CHARACTERIZATION OF THE PREDUAL AND IDEAL STRUCTURE OF A JBW*-TRIPLE

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In recent years, a certain category of normed Jordan triple systems called JB*-triples has been an object of study in both complex analysis and functional analysis. A standard example of a JB*-triple system is a norm-closed subspace of the space of all bounded linear operators on a complex Hilbert space which is also closed under the Jordan triple product \( \{xy^*z\} := \frac{1}{2}(xy^*z + zy^*x) \). Therefore, JB*-triples generalize C*-algebras. The importance of the category of JB*-triples in complex analysis stems from its equivalence with the category of bounded symmetric domains with base point in complex Banach spaces [17]. A detailed presentation of this theory is contained in [29]. In the context of functional analysis, JB*-triples arise naturally in the solution of the contractive projection problem for C*-algebras in [9] (see also [18]).

In this paper, we study JBW*-triples, i.e., JB*-triples which are dual Banach spaces, in analogy to the theory of JBW*-algebras [13]. After the presentation of preparatory material in section 1 and section 2 we will characterize the predual of a JBW*-triple by various conditions and prove its uniqueness in section 3. The main result of section 4 relates the ideal structure of a JBW*-triple to that of the JBW*-algebra determined by a complete tripotent.

In a forthcoming paper, we will prove a coordinatization theorem for JBW*-triples and we will use it to obtain a classification of JBW*-triples of type I.

Since the completion of this study which is contained in the author's dissertation (1984), great progress has been made in the theory of JB*-triples in [7], [2] (separate weak-*-continuity of the product of a JBW*-triple) and [11]. The results in [2] and [11], however, may not be used to simplify proofs given here as they in turn make use of results presented here and in the mentioned forthcoming paper.

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1. Jordan-*-triples

Our standard reference for Jordan algebras is [15], for Jordan triple systems it is [22] and [23].

A Jordan-*-triple is a complex vector space $U$ with a sesquilinear map $U \times U \to \text{End}(U) : (x, y) \to x \otimes y^*$ such that

\begin{equation}
(1.1) \quad \text{the triple product } \{xy^*z\} := x \otimes y^*(z) \text{ is symmetric in } x \text{ and } z,
\end{equation}

\begin{equation}
(1.2) \quad \{uv^*\{xy^*z\}\} = \{\{uv^*x\}y^*z\} - \{x\{vu^*y\}^*z\} + \{xy^*\{uv^*z\}\}
\end{equation}

for all $u,v,x,y,z \in U$.

Let $Q(x, z)y := \{xy^*z\}$, $Q(x) := Q(x, x)$ for all $x, y, z \in U$. Every Jordan-*-algebra is a Jordan-*-triple in the product

\begin{equation}
(1.3) \quad \{xy^*z\} := (x \circ y^*) \circ z - (x \circ z) \circ y^* + (y^* \circ z) \circ x.
\end{equation}

(1.4) A Jordan-*-triple $U$ is called abelian if $x \otimes y^*$ and $u \otimes v^*$ commute or equivalently, if $\{xy^*\{uv^*w\}\} = \{(xy^*u)v^*w\}$ for all $x, y, u, v, w \in U$.

A subtriple generated by a single element is always abelian.

(1.5) A non-zero tripotent (i.e., an element $e$ with $Q(e)e = e$) in a Jordan-*-triple $U$ induces a decomposition of $U$ into the eigenspaces of $e \otimes e^*$, the Peirce decomposition $U = U_1(e) \oplus U_{1/2}(e) \oplus U_0(e)$ where $U_k(e) := \{z \in U | \{ee^*z\} = \{ee^*z\} \} \text{ for } k = 0, \frac{1}{2}, 1$.

$U_k := U_k(e)$ is called the Peirce-$k$-space of $e$.

For Peirce-$k$-spaces, the following multiplication rules hold:

\begin{equation}
(1.6) \quad \{U_1, U_0\} = \{U_0, U_1\} = 0,
\end{equation}

\begin{equation}
(1.7) \quad \{U_i, U_j\} \subset U_{i-j+k}, \text{ where } i, j, k \in \{0, \frac{1}{2}, 1\} \text{ and }
\end{equation}

\begin{equation}
U_l := \text{ for } l \neq 0, \frac{1}{2}, 1.
\end{equation}

In particular, Peirce-$k$-spaces are subtriples.

(1.8) The projection $p_k^e = p_k$ of $U$ onto $U_k$ with $p_k(z) = 0$ for $z \in U_j, j \neq k$ is called the Peirce-$k$-projection of $e$.

$p_k^e$ is a polynomial over the integers in $e \otimes e^*$. Furthermore, $p_1^e = Q(e)^2$. This and (1.7) yield $U_1(e) = Q(e)U$. 

(1.9) Two tripotents \( e \) and \( f \) are called *compatible* if \( p_j^e \) and \( p_k^f \) commute for all \( j,k \in \{0, \frac{1}{2}, 1\} \).

(1.10) If \( f \in U_k(e) \) for some \( k \in \{0, \frac{1}{2}, 1\} \), then \( e \) and \( f \) are compatible. This follows from (1.8) together with (1.2).

In particular, \( e \) and \( f \) are compatible if they are orthogonal, i.e., if \( \{ee*f\} = 0 \) which implies \( e \square f^* = f \square e^* = 0 \).

A finite compatible family \( e_1, \ldots, e_n \) of tripotents has a joint Peirce decomposition

\[
U = \bigoplus_{k_i = \{0, \frac{1}{2}, 1\}} U_{k_1}(f_1) \cap \cdots \cap U_{k_n}(f_n).
\]

(1.11) A tripotent \( e \) is called *complete* if \( U_0(e) = 0 \). \( e \) is called *unitary* if \( U = U_1(e) \).

