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LOGARITHMIC CONCAVITY OF THE INVERSE INCOMPLETE BETA FUNCTION WITH RESPECT TO THE FIRST PARAMETER

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Abstract

The beta distribution is a two-parameter family of probability distributions whose distribution function is the (regularised) incomplete beta function. In this paper, the inverse incomplete beta function is studied analytically as a univariate function of the first parameter. Monotonicity, limit results and convexity properties are provided. In particular, logarithmic concavity of the inverse incomplete beta function is established. In addition, we provide monotonicity results on inverses of a larger class of parametrised distributions that may be of independent interest.

1. Introduction

Let a probability distribution on $I \subset \mathbb{R}$ have a cumulative distribution function F. Given a number $p \in (0, 1)$, any number $q \in I$ that satisfies $F(q-) \leq p$ and $F(p) \geq p$ is called a p-quantile. Particularly interesting is the case where F is continuous and strictly increasing and supported on an interval, as then the p-quantile is unique, satisfies $F(q) = p$ and defines a function, which is the classic inverse distribution function $q = F^{-1}(p)$. This pertains in particular the cases we will focus on, where we have a strictly positive density.

We are interested in parametrised families of probability distributions and the behaviour of the p-quantile with respect to the parameter, with p being fixed. In case we have a family of continuous cumulative distribution functions F_a, with a being the parameter of the family, such that for each a the corresponding p-quantile is unique, we may define it as a function of a implicitly through the functional equation $F_a(q_p(a)) = p$.

In the case of the median (i.e. the $1/2$-quantile) of the gamma distribution, such studies have been done in several occasions, e.g. in [2], [7] and [8]. In [1], Adell and Jodrá explore a very interesting connection with a sequence due to Ramanujan. In [4] and [5], Berg and Pedersen give a proof of the continuous version of the Chen-Rubin conjecture, originally stated in [7], and they moreover prove convexity and find asymptotic expansions.

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In the present article, the main focus is on the p-quantile of the beta distribution, or equivalently the inverse of the (regularised) incomplete beta function (1.1), as a function of the parameter a. For a standard reference on the beta distribution see [10, Chapter 25]. This inverse has also been considered by Temme [18] who studied its uniform asymptotic behaviour. In particular, his results give a very accurate approximation for the inverse for $a + b > 5$. This is widely used in computer algorithms approximating the inverse incomplete beta function. See also [17] for some interesting inequalities for the median. In [11], logarithmic convexity/concavity results are proved for the regularised incomplete beta function with respect to to parameters, though the methods employed there are quite different, and there does not seem to be any direct connection with the results in the present article. In applications, (strict) logarithmic concavity is an important property, as it ensures the uniqueness of a minimum and it is invariant under taking products.

The beta function is defined for $\Re a, \Re b > 0$ as the integral

$$\mathrm{B}(a, b) := \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t.$$

It also has the following representation as a ratio of gamma functions

$$\mathrm{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

which gives a meromorphic continuation of the beta function in $\mathbb{C}^2$. More information on the beta function can be found in [3]. The incomplete beta function is defined for $x \in [0, 1]$ and $\Re a, \Re b > 0$ by

$$\mathrm{B}(x; a, b) := \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t.$$

The beta distribution is the 2-parameter family of probability distributions, whose cumulative distribution function is the regularised incomplete beta function

$$I(x; a, b) := \frac{\mathrm{B}(x; a, b)}{\mathrm{B}(a, b)}. \tag{1.1}$$

We fix $p \in (0, 1)$ and $b > 0$, and we consider the first parameter a as a variable. We shall see in the Appendix that, due to a reflection formula for the regularised incomplete beta function, we can translate some results to the case where we fix the other parameter instead. We consider the p-quantile of the beta distribution, which in the literature is often also called the inverse incomplete beta function, as a function of a. We denote it by $q\colon (0,\infty) \to (0,1)$ and define it implicitly by the equation $I(q(a); a, b) = p$, or equivalently by

$$\int_0^{q(a)} t^{a-1}(1-t)^{b-1}\,\mathrm{d}t = p\int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t. \tag{1.2}$$

In the literature this value is often denoted by $I_p^{-1}(a,b)$, and in our case q is the function $a \mapsto I_p^{-1}(a,b)$. Moreover, we consider the function

$$\phi(a) := -a\log q(a), \tag{1.3}$$

which turns out to contain further information on q. The Figures 1, 2, and 3 illustrate how the median of the beta distribution behaves with respect to a.

