \mathcal{V}$ and suppose that there exist $x \in V \setminus W$ and $y \in W \setminus V$. We can assume that $x, y \in S$. Then $x \in S \cap V$, $y \in S \setminus V$ and hence $x < y$. By symmetry we get $y < x$, contradiction. The rest of the proof is immediate.

Recall [1] that a group or a group ordering $S$ is called Archimedean if for every $x, y \in S$ we have $N x \not< y$ (that is, there exists an integer $n > 0$ such that $nx > y$). Let us assume in the sequel that the field $K$ (i.e., the ordering $K^+$) is Archimedean. A well-known theorem of H"older (see [1, pp. 45 and 126]) states that each Archimedean group (respectively field) is an ordered subgroup (respectively subfield) of the group (respectively field) of real numbers $\mathbb{R}$. Consequently, we can assume that $K \subset \mathbb{R}$ and $K^+ = \mathbb{R}^+ \cap K$.

**Proposition 1.4.** If $L$ is a linear space over an Archimedean field $K$ and $S$ is a $K$-ordering on $L$, then the following conditions are equivalent:

1. $S$ is indecomposable,
2. $S$ is Archimedean,
3. $S = f^{-1}(\mathbb{R}^+)$ for some $K$-linear imbedding $f : L \to \mathbb{R}$.

In particular, indecomposable real ordered spaces are at most one-dimensional.

**Proof.** (1) $\iff$ (2). We show a more general equivalence for $K$ being an arbitrary ordered field: $S$ is decomposable if and only if $K^+ x < y$ for some
x, y ∈ S. The condition 0 < K⁺x < y gives us a proper non-zero subspace

\[ V = \{ z ∈ L ; |z| ≤ λx \text{ for some } λ ∈ K^+ \} \]

satisfying \( S \cap V < S \setminus V \). Conversely, for any such subspace \( V \), if \( x ∈ S \cap V \) and \( y ∈ S \setminus V \), then \( K⁺x < y \).

(2) ⇔ (3). This is a version of Hölder's theorem and the proof is similar. For a fixed \( x ∈ S \) and \( y ∈ L \) the real number \( f(y) \) is defined by the section \( (L(y)|U(y)) \), where

\[ L(y) = \{ λ ∈ K ; λx ≤ y \} \]

and

\[ U(y) = \{ λ ∈ K ; λx > y \}. \]

The mapping \( f \) is \( K \)-linear since \( L(y+z) = L(y)+L(z) \) and \( L(μy) = μL(y) \) for \( y, z ∈ L \) and \( μ ∈ K^+ \).

The above propositions allow us to prove two extension theorems. Note that the first of them is true also in the more general situation when \( K \) and \( F \) are arbitrary ordered fields, not necessarily Archimedean (see Theorem 2.5). However, the proof of the general case uses methods of convexity.

**Theorem 1.5.** Let \( K \) be a subfield of an Archimedean field \( F \) such that \( K^+ = F^+ \cap K \). For any \( K \)-preordering \( S \) on \( L \) there exists an \( F \)-preordering \( T \) on \( FL = F \otimes_K L \) such that \( S = T \cap L \) and \( T₀ = FS₀ \). In particular, an ordering on \( L \) extends to an ordering on \( FL \).

**Proof.** (a) Reduction. Assume that our theorem holds for orderings. Let \( S₀ = M \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
L & \xrightarrow{\varphi} & FL \\
v \downarrow & & \downarrow \mu \\
L/M & \xleftarrow{\psi} & F(L/M) = FL/FM
\end{array}
\]

It follows from Lemma 1.1 that \( S = v^{-1}(S) \) where \( S \) is a \( K \)-ordering on \( L/M \). Since \( S = ψ^{-1}(T) \) for an \( F \)-ordering \( T \) on \( F(L/M) \), we see that

\[ S = v^{-1}ψ^{-1}(T) = ϕ^{-1}μ^{-1}(T) = T \cap L, \]

where \( T = μ^{-1}(T) \) is an \( F \)-preordering on \( FL \) satisfying \( T₀ = FM \).