(1.12) The *Peirce spaces with respect to an orthogonal family* \( \mathcal{E} = (e_i)_{i \in I} \) of tripotents are defined by

\[
U_{ii} := U_1(e_i), \; U_{ij} := U_{1/2}(e_i) \cap U_{1/2}(e_j) \quad (i \neq j),
\]

\[
U_{io} := U_{0i} := U_{1/2}(e_i) \cap \bigcap_{j \neq i} U_0(e_j), \; U_{00} := \bigcap_{i \in I} U_0(e_i).
\]

\( \mathcal{E} \) is called *complete* if \( U_{00} = 0 \).

The sum \( P \) of the Peirce spaces is direct. If \( \mathcal{E} \) is finite then \( P = U \).

(1.13) \( \{U_{ij}U_{jk}^*U_{kl}\} \subseteq U_{ii} \quad (i,j,k,l \in I \cup \{0\}) \).

Products of Peirce spaces which cannot be written in this form vanish.

1.14) *Lemma. Let* \( e, f \) *be tripotents in* \( U \).

1. \( f \in U_1(e) \) *implies* \( U_1(f) \subseteq U_1(e) \) and \( U_0(e) \subseteq U_0(f) \).

2. \( f \in U_1(e) \) and \( e \in U_1(f) \) imply \( U_k(e) = U_k(f) \) for every \( k = 0, \frac{1}{2}, 1 \).

*Proof.* (1) \( U_1(f) = Q(f)U \subset U_1(e) \) by (1.7), \( (f \square f^*)U_0(e) = 0 \) by (1.6).

(2) By (1), \( U_k(e) = U_k(f) \) for \( k = 0, 1 \). The compatibility of \( e \) and \( f \) (1.10) then yields \( U_{1/2}(e) = U_{1/2}(f) \).

2. J\( \star \)-triples.

A J\( \star \)-*triple* is a Jordan-\*-*triple* \( U \) endowed with a complete norm such that the triple product is jointly continuous, \( z \square z^* \) is a hermitian operator.
with positive spectrum and

\[(2.1) \quad \|\{zz^*z\}\| = \|z\|^3 \quad \text{for all } z \in U.\]

\[(2.1) \quad \text{is equivalent to}\]

\[(2.2) \quad \|z \boxtimes z^*\| = \|z\|^2 \quad \text{for all } z \in U \quad (\text{cf. } [17, (5.3)]).\]

Closed subtriples of JB*-triples and \(l^\infty\)-sums of JB*-triples are again JB*-triples.

Any JB*-algebra (cf. [30], is a JB*-triple in the product (1.3) ([29, 20.35]). So in particular, every C*-algebra is a JB*-triple in the product \(\{xy^*z\} := \frac{1}{2}(xy^*z + zy^*x).\)

(2.3) Conversely, if \(e\) is a tripotent in a JB*-triple \(U\), then \(U_1(e)\) is a JB*-algebra with product \(x \circ y := \{xe^*y\}\) and involution \(x^* := \{ex^*e\}.\) (Cf. [6, (2.2)] and [19, (3.7)]).

(2.4) Proposition. The surjective isometries of JB*-triples are precisely the algebraic isomorphisms.

Proof. Let \(f : U_1 \to U_2\) be an algebraic isomorphism of the JB*-triples \(U_1\) and \(U_2\) (no continuity assumed). Then \(\sigma(z \boxtimes z^*) = \sigma(f(z) \boxtimes f(z)^*)\) and so

\[\|z\|^2 = \|z \boxtimes z^*\| = \sup \sigma(z \boxtimes z^*) = \sup \sigma(f(z) \boxtimes f(z)^*) = \cdots = \|f(z)\|^2\]

for all \(z \in U_1\) because for hermitian operators norm and spectral radius coincide ([27, Proposition 2]). The converse follows from [17, (5.5)].

(2.5) The complete tripotents of a JB*-triple \(U\) coincide with the complex and the real extreme points of the closed unit ball of \(U\) (cf. [19, (3.5)] and [6, (14.1)]).

(2.6) Let \(e\) be a tripotent in a JB*-triple. Then the Peirce projections of \(e\) are contractive. If \(e\) is complete they are hermitian.

(Use the fact that \(\exp(it e \boxtimes e^*)\) is an isometry for all real \(t\) (cf. [10, 1.2]).)

(2.7) Let \(e\) and \(f\) be tripotents in \(U\). \(f\) is said to be an \(e\)-projection if \(f\) is a projection in the Jordan-*-algebra \(U_1(e)\) (in the sense of (2.3)). If \(f\) is in the center of \(U_1(e)\) it is called a central \(e\)-projection.

3. JBW*-triples – characterizations of the predual.

A JB*-triple need not have any tripotents. However, if the JB*-triple is a dual Banach space then it follows from (2.5) and the Krein-Milman theorem that there exist “many” tripotents (cf. (3.11)).
(3.1) Definition. A JB*-triple $U$ is a JBW*-triple if $U$ (as a Banach space) has a predual $U_*$ such that

(3.2) the triple product is separately $\sigma(U, U_*)$-continuous.

$\sigma(U, U_*)$ will be also denoted by $w^*$.

In (3.21) it will be shown that a JBW*-triple has a unique predual in the following sense:

(3.3) A Banach space $E$ is said to have a \textit{unique} predual $F \subset E^*$, if $F$ is the only closed subspace of $E^*$ which is a predual of $E$ in the canonical duality. It should be noted, however, that a weaker notion of uniqueness of the predual is also used in the literature (see e.g. [12]).