In the rest of the paper we fix $p \in (0,1)$. We first get the following two propositions, regarding monotonicity and first order asymptotics:

Proposition 1.1. *The function q in* (1.2) *is a real analytic and increasing function on* $(0,\infty)$. *It has limits*

$$\lim_{a\to 0} q(a) = 0$$

and

$$\lim_{a\to\infty} q(a) = 1.$$

Proposition 1.2. *The function ϕ in* (1.3) *is real analytic on* $(0,\infty)$. *It is decreasing if $b < 1$, constant if $b = 1$ and increasing if $b > 1$. It has limits*

$$\lim_{a\to 0}\phi(a) = -\log p$$

and

$$\lim_{a\to\infty}\phi(a) = \gamma_b,$$

where γ_b is the $(1-p)$*-quantile of the gamma distribution with parameter b.*

Then, we investigate the analytic properties of the inverse incomplete beta function deeper. In particular, investigating its logarithm, we obtain the following two results, which constitute the main contribution of this paper:

Theorem 1.3. *For fixed $b \in (0,1)$, ϕ in* (1.3) *is* (*strictly*) *convex.*

Theorem 1.4. *For fixed $b \in (0,\infty)$, q in* (1.2) *is* (*strictly*) *log-concave.*

Remark 1.5. One can infer from Figure 1 that q is neither concave nor convex; its reciprocal $1/q$, though, is logarithmically convex by Theorem 1.4,

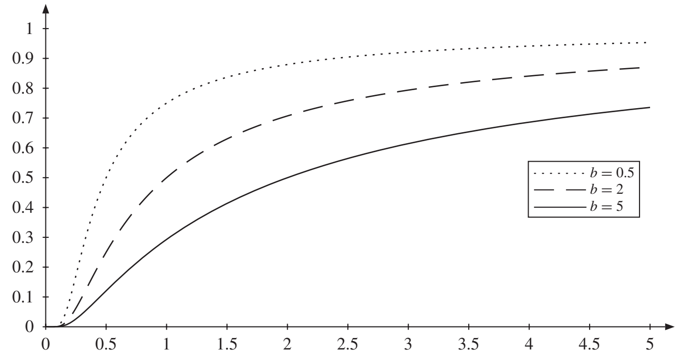

FIGURE 1. Plot of q for $p = 1/2$.

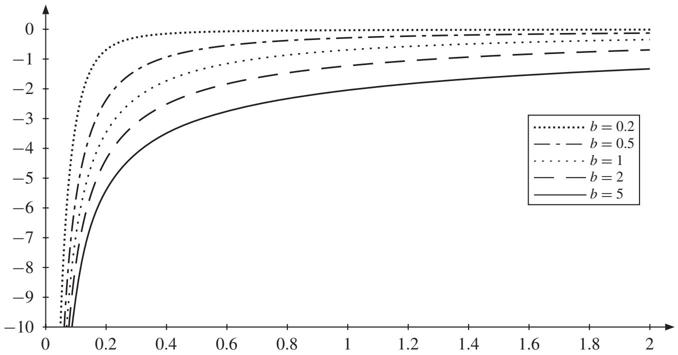

FIGURE 2. Plot of $\log q$ for $p = 1/2$.

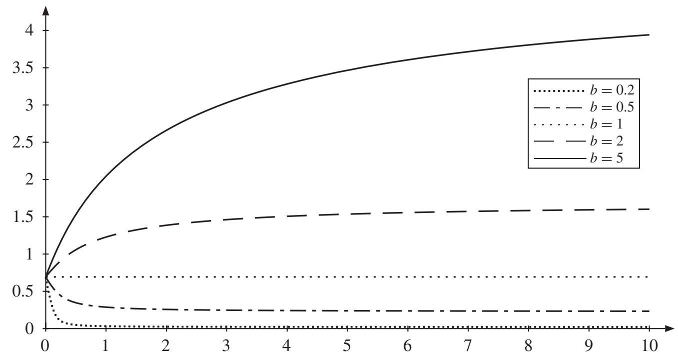

FIGURE 3. Plot of ϕ for $p = 1/2$.

hence also convex. Moreover, based on Figure 3, as well as numerical results, for $b > 1$ we conjecture that ϕ is concave.