(b) The case of orderings. Let \( S \) be a \( K \)-ordering. Since \( L ⊂ FL ⊂ RL \) it can be assumed that \( F = R \). If \( S \) is indecomposable, then, in virtue of
Proposition 1.4, we have the following commutative diagram:

\[
\begin{array}{c}
L & \xleftarrow{f} & R \\
\cap & \xrightarrow{g} & RL \\
\end{array}
\]

where \( S = f^{-1}(R^+) \) and \( g \) is \( R \)-linear. Take some (for example, lexicographic) \( R \)-ordering \( t \) on \( \text{Ker}(g) \). Proposition 1.2 gives an \( R \)-ordering \( T = t \cup g^{-1}(R^+) \) on \( RL \) satisfying \( T \cap L = S \). For an arbitrary \( K \)-ordering \( S \) consider the skeleton \( (S_V; V \in \mathcal{W}) \) over \( K \). As above, we obtain a family \( (\tilde{T}_V; V \in \mathcal{W}) \), where \( \tilde{T}_V \) is an \( R \)-ordering on \( R(V/\tilde{V}) = RV/R\tilde{V} \) satisfying \( \tilde{T}_V \cap (V/\tilde{V}) = S_V \). Proposition 1.2 gives us an \( R \)-ordering \( T = \cup_{V \in \mathcal{W}} T_V \) on \( RL \). As in (a), \( T_V \cap V = S_V \), and finally \( T \cap L = S \).

**Theorem 1.5'**. Let \( S \) be a strong ordering on an abelian group \( A \). Then \( S = T \cap A \) for some \( R \)-ordering \( T \) on \( RA = R \otimes \mathbb{Z} A \).

**Proof.** By Levi Theorem (see [1, p. 36]), the group \( A \) is torsion-free. Hence it is contained in \( RA = R \otimes \mathbb{Q} A_{(0)} \), where \( A_{(0)} \) denotes the linear space over the field \( \mathbb{Q} \) of rationals, being the localization of \( A \). The ordering \( S \) can be extended to a \( \mathbb{Q} \)-ordering \( S_{(0)} = \{ s/t; s \in S, t \in \mathbb{N} \} \) on \( A_{(0)} \), and we are in position to apply Theorem 1.5. Since \( R \) is a flat \( \mathbb{Z} \)-module, it is possible to give also a direct proof similar to the part (b) of the proof of Theorem 1.5; it is based on original results from [1] instead of Propositions 1.2–1.4.

The above theorems reduce our investigation to the real field \( R \). Propositions 1.3 and 1.4 give us the following description of this case:

**Proposition 1.6.** Every \( R \)-preordering has the form

\[
S = \bigcup_{V \in \mathcal{W}} (R^+ e_V + \tilde{V}),
\]

where \( \mathcal{W} = \mathcal{V} \setminus \{ V_0 \} \) for some admissible chain \( \mathcal{V} \), \( \dim(V/\tilde{V}) = 1 \) and \( e_V \in V \setminus \tilde{V} \) for each \( V \in \mathcal{W} \). In this case, \( S_0 = V_0 \).

**Proof.** Lemma 1.1 reduces the proof to the case when \( S \) is an \( R \)-ordering. In virtue of Proposition 1.4, the skeleton of \( S \) consists of one-dimensional orderings, each of the form \( R^+ e \). The rest is immediate.

A more familiar description of \( R \)-orderings (or, equivalently, \( R \)-semispaces at \( 0 \)) is known as the so-called "Klee representation" (see [3], [4]). Jamison [3] pointed out that this representation is possible only over the field \( R \).
Here we give a relative generalization for $K$-preorderings over an Archimedean field $K$.

Let $K$ denote a subring of $R$, for example the ring of integers or an Archimedean field. A subset $\Phi$ of $\text{Hom}_K(L, R)$ totally ordered by $<\text{ is called admissible with respect to a subspace } V_0 \text{ of } L$, if for every $x \in L \setminus V_0$ there exists a smallest member $\varphi_x \in \Phi$ with $\varphi_x(x) \neq 0$. It is called simply admissible if $V_0 = 0$. If $\Phi$ is admissible with respect to $V_0$ then

$$S = S(\Phi, <) = \{x \in L \setminus V_0; \varphi_x(x) > 0\}$$

is a $K$-preordering on $L$ with $S_0 = V_0$. Conversely, we have

**Theorem 1.7.** If $K$ is an Archimedean field, then any $K$-preordering $S$ has the form $S = S(\Phi, <)$, where $\Phi$ is admissible with respect to $S_0$. Every abelian group ordering has the form $S(\Phi, <)$ for some admissible $\Phi$.