(3.4) If $E_i$ are Banach spaces with unique preduals $F_i (i = 1, 2)$ then every surjective isometry $j : E_1 \rightarrow E_2$ is $\sigma(E_1, F_1) - \sigma(E_2, F_2)$-continuous.

(3.5) Remark. Barton and Timoney have recently shown [2] that (3.2) is a consequence of (3.1). However, in their proof they use (3.20) so that (3.2) cannot be omitted at this stage.

(3.6) A JBW*-algebra (i.e., a JB*-algebra with a predual) is a JBW*-triple in the product (1.3) as follows from [26, Lemma 2.2] and [8, Corollary 3.3].

Further examples of JBW*-triples can be obtained from

(3.7) A $\sigma(U, U_*)$-closed subtriple $V$ of a JBW*-triple $U$ (with predual $U_*$) is a JBW*-triple with predual $U_*/V^\circ$ (where $V^\circ$ is the polar of $V$ in $U_*$), and from

(3.8) If $(U_i)_{i \in I}$ is a family of JBW*-triples then $U := \bigoplus_{i \in I} U_i$ is a JBW*-triple.

(3.9) Lemma. If $e$ is a tripotent in a JBW*-triple $U$ then the Peirce projections of $e$ are $\sigma(U, U_*)$-continuous.

Proof. The Peirce projections of $e$ are polynomials in $e \Box e^*$. So (3.9) follows from (3.2).

(3.10) If $e$ is a tripotent in a JBW*-triple $U$ then $U_1(e)$ is a JBW*-algebra (by means of (2.3)).

Proof. This follows from (3.9).

With respect to the local properties of a JBW*-triple one obtains the following lemma:
\section*{(3.11) Lemma.} An abelian, $\sigma(U, U_*)$-closed subtriple $W$ of a JBW*-triple $U$ is isometrically isomorphic to a commutative $W^*$-algebra (endowed with the product (1.3)). In particular, the set of tripotents is norm-total in $U$.

\textbf{Proof.} It follows from (2.5) and the Krein-Milman theorem that $W$ contains a tripotent $e$ which is complete in $W$. Because $W$ is abelian one has
\[
\{ee^*\{ee^*z\} = \{ee^*e\}e^*z = \{ee^*z\} \quad \text{for all } z \in W,
\]
i.e., $e \triangleq e^*\vert_W$ is an idempotent map and therefore $W \cap U_{1/2}(e) = 0$. So $W$ is an associative $w^*$-closed $-subalgebra$ of the JBW*-algebra $U_1(e)$ (in the sense of (2.3)), i.e., $W$ is a commutative $W^*$-algebra with unit $e$. The $w^*$-closed subtriple generated by a single element is abelian by (1.4) and (3.4). In $W^*$-algebras, the set of projections is norm-total ([24, 11.1.3]). This proves the second assertion.

\section*{(3.12) Lemma.} If $U$ is a JBW*-triple then

1. for every $z \in U$ there is a complete tripotent $e \in U$ such that $z \in U_1(e)$ and $z = \{ez^*e\}$,

2. for every orthogonal family $(f_j)_{j \in J}$ of tripotents in $U$ there is a complete tripotent $f$ in $U$ such that $f_j$ is a $f$-projection for all $j \in J$.

\textbf{Proof.} The proofs of (1) and of (2) are parallel: The subtriple $V_z$ ($V$ respectively) generated by $z$ (by $(f_j)_{j \in J}$ respectively) is abelian. By Zorn's lemma there is an abelian subtriple $W_z$ ($W$ respectively) containing $V_z$ ($V$ respectively) which is maximal with respect to inclusion. By (3.2), $W_z$ ($W$ respectively) is $w^*$-closed.

By (3.11) we can assume that $W_z$ and $W$ are commutative $W^*$-algebras.

Let $z = u\hat{z}$ be the polar decomposition of $z$ in the $W^*$-algebra $W_z$. Then one checks immediately that $e := 1_{W_z} - uu^* + u$ is a tripotent which is unitary in $W_z$ and satisfies $z = \{ez^*e\}$. Let
\[
f := 1_{W} - \sum_{j \in J} f_j f_j^* + \sum_{j \in J} f_j.
\]

The sums exist in the $W^*$-algebra $W$ with respect to the $w^*$-topology. Using (3.4), it is easily checked that $f$ is a tripotent which is unitary in $W$ and that $f_j$ is a $f$-projection for all $j \in J$.

We show finally that $e$ and $f$ are complete tripotents in $U$: Suppose this is false. Then there is a $0 \neq x_z \in U_0(e)$ (a $0 \neq x \in U_0(f)$ respectively). But by (1.4) and (1.6) the subtriple generated by $\{x_z\} \subset W_z$ (by $\{x\} \subset W$ respectively) is abelian. This contradicts the maximality of $W_z$ (of $W$ respectively).

\section*{(3.13) Corollary.} An orthogonal family $\mathscr{F} := (f_j)_{j \in J}$ of tripotents in a
JBW*-triple $U$ is summable with respect to $\sigma(U, U_*)$. $g := \sum_{j \in J} f_j$ is a tripotent and $f_j$ is a $g$-projection for all $j \in J$.

$\mathcal{F}$ is a complete orthogonal family if and only if $g$ is a complete tripotent.

**Proof.** By (3.12) there is a tripotent $f$ in $U$ such that $\mathcal{F}$ is an orthogonal family of projections in the JBW*-algebra $U_1(f)$. Therefore $\mathcal{F}$ is summable in the $w^*$-topology. The next two statements follow from (3.2).

Finally, if $g$ is complete and $z \in U_{00}$ (for the notations see (1.12)) then

$$\{gg^*z\} = \sum_{j \in J} \{f_j f_j^* z\} = 0,$$

so $z = 0$.