The article is organised as follows. In §2 we present some general results regarding p-quantiles of more general probability distributions, that may be of independent interest. For instance, Lemma 2.1 is a generalisation of results concerning monotonicity properties of ratios of power series and polynomials to ratios of integrals. In §3 we study the monotonicity and limit properties of q and ϕ and prove Propositions 1.1 and 1.2. In §4 we prove convexity of ϕ for $b < 1$, while in §5 we prove logarithmic concavity of q. In the Appendix, we look into the dependence on the parameter b with a being fixed and translate some of the results to this case.

2. General results on p-quantiles of probability distributions

In the rest of this article, whenever x is a variable, ∂_x denotes differentiation with respect to the x, while when $j \in \mathbb{N}$ and f is a function of at least j variables, $\partial_j f$ denotes the jth partial derivative of f.

LEMMA 2.1. *Let $I \subset \mathbb{R}$ be an open interval, $A \subset \mathbb{R}$ a non-empty Borel set, μ a σ-finite Borel measure on A such that $\mu(A) > 0$, and $u, v \colon A \to [0, +\infty)$ measurable functions, not simultaneously* 0. *Let $f \colon I \times A \to (0, +\infty)$ be such that*

(i) *$a \mapsto f(a, t)$ is differentiable for μ-a.e. $t \in A$,*

(ii) *$t \mapsto u(t) f(a, t)$ and $t \mapsto v(t) f(a, t)$ are μ-integrable for all $a \in I$, and*

(iii) *for each compact subset $K \subset I$, there exists a function $g_K \colon A \to [0, +\infty)$ such that $u g_K$, $v g_K$ are μ-integrable and $|\partial_1 f(a, t)| \le g_K(t)$ for all $a \in K$ and μ-a.e. $t \in A$.*

Let $F \colon I \to \mathbb{R}$ be defined by:

$$F(a) := \frac{\int_A f(a, t) u(t) \, \mathrm{d}\mu(t)}{\int_A f(a, t) v(t) \, \mathrm{d}\mu(t)}.$$

Then, the following holds. If for all $a \in I$ and for all $t \in A \setminus N_a$, for some μ-null set N_a:

(I) *$\partial_1 f(a, t)/f(a, t)$ and $u(t)/v(t)$ both increase or both decrease with respect to t, then F is increasing;*

(II) *$\partial_1 f(a, t)/f(a, t)$ increases (decreases) with respect to t and $u(t)/v(t)$ decreases (increases), then F is decreasing.*

PROOF. Let $U(a)=\int_A f(a,t)u(t)\,\mathrm{d}\mu(t)$ and $V(a)=\int_A f(a,t)v(t)\,\mathrm{d}\mu(t)$. As $u(t)\partial_1 f(a,t)$ and $v(t)\partial_1 f(a,t)$ are dominated on compact subsets of I by a μ-integrable function of t, both U and V are differentiable, and the derivatives can be given by differentiating the integrands (see [12, Theorem 6.28]). Then, F' also exists and hence we need to investigate the derivative

$$F'(a)=\frac{U'(a)V(a)-U(a)V'(a)}{V^2(a)}.$$

We find

$$\begin{aligned}
&U'(a)V(a)-U(a)V'(a)\\
&=\int_A\int_A u(s)v(t)\big(\partial_1 f(a,s)f(a,t)-\partial_1 f(a,t)f(a,s)\big)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\
&=\int_A\int_{A\cap\{s<t\}} u(s)v(t)\big(\partial_1 f(a,s)f(a,t)-\partial_1 f(a,t)f(a,s)\big)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\
&\quad+\int_A\int_{A\cap\{s>t\}} u(s)v(t)\big(\partial_1 f(a,s)f(a,t)-\partial_1 f(a,t)f(a,s)\big)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\
&=\int_A\int_{A\cap\{s<t\}} u(s)v(t)\big(\partial_1 f(a,s)f(a,t)-\partial_1 f(a,t)f(a,s)\big)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\
&\quad+\int_A\int_{A\cap\{s<t\}} u(t)v(s)\big(\partial_1 f(a,t)f(a,s)-\partial_1 f(a,s)f(a,t)\big)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\
&=\int_A\int_{A\cap\{s<t\}} (u(s)v(t)-u(t)v(s))\big(\partial_1 f(a,s)f(a,t)\\
&\qquad\qquad -\partial_1 f(a,t)f(a,s)\big)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t),
\end{aligned}$$

where in the penultimate equality we made use of Fubini's theorem. The last integrand, and hence $F'(a)$, is non-negative (non-positive) if $\partial_1 f/f$ and u/v have the same (opposite) monotonicity properties μ-a.e., which proves the lemma.