**Proof.** By Theorem 1.5 or 1.5' we can assume that $K = R$. In this case, the description of $S$ presented in Proposition 1.6 gives us a family

$$\{\psi_V : V \to R ; V \in \mathcal{V}\}$$

of linear functionals defined by the conditions $\psi_V(e_V) = 1$ and $\psi_V(\tilde{V}) = 0$. Let $\varphi_V$ denote an extension of $\psi_V$ on $L$. The family

$$\Phi = \{\varphi_V ; V \in \mathcal{V}\}$$

is admissible with respect to $V_0 = S_0$ if the ordering $<$ is defined as follows:

$$\varphi_w < \varphi_V \iff W \supseteq V.$$ 

In fact, if $x \in L \setminus V_0$, then $x \in V_x \setminus \tilde{V}_x$ with $V_x \in \mathcal{V}$, and therefore $\varphi_x = \varphi_{V_x} \in \Phi$. Since $\varphi_x(x) > 0$ if and only if $x \in R^+ e_{V_x} + \tilde{V}_x$, we obtain $S = S(\Phi, <)$.

2. Convexity under extension of the space.

Let $K_2$ be an ordered field and let $K_1$ be a subfield of $K_2$. Consider a linear space $L_2$ over $K_2$ and a subspace $L_1$ of $L_2$ over $K_1$ such that every subset of $L_1$ linearly independent over $K_1$ is also linearly independent over $K_2$. For instance, if $L_1$ is a linear space over $K_1$, then in the part of $L_2$ we can take the space $K_2 \otimes_{K_1} L_1$ over $K_2$. The spaces $\mathbb{Q}^n$ and $\mathbb{R}^n$ are simple examples of such $L_1$ and $L_2$.

The symbols $\text{aff}_i A$, $\text{conv}_i A$ mean the affine and the convex hulls of a set $A$ of the space $L_i$ over $K_i$, $i = 1, 2$.

**Lemma 2.1.** For every $A \subset L_1$ we have

$$\text{conv}_1 A = L_1 \cap \text{conv}_2 A.$$ 

**Proof.** Let $a \in L_1 \cap \text{conv}_2 A$. There exists a finite minimal set $\{a_0, \ldots, a_n\} \subset A$ for which $a \in \text{conv}_2 \{a_0, \ldots, a_n\}$. Hence $a = \alpha_0 a_0 + \ldots + \alpha_n a_n$ for some $\alpha_i \in K_2$. 

such that $0 \leq \alpha_i \leq 1$, $i = 0, \ldots, n$, and $\alpha_0 + \ldots + \alpha_n = 1$. We can assume that $a_0 = 0$. Since $a_1, \ldots, a_n$ are linearly independent and $a = \alpha_1 a_1 + \ldots + \alpha_n a_n$ over $K_2$ it follows that $a$ is a linear combination of $a_1, \ldots, a_n$ over $K_1$, that is, $\alpha_1, \ldots, \alpha_n \in K_1$. From $\alpha_0 + \ldots + \alpha_n = 1$ we get $\alpha_0 \in K_1$. Consequently,

$$a \in \text{conv}_1 \{a_0, \ldots, a_n\} \subseteq \text{conv}_1 A.$$  

We see that $L_1 \cap \text{conv}_2 A \subseteq \text{conv}_1 A$. The inverse inclusion is obvious.

From the above Lemma we obtain that a subset of $L_1$ is convex if and only if it is the intersection of $L_1$ with a convex subset of $L_2$. More exactly: $A = L_1 \cap \text{conv}_2 A$ for every convex subset $A$ of $L_1$.