Conversely, let $\mathcal{F}$ be complete. Because $f_j \in U_1(g)$ for all $j \in J$ one has $U_0(g) \subset \bigcap_{j \in J} U_0(f_j) = 0$ by (1.14)(1).

(3.14) A bounded linear map $p$ on a Banach space $U$ is called a projection on $U$ if $p^2 = p$. Two projections $p$ and $q$ on $U$ are orthogonal if $pq = qp = 0$.

(3.15) **Lemma.** Let $U$ be a JBW*-triple, let $\mathcal{F} := (f_j)_{j \in J}$ be an orthogonal family of tripotents in $U$. Then the Peirce sum with respect to $\mathcal{F}$ (see (1.12)) is $\sigma(U, U_*)$-dense in $U$.

More precisely: There are unique weak-$*$-continuous, pairwise orthogonal projections $p_{ij}$ on $U$ onto the Peirce spaces $U_{ij}$ $(i,j) \in J \cup \{0\}$ which are given by

$$p_{kk} = p_k^f = Q(f_k)^2,$$

where $k, l \in J$, $k = 1$,

$$p_{k1} = p_{1/2}^f p_{1/2}^l = 4Q(f_k, f_l)^2,$$

and $f := \sum_{j \in J} f_j$,

$$p_{k0} = p_{1/2}^f p_{1/2}^l,$$

$$p_{00} = p_0^f.$$

Every $z \in U$ lies in the $w^*$-closed subspace spanned by

$$\{p_{ij}(z) \mid i, j \in J \cup \{0\}\}.$$

**Proof.** $f$ exists by (3.13). Because $\mathcal{F} \cup \{f\}$ is a compatible family (1.9), $p_{ij}$ is a projection for all $i, j \in J \cup \{0\}$ which is $w^*$-continuous by (3.9).

Obviously, $p_{kl}(U) = U_{kl}$ holds for all $k, l \in J$.

"$p_{10}(U) \subset U_{10}":$ For $z \in p_{10}(U), g := \sum_{k \in J, k \neq l} f_k$ we have

$$\{gg^*z\} = \{ff^*z\} - \{f_l f_l^* z\} = \frac{1}{2}z - \frac{1}{2}z = 0,$$

so $z = 0$.
therefore $z \in U_0(f_k)$ for all $k \in J$, $k \neq l$ by (1.14)(1).

"$U_{10} \subset p_{10}(U)$": \quad $\{ff^*z\} = \{f_i f_i^* z\} = \frac{1}{2} z$ holds for all $z \in U_{10}$.

"$p_0(U) \subset U_{00}$": \quad Holds by (1.14)(1).

"$U_{00} \subset p_0(U)$": \quad $\{ff^*z\} = \sum_{j \in J} \{f_j f_j^* z\} = 0$ by (3.2).

This shows that $p_{kl}(U) = U_{kl}$ for all $k, l \in J \cup \{0\}$. The projections $p_{ij}$ ($i, j \in J \cup \{0\}$) commute and the Peirce-sum is direct, so the projections are pairwise orthogonal.

For the last assertion we may assume without loss of generality that $z \in U_m(f)$ for some $m \in \{0, \frac{1}{2}, 1\}$.

"$m = 1$": From (1.13) (for the orthogonal family $\{f_i, f_j\}$) follows $Q(f_i f_j) U \subset U_{ij}$, so $Q(f_k f_i) Q(f_i f_j) = 0$ for $\{k, l\} \neq \{i, j\}$ again by (1.13). Therefore

$$z = Q(f)^2 z = \sum_{i \in J} \sum_{j \in J} c_{ij} Q(f_i f_j)^2 z = \sum_{i \in J} \sum_{j \in J} c_{ij}^{-1} p_{ij}(z)$$

where $c_{ij} = 1$ for $i = j$ and $c_{ij} = 2$ for $i \neq j$ (summation with respect to the $w^*$-topology).

"$m = \frac{1}{2}$": \quad $p_1^*(z) \in U_1(f_j) \cap U_{1/2}(f) = 0$ for all $j \in J$ because $f$ and $f_j$ are compatible, so $p_{j0}(z) = p^*_{1/2}(z) = 2 \{f_j f_j^* z\}$. By (3.2),

$$z = 2\{ff^*z\} = 2 \sum_{j \in J} \{f_j f_j^* z\} = \sum_{j \in J} p_{j0}(z).$$

"$m = 0$": \quad Here nothing remains to be shown.

The uniqueness of the projections follows from the weak-$*$-density of the Peirce-sum.

In the following, $U$ always is a JBW$^*$-triple with a predual $U_*$ which satisfies (3.2).

(3.16) If $e$ is a tripotent in $U$ and if $f \in U_*$ then $f|_{U_1(e)} \in U_1(e)_*$ as a consequence of (3.9) and the uniqueness of the predual of the JBW$^*$-algebra $U_1(e)$ ([8, Corollary 3.7]).

The converse also holds:

(3.17) **Proposition.** If $f \in U^*$ and if $f|_{U_1(e)} \in U_1(e)_*$ for all complete tripotents $e$ in $U$ then $f \in U_*$. 

PROOF. Let $U^1$ be the closed unit ball of $U$. By the Krein-Šmulian theorem it suffices to show that $f|_{U^1}$ is continuous with respect to the topology induced by $\sigma(U, U_*)$.

Let $(z_i)_{i \in I}$ be a $w^*$-convergent net in $U^1$ with $\lim (z_i)_{i \in I} = : z_0$. Let
$$A := \{ g \in U^* \mid \text{there exists } x \in U^1 \text{ such that } \|g\| = g(x) \}. $$

By [3], $A$ is norm-dense in $U^*$.

Let $\epsilon > 0$. Choose $g \in A$, $x \in U^1$ and a complete tripotent $e$ in $U$ such that $\|f - g\| < \frac{\epsilon}{2}$, $\|g\| = g(x)$ and $x \in U_1(e)$ (3.12). For $i \in I \cup \{0\}$ let $z_i = z_i^l + z_i^{1/2}$ where $z_i^k \in U_k(e)$ for $k = 1, \frac{1}{2}$.

By (3.9) and the assumption for $f$ there is an $i_0 \in I$ such that $|f(z_0^l - z_i^l)| < \frac{\epsilon}{4}$ for all $i \geq i_0$. Furthermore we have $\|z_0^{1/2} - z_i^{1/2}\| \leq 2$ for all $i \in I$ by (2.6). By [10, Proposition 1a]], $g(U_{1/2}(e)) = 0$. Therefore $|f(z_0^{1/2} - z_i^{1/2})| < \frac{\epsilon}{4}$ for all $i \in I$. It follows that $|f(z_0 - z_i)| < \epsilon$ for all $i \geq i_0$.

REMARK. (3.16) and (3.17) show that a JB*-triple $U$ has at most one predual $U_*$ such that (3.2) is satisfied. It cannot be inferred from this, however, that $U$ has a unique predual.

Using (3.17), it is possible to generalize to JBW*-triples a known result about W*-algebras ([28, III 3.11]). Let us first state the result for JBW*-algebras.

(3.18) PROPOSITION. Let $A$ be a JBW*-algebra, let $f \in A^*$. Then the following conditions are equivalent

1. $f \in A_*$,
2. $f \left( \sum_{i \in I} e_i \right) = \sum_{i \in I} f(e_i)$ for every orthogonal family $(e_i)_{i \in I}$ of projections.

PROOF. By [8, 3.7] and [13, 4.4.15] there is a central projection $e$ in $A^{**}$ with $(A_*)^e = (1 - e)A^{**}$. If one defines the normal part of a functional $g \in A^*$ to be $g_n := eg$ (where $eg(z) := g(ez)$ for all $z \in A$) and the singular part of $g$ to be $g_s := g - g_n$, the proofs of [28, III Theorem 3.8, "(i) = \geq (ii)""] and [28, Theorem 3.11] carry over literally.

(3.19) PROPOSITION. Let $f \in U^*$. Then the following conditions are equivalent

1. $f \in U_*$,
2. $f \left( \sum_{i \in I} e_i \right) = \sum_{i \in I} f(e_i)$ for every orthogonal family $(e_i)_{i \in I}$ of tripotents.

PROOF. (2) follows from (1) by (3.13). Conversely, by (3.17) it suffices to
show that $f|_{U_1(g)} \in U_1(g)_*$ for every tripotent $g$ in $U$. So (1) follows from (2) by (3.17) and (3.18).

We show next that a predual of a JBW*-triple enjoys the property of being well-framed ("bien encadré") (cf. [12, Definition 14]) which will imply the uniqueness of the predual.

(3.20) **Proposition.** A predual $U_*$ of a JBW*-triple is well-framed.

**Proof.** By (3.19), for every $f \in U^* \setminus U_*$ there is an orthogonal family $(e_i)_{i \in I}$ of tripotents with $\sum_{i \in I} e_i = e$ and $\sum_{i \in I} f(e_i) \neq f(e)$.

Let $W$ be the norm closed subspace of $U$ spanned by $\{e, e_i | i \in I\}$. By (3.13), $(e_i)_{i \in I}$ is an orthogonal family of projections in the JBW*-algebra $U_1(e)$, so $W$ is (isometrically) isomorphic to a commutative C*-algebra with unit $e$. We show that $W$ does not contain a subspace isomorphic to $l_1^1(N)$: Because $l_1^1(N)$ is separable such a subspace would be contained in a closed subspace $W'$ of $W$ spanned by a countable subset of $\{e, e_i | i \in I\}$. It is easily checked that a closed subset of a commutative C*-algebra spanned by the unit and a countably infinite orthogonal family of projections is isomorphic to $c_0(N)$ which in turn is isomorphic to $c_0(N)$. But $c_0(N)$ does not contain a subspace isomorphic to $l_1^1(N)$ ([21, Theorem I 2.7]) so $W'$ and hence $W$ does not contain such a subspace.

This, together with the proof of [12, Proposition 3], shows that the closed unit ball of $W$ is "*-admissible" ([12, Definition 13]). Using [12, Proposition 17], we obtain the desired result.

(3.21) **Theorem.** The predual of a JBW*-triple is unique.

**Proof.** (3.20) and [12, Theorem 15] show that $U_*$ is unique in the sense of [12] and that every surjective isometry on $U$ is $\sigma(U, U_*)$-$\sigma(U, U_*)$-continuous which implies the uniqueness of $U_*$ in the sense of (3.3).

(3.22) **Corollary.** An (algebraic) isomorphism of JBW*-triples is weak-*-continuous.

**Proof.** By (2.4), every isomorphism of a JB*-triple is isometric. So the result follows from (3.21) and (3.4).

Summing up, one obtains the following characterization of the predual of a JBW*-triple:

(3.23) **Theorem.** Let $U$ be a JBW*-triple with predual $U_*$, let $f \in U^*$. Then the following conditions are equivalent

1) $f \in U_*$.
(2) there is a complete tripotent $e$ in $U$ such that 
\[ f|_{U_1(e)} \in U_1(e)_\star \quad \text{and} \quad f(U_{1/2}(e)) = 0. \]

(3) $f|_{U_1(e)} \in U_1(e)_\star$ for every complete tripotent in $U$.

(4) $f\left( \sum_{i \in I} e_i \right) = \sum_{i \in I} f(e_i)$ for every orthogonal family $(e_i)_{i \in I}$ of tripotents.