**Lemma 2.2.** If $A_i$, $i = 1, \ldots, n$, are convex subsets of $L_1$ and $\bigcap_{i=1}^n A_i = \emptyset$, then $\bigcap_{i=1}^n \text{conv}_2 A_i = \emptyset$.

**Proof.** Suppose that $\bigcap_{i=1}^n \text{conv}_2 A_i \neq \emptyset$. Thus $\bigcap_{i=1}^n \text{conv}_2 F_i \neq \emptyset$ for some finite $F_i \subseteq A_i$, $i = 1, \ldots, n$. Consequently, there exist minimal finite subsets $G_i$ of $A_i$, $i = 1, \ldots, n$, such that $\bigcap_{i=1}^n \text{conv}_2 G_i \neq \emptyset$. Let $w$ be a point of this set.

Suppose that there exists a point $z \in \bigcap_{i=1}^n \text{aff}_2 G_i$ different from $w$. Let $G_1 = \{c_1, \ldots, c_m\}$. Obviously, $w$ and $z$ have the forms

$$w = \alpha_1 c_1 + \ldots + \alpha_m c_m, \quad z = \beta_1 c_1 + \ldots + \beta_m c_m,$$

where $\alpha_1 + \ldots + \alpha_m = 1$, $\beta_1 + \ldots + \beta_m = 1$, $\alpha_j \geq 0$ and $\alpha_j, \beta_j \in K_2$ for $j = 1, \ldots, m$. Since $w \neq z$ and $(\alpha_1 - \beta_1) + \ldots + (\alpha_m - \beta_m) = 0$, at least one of the scalars

$$\lambda_j = \alpha_j/(\alpha_j - \beta_j), \quad j = 1, \ldots, m,$$

is defined and non-negative. Let $\lambda_j = \lambda_j$ be the smallest non-negative one. We have $\alpha_j + \lambda_j (\beta_j - \alpha_j) \geq 0$ for $j = 1, \ldots, m$ with the equality for $j = j_0$. Moreover,

$$\sum_{j=1}^m [\alpha_j + \lambda_j (\beta_j - \alpha_j)] = \lambda_j \sum_{j=1}^m \beta_j + (1 - \lambda_j) \sum_{j=1}^m \alpha_j = 1.$$

Therefore the point

$$g_1 = \sum_{j=1}^m [\alpha_j + \lambda_j (\beta_j - \alpha_j)] c_j = w + \gamma_1 (z - w)$$

of the half-line with the end-point $w$ through $z$ belongs to the convex hull of a proper subset of $G_1$. Generally, a point

$$g_i = w + \gamma_i (z - w), \quad \text{where} \quad \gamma_i \geq 0,$$

of this half-line belongs to the convex hull of a proper subset of $G_i$ for
every $i = 1, \ldots, n$. Let $\gamma_i = \min\{\gamma_1, \ldots, \gamma_m\}$. Obviously, $g_{i_0} \in \text{conv}_2 G_i$ for $i = 1, \ldots, n$. Since $g_{i_0}$ belongs to the convex hull of a proper subset of $G_{i_0}$, we obtain a contradiction with the minimality of $G_{i_0}$. Consequently, $\bigcap_{i=1}^m \text{aff}_2 G_i = \{w\}$. This means that $w$ is the only solution over $K_2$ of a system of linear equations with coefficients from the field $K_1$. So $w \in L_1$. Since $w \in \text{conv}_2 G_i$, from Lemma 2.1 we obtain $w \in \text{conv}_1 G_i$, $i = 1, \ldots, n$. Hence $w \in \bigcap_{i=1}^n A_i$. A contradiction with the assumption.