(5) $f|_W$ is weak-*-continuous for every maximal abelian subtriple $W$.

PROOF. The equivalence of (1), (3), and (4) was shown in (3.16), (3.17), and (3.19).

"(1) implies (2)"' The closed unit ball $U^1$ of $U$ is $w^*$-compact. Therefore there is a $w \in U^1$ with $f(w) = \|f\|$ and by (3.12) there is a complete tripotent $e$ with $w \in U_1(e)$. So $f(U_{1/2}(e)) = 0$ by [10, 1.2].

"(2) implies (1)"' Because $U_1(e)$ is $w^*$-closed and has a unique predual there is a $g \in U_1$ such that $f|_{U_1(e)} = g|_{U_1(e)}$. Let $p$ be the Peirce projection onto $U_1(e)$. Then $f = f \circ p = g \circ p \in U_1$ by (3.9).

Obviously, (1) implies (5).

"(5) implies (4)"' By (3.13) an orthogonal family $(e_i)_{i \in I}$ of tripotents is summable in the weak-*-topology. The subtriple $V$ spanned by $(e_i)_{i \in I}$ is abelian. Choose any maximal abelian (necessarily $w^*$-closed) subtriple $W$ of $U$ which contains $V$ and apply (5).

The following proposition is, of course, a consequence of the above mentioned result of [2]. It is used in their proof, however, and is therefore not omitted.

(3.24) PROPOSITION. If a JB*-triple $U$ has a unique predual $U_\star$ then the triple product is separately weak-*-continuous.

PROOF. $x \square x^*$ is a hermitian operator on $U$ for all $x \in U$. By [31, 3.4] and the uniqueness of the predual, $x \square x^*$ is $w^*$-continuous. By the polarization formula [16, (1.4)] $x \square y^*$ is $w^*$-continuous for all $x, y \in U$. In particular, the Peirce projections are $w^*$-continuous (see (1.8)).

Let $e$ be a tripotent in $U$. Then $Q(e)^2$ is the Peirce projection onto $U_1(e)$, $Q(e)|_{U_1(e)}$ is the involution of the JBW*-algebra $U_1(e)$ and is therefore $w^*$-continuous, so $Q(e) = Q(e)^3$ is $w^*$-continuous.
Let \( x \in U \). \( Q(x) \) is \( w^* \)-continuous if and only if \( f \circ Q(x) \in U_* \) for every \( f \in U_* \). Let \( f \in U_* \). By \([10, \text{Proposition 2}]\) and the above argument, there is a tripotent \( e \) in \( U \) such that \( f = f \circ Q(e)^2 \). Hence it suffices to show that \( Q(e)^2Q(x) \) is \( w^* \)-continuous. We have
\[
Q(y)Q(z) = 2(y \square z^*)^2 - y \square \{zy^*z\}^* \quad \text{for all } y, z \in U,
\]
so \( Q(y)Q(z) \) is \( w^* \)-continuous for all \( y, z \in U \). Hence, \( Q(e)^2Q(x) = Q(e)(Q(e)Q(x)) \) is \( w^* \)-continuous.

4. Ideals in JBW*-triples.

(4.1) A subspace \( J \) of a Jordan-*-triple \( U \) is called an ideal if \( \{UU^*J\} + \{UJ^*U\} \subset J \). Two ideals \( I \) and \( J \) are said to be orthogonal if \( I \cap J = 0 \). In this case, \( I \square J^* = J \square I^* = 0 \).

For a subset \( X \) of a JBW*-triple \( U \) let \( U(X) \) denote the weak-*-closed ideal in \( U \) generated by \( X \). For \( x \in U \) we write \( U(x) \) instead of \( U(\{x\}) \). The weak-*-closed linear span of the union of a family \( (X_k)_{k \in K} \) of subsets of \( U \) is denoted by \( \sum_{k \in K} X_k \). We recall that for any tripotent \( e \) in \( U \), \( U_1(e) \) naturally carries the structure of a JBW*-algebra.

(4.2) **Theorem.** Let \( U \) be a JBW*-triple, \( e \) a complete tripotent in \( U \). Then the map \( I \to U(I) \) is a bijection from the set \( \mathcal{J}_e \) of all weak-*-closed *-ideals of the JBW*-algebra \( U_1(e) \) onto the set \( \mathcal{J} \) of all weak-*-closed ideals of \( U \), with inverse \( J \to J \cap U_1(e) \). One has the following properties

1. \( U(I \cap J) = U(I) \cap U(J) \quad (I, J \in \mathcal{J}_e) \)
2. \( U \left( \sum_{k \in K} I_k \right) = \sum_{k \in K} U(I_k) \quad (I_k \in \mathcal{J}_e) \)
3. \( U(z) = U_1(z) + U_{1/2}(z) \) for every central e-projection \( z \), and every weak-*-closed ideal in \( U \) can be uniquely written in this form.
4. To every \( J \in \mathcal{J} \) there is a unique complementary ideal \( J^\perp \in \mathcal{J} \).
5. \( J = (J \cap I) \oplus (J \cap I^\perp) \) for all \( I, J \in \mathcal{J} \).

**Proof.** We first show (3): Let \( z \) be a central e-projection, \( w := e - z \). Consider the Peirce decomposition of \( U \) with respect to the orthogonal family \( \{z, w\} \). \( U_{z, w} := U_{1/2}(z) \cap U_{1/2}(w) = 0 \) because \( z \) is a central e-projection. \( U_{00} := U_0(z) \cap U_0(w) = 0 \) because \( z + w \) is a complete tripotent. Let

\[
U_{z, 0} := U_{1/2}(z) \cap U_0(w) = U_{1/2}(z), \quad U_{w, 0} := U_{1/2}(w) \cap U_0(z) = U_{1/2}(w).
\]

We show that \( U_{z, 0} \square U_{w, 0}^* = U_{w, 0} \square U_{z, 0}^* = 0 \). It then follows from the
multiplication rules (1.13) that $U_1(z) + U_{1/2}(z)$ is an ideal with complement $U_0(z) = U_1(w) + U_{1/2}(w)$. Suppose, $a \in U_{z,0}$, $b \in U_{w,0}$ and $a \square b^* \neq 0$. By (3.11), we can assume that $a$ is a tripotent. We have $a \perp w$, so $c := a + w$ is a tripotent. Let $b = b_0 + b_{1/2} + b_1$, where $b_k \in U_1(a)$ for $k = 0, \frac{1}{2}, 1$. Then $b_1 = Q(a)^\frac{1}{2} b = 0$ by (1.13), $b_{1/2} \in U_{w,0} \cap U_1(c)$ (because $a, z$ and $w$ are compatible) and $b_{1/2} \neq 0$ because $a \square b^* \neq 0$. So

$$0 \neq \{cb_{1/2}^*c\} = 2\{ab_{1/2}^*w\} \in U_{z,w}$$

by (1.13), a contradiction. Similarly, one shows $U_{w,0} \square U_{z,0}^* = 0$. Clearly, $U_1(z) + U_{1/2}(z) \subset U(z)$, so the first assertion of (3) follows.

Conversely, let $J$ be a $w^*$-closed ideal in $U$, let $I := J \cap U_1(e)$. $I$ is a $w^*$-closed *-ideal in the JBW*-algebra $U_1(e)$, so $I = U_1(z)$ for some central e-projection $z$ (cf. [8,4.3]). Clearly $U(z) \subset J$. If $U(z) \neq J$ then $U_{1/2}(e-z) \cap J \neq 0$ because $J$ is an ideal. This implies $U_1(e-z) \cap J \neq 0$ by [10,1.5], a contradiction. If $z'$ is a central e-projection with $U(z) = U(z')$, then $U_1(z) = U_1(z')$, so $z = z'$. This shows (3) and (4). (5) now follows from (3) and (4). If $z_I$ is the unique central e-projection associated with $I \in \mathcal{F}_e$ by [8,4.3], then $U(I) = U(z_I)$, so by (3), $I \to U(I)$ is the composition of the two bijections $I \to z_I$ and $z_I \to U(z_I)$. Its inverse is $J \to J \cap U_1(e)$ as shown above.

Let $I, J \in \mathcal{F}_e$. Then

$$U_1(e) \cap (U(I) \cap U(J)) = (U_1(e) \cap U(I)) \cap (U_1(e) \cap U(J)) = I \cap J,$$

so (1) follows.

Let $I_k \in \mathcal{F}_e(k \in K)$. Then $\sum_{k \in K} I_k$ and $\sum_{k \in K} U(I_k)$ are ideals by (3.2), $U(\sum_{k \in K} I_k) \supset U(I_j)$ for all $j \in K$ and $\sum_{k \in K} I_k \subset \sum_{k \in K} U(I_k)$, so (2) follows.

(4.2)(4) has the following converse:

(4.3) **Lemma.** Let $U$ be a JBW*-triple, let $I$ and $J$ be ideals in $U$ with $I \oplus J = U$. Then $I$ and $J$ are weak-*-closed.

**Proof.** Let $e$ be a complete tripotent in $U$ (2.5), let $f + g = e$ with $f \in I$, $g \in J$. Then $f$ and $g$ are tripotents with $I = U_0(g)$ and $J = U_0(f)$. So (4.3) follows from (3.9).

(4.4) **Lemma.** Let $U$ be a JB*-triple, let $I$ and $J$ be closed subtriples in $U$ with $I \oplus J = U$. Then $\|z + w\| = \max(\|z\|, \|w\|)$ for all $z \in I$, $w \in J$ if and only if $I$ and $J$ are ideals.

**Proof.** $I \oplus \infty J$ is a JB*-triple (operations defined componentwise). So (4.4) follows from the fact that the algebra isomorphisms of a JB*-triple are precisely the surjective isometries (2.4).

(4.5) **Lemma.** Let $U$ be a JBW*-triple, let $(U_k)_{k \in K}$ be an orthogonal family
of weak-*-closed ideals in $U$. Then $\sum_{k \in K} U_k$ is canonically isometrically isomorphic to $\bigoplus_{k \in K} U_k$.

**Proof.** Let $p_k$ be the canonical projection of $U$ onto $U_k$ ($k \in K$). $p_k$ is contrative by (4.2)(4) and (4.4), so $\phi(x) = (p_k(x))_{k \in K}$ defines a map from $\sum_{k \in K} U_k$ into $\bigoplus_{k \in K} U_k$. By (2.4), it suffices to show that $\phi$ is an algebraic triple isomorphism. Clearly, $\phi$ is an injective triple homomorphism. To show that $\phi$ is surjective, let $(x_k)_{k \in K}$ be bounded, $x_k \in U_k$ for every $k \in K$. Then the $w^*$-closed subtriple generated by $(x_k)_{k \in K}$ is abelian, therefore it is isomorphic to a commutative $W^*$-algebra by (3.11). But a bounded family of elements of a $W^*$-algebra which lie in pairwise orthogonal $w^*$-closed ideals is summable in the $w^*$-topology. So $\phi$ is surjective.

A Jordan-*-triple $U$ is called indecomposable if $U = I \oplus J$ for ideals $I$ and $J$ in $U$ implies $I = 0$ or $J = 0$.