The following consequence of Lemma 2.2 is of independent interest:

**Proposition 2.3.** For arbitrary convex sets $A_i$ of $L_1$, $i = 1, \ldots, n$, we have

$$\text{conv}_2 \bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \text{conv}_2 A_i.$$  

**Proof.** It is sufficient to consider the case $n = 2$.

Let $x \in \text{conv}_2 A_1 \cap \text{conv}_2 A_2$. For some finite $F_1 \subset A_1$ and $F_2 \subset A_2$ we have $x \in \text{conv}_2 F_1 \cap \text{conv}_2 F_2$. Consider an extreme point $y$ of $\text{conv}_2 F_1 \cap \text{conv}_2 F_2$. We can find minimal sets $M_1 \subset F_1$ and $M_2 \subset F_2$ such that $y \in \text{conv}_2 M_1 \cap \text{conv}_2 M_2$. Since $y$ is extreme, $\text{conv}_2 M_1 \cap \text{conv}_2 M_2 = \{y\}$. From Lemma 2.2 we get $y \in \text{conv}_2 (M_1 \cap M_2)$. The arbitrariness of the extreme point $y$ implies $\text{conv}_2 F_1 \cap \text{conv}_2 F_2 \subset \text{conv}_2 (M_1 \cap M_2)$. Consequently, $x \in \text{conv}_2 (M_1 \cap M_2) \subset \text{conv}_2 (A_1 \cap A_2)$, which proves the inclusion $\supset$. The inverse inclusion is obvious.

The example of intervals with end-points $a_i, b_i \in \mathbb{Q}$ such that $a_i < \sqrt{2} < b_i$, $i = 1, 2, \ldots$ and $\lim a_i = \sqrt{2} = \lim b_i$ shows that the above equality and Lemma 2.2 do not hold for infinite intersections.

Let us observe that Lemma 2.2 and Proposition 2.3 enable simple transfers of some theorems on intersection of convex sets (e.g. of Helly-type theorems) from linear spaces over the field of reals into linear spaces over subfields of reals, and consequently, over Archimedean fields.

**Theorem 2.4.** A subset of $L_1$ is a convex half-space of $L_1$ if and only if it is the intersection of $L_1$ with a convex half-space of $L_2$.

**Proof.** Obviously, the intersection of $L_1$ with a convex half-space of $L_2$ is a convex half-space of $L_1$.

Let $G$ be a convex half-space of $L_1$. From Lemma 2.2 we obtain $\text{conv}_2 G \cap \text{conv}_2 (L_1 \setminus G) = \emptyset$. By a known result of Kakutani (comp. Theorem 2.3 in [7]): the proof is correct also in the general situation of linear spaces over ordered fields) there exists a convex half-space $H$ of $L_2$ such that $\text{conv}_2 G \subset H$ and $H \subset \text{conv}_2 (L_1 \setminus G) = \emptyset$. So $G \subset H$ and $H \cap (L_1 \setminus G) = \emptyset$. Since $G \subset L_1$, we get $G = H \cap L_1$. 


THEOREM 2.5. A subset of $L_1$ is a semispace of $L_1$ at a subspace $M$ of $L_1$ if and only if it is the intersection of $L_1$ with a semispace of $L_2$ at the subspace $\text{conv}_2 M = \text{aff}_2 M$.

PROOF. It follows from Lemma 2.1 that $M = L_1 \cap \text{conv}_2 M$. Then the "if" part of our Theorem is evident in the language of preorderings. However, it can be shown without preorderings using the following characterization of semispaces (see [6, Proposition 2.2]): a subset $S$ of a linear space $L$ over an ordered field is a semispace of $L$ at a subspace $M$ if and only if $S$ is convex and $L = S \cup M \cup (-S)$.

Conversely, let $S$ be a semispace of $L_1$ at $M$. Thanks to Lemma 2.2 the set $\text{conv}_2 S$ is disjoint with the subspace $\text{conv}_2 M$. By Zorn's Lemma there exists a semispace $T$ of $L_2$ at $\text{conv}_2 M$ containing $\text{conv}_2 S$. From the "if" part it follows that $T \cap L_1$ is a semispace of $L_1$ at $M$. Since $S \subset T \cap L_1$ is also a semispace of $L_1$ at $M$, we have $S = T \cap L_1$.

As we pointed out in the introduction, Theorem 2.5 together with Klee classification enables a classification of semispaces in any linear space over an Archimedean field. Similarly, Theorem 2.4 together with part 1 of Theorem 1 of [5] gives a description of convex half-spaces in a finite-dimensional linear space over an Archimedean field. The analogous description does not concern infinite-dimensional spaces because of the existence of convex half-spaces which are not translates of semispaces (see Remark 2.5 in [6]).

REFERENCES