(4.6) **Lemma.** Let $U$ be a JBW*-triple, $e$ a complete tripotent in $U$. Then the following conditions are equivalent:

1. $U$ is indecomposable,
2. $U$ and $0$ are the only weak-*-closed ideals in $U$,
3. the JBW*-algebra $U_1(e)$ is a factor (i.e., has trivial center).

**Proof.** (2) implies (1) by (4.3). The other implications follow from (4.2).

(4.7) **Definition.** A JBW*-triple which satisfies one of the conditions in (4.6) is called a JBW*-triple factor.

If one is interested in classifying JBW*-triples then one is naturally led to the following definitions:

(4.8) **Definition.** Let $U$ be a Jordan-*-triple, $p$ a tripotent in $U$. $p$ is called abelian if $U_1(p)$ is abelian in the sense of (1.4). $p$ is called minimal if $U_1(p) = \mathbb{C} \cdot p$.

(4.9) **Lemma.** An abelian tripotent $p$ in a JBW*-triple factor $U$ is minimal.

**Proof.** By (3.12), there is an $e$-projection $e$ in $U$ such that $p$ is an $e$-projection. $U_1(e)$ is a JBW*-algebra factor by (4.6), so $p$ is minimal in $U_1(e)$ ([13, 5.2.17]). Because $U_1(p) \subset U_1(e)$ $p$ is also minimal in $U$.

If $Z$ is the center of a JBW*-algebra $A$ and if $p$ is an abelian projection in $A$ then $U_1(p) = Zp$ by [13, 5.2.17]. For JBW*-triples we have the following weak analogue:
(4.10) **Lemma.** Let $U$ be a JBW*-triple, let $p$ be an abelian tripotent in $U$ with $U(p) = U$, let $(p_i)_{i \in I}$ be an orthogonal family of tripotents in $U$ with $\sum_{i \in I} p_i = p$. Then $U = \bigoplus_{i \in I} U(p_i)$.

**Proof.** By (3.12), there is a complete tripotent $e$ in $U$ such that $p$ is an $e$-projection. The central carrier of $p$ in the JBW*-algebra $(U_1(e), \circ, *)$ equals $e$ and $z \mapsto z \circ p$ maps the center $Z$ of $(U_1(e), \circ, *)$ isomorphically onto $U_1(p)$ by [13, 5.2.17]. So there is an orthogonal family $(z_i)_{i \in I}$ of central $e$-projections with $p_i = z_i \circ p$. Clearly, $Uz_i = U(p_i)$ and by (4.2) and (4.5), the result follows.

A JBW*-algebra is of type I if its self-adjoint part is a JBW-algebra of type I i.e., if there is an abelian projection with central carrier 1. This is equivalent to the existence of an abelian projection which generates the JBW*-algebra as a $w*$-closed ideal. In analogy to these notions one defines

(4.11) **Definition.** A JBW*-triple $U$ is of type I if there is an abelian tripotent $p$ in $U$ with $U(p) = U$.

(4.12) **Lemma.** Every $w*$-closed ideal $J$ of a JBW*-triple $U$ of type I is of type I.

**Proof.** Let $p$ be an abelian tripotent in $U$ with $U(p) = U$. Then the canonical projection of $U$ onto $J$ (4.2)(4) maps $p$ onto an abelian tripotent $q$ with $J = J(q)$.

(4.13) **Proposition.** Let $U$ be a JBW*-triple. Then there is a unique decomposition $U = U_1 \oplus U_0$ where $U_1$ and $U_0$ are ideals in $U$ such that $U_1$ is of type I and $U_0$ contains no non-zero abelian tripotents.

**Proof.** By Zorn's lemma, there is a maximal family $(p_i)_{i \in I}$ of abelian tripotents in $U$ such that the ideals $U(p_i)$ ($i \in I$) are pairwise orthogonal.

$$U_1 := \sum_{i \in I} U(p_i) = U \left( \sum_{i \in I} p_i \right)$$

is of type I and by maximality, $U_0 := U_1^\perp$ contains no abelian tripotents.

Suppose $U = V \oplus W$ where $V$ is of type I and $W$ contains no non-zero abelian tripotents. Then $U_0 = (U_0 \cap V) \oplus (U_0 \cap W)$ by (4.2)(5). By (4.12), $U_0 \cap V = 0$, that is, $U_0 = W$. By the uniqueness of the complement in (4.2)(4), also $U_1 = V$.

**Remark.** $U_0$ is isomorphic to a JW*-triple, i.e., an ultra-weakly closed $J^*$-algebra in the sense of [14]. It will follow from this, together with the classification of JBW*-triples of type I, that $U$ can be uniquely decomposed
into a special and an exceptional part. These facts cannot be proved at this stage, however, and will be shown in forthcoming papers.

(4.14) Proposition. Let $U$ be a JBW*-triple. Then the following conditions are equivalent

1. $U$ is of type I,
2. every non-zero, weak-*-closed ideal of $U$ contains a non-zero abelian tripotent,
3. there is a complete tripotent $e$ in $U$ such that $U_1(e)$ is a JBW*-algebra of type I.

Proof. The equivalence of (1) and (2) follows from (4.13) and (4.12).

"(1) implies (3)": Let $p$ be an abelian tripotent in $U$ with $U(p) = U$, let $e$ be a complete tripotent in $U$ such that $p$ is an (abelian) $e$-projection (cf. (3.12)). By (4.2), the $w^*$-closed ideal generated by $p$ in the JBW*-algebra $U_1(e)$ is $U_1(e)$, so (3) follows.

"(3) implies (1)": Let $p$ be an abelian projection in the JBW*-algebra $U_1(e)$ with central carrier $e$. Then $p$ is an abelian tripotent in $U$ with $U(p) = U$ by (4.2).

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