REMARK 2.2. In the proceeding Lemma, the same conclusion holds if we allow u, v to assume the value zero at the same time, as then, without altering the values, we can just integrate over the set $A'=A\setminus(\{u(t)=0\}\cap\{v(t)=0\})$, which is again a Borel set, and we just consider the condition u/v being increasing (or decreasing) in A'.

REMARK 2.3. Lemma 2.1 generalises results concerning monotonicity properties of ratios of power series and polynomials. For instance, it gives [13, Lemma 2.2], if we set μ to be the counting measure on $\mathbb{N}$.

LEMMA 2.4. *Let I, J be two open intervals. Let $f\colon I \times J \to (0, \infty)$ be such that:*

(i) *$a \mapsto f(a, x)$ is differentiable for a.e. $x \in J$,*

(ii) *$x \mapsto f(a, x)$ is integrable for all $a \in I$, and*

(iii) *for each compact subset $K \subset I$, there exists an integrable function $g_K\colon J \to [0, +\infty)$ such that $|\partial_1 f(a, x)| \le g_K(x)$ for all $a \in K$ and a.e. $x \in J$.*

Let $q(a)$ be the p-quantile of the probability distribution with density

$$\frac{f(a, x)}{\int_J f(a, t)\,\mathrm{d}t}.$$

If for all $a \in I$ and for all $t \in J \setminus N_a$, for some null set $N_a \subset J$, $\partial_1 f(a, x)/f(a, x)$ increases (decreases) with respect to x, then $q(a)$ is increasing (decreasing).

PROOF. We will deal with the case that the logarithmic derivative of f is increasing, and the other case, that it is decreasing, is analogous. Let $x \in J = (c, d)$, where $-\infty \le c < d \le +\infty$. Then the cumulative distribution function is

$$F(a; x) = \frac{\int_c^x f(a, t)\,\mathrm{d}t}{\int_c^d f(a, t)\,\mathrm{d}t} = \frac{\int_c^d f(a, t) 1_{[c,x]}(t)\,\mathrm{d}t}{\int_c^d f(a, t)\,\mathrm{d}t}.$$

We set $u(t) = 1_{[c,x]}(t)$ and $v(t) = 1$. As $u/v = u$ decreases and $\partial_1 f/f$ increases with respect to t everywhere, except possibly on a null set depending on a, Lemma 2.1 gives that F decreases with respect to a for any fixed $x \in J$. This means

$$\frac{\int_c^{q(a+h)} f(a, t)\,\mathrm{d}t}{\int_c^d f(a, t)\,\mathrm{d}t} \ge \frac{\int_c^{q(a+h)} f(a+h, t)\,\mathrm{d}t}{\int_c^d f(a+h, t)\,\mathrm{d}t} = p = \frac{\int_c^{q(a)} f(a, t)\,\mathrm{d}t}{\int_c^d f(a, t)\,\mathrm{d}t},$$

so $q(a+h) \ge q(a)$. Hence the p-quantile is increasing, which ends the proof.

REMARK 2.5. In Lemma 2.1, if the logarithmic derivative $\partial_1 f/f$ is strictly monotone (and $u \not\equiv v$), it is easy to see from the proof that the ratio of the integrals in the conclusion should also be strictly monotone. Hence, also in Lemma 2.4, if the logarithmic derivative is strictly increasing (decreasing), then the p-quantile is also strictly increasing (decreasing).

The following lemma deals with the question of convergence of p-quantiles of a convergent sequence of probability distributions. We denote the two point compactification of $\mathbb{R}$ by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, with its usual topology.

Lemma 2.6. *Let F_n be a sequence of cumulative distribution functions on $\mathbb{R}$, converging pointwise to a function F_∞. Fix some $p \in (0, 1)$, and let q_n be a p-quantile of F_n for each n. Then, if (q_n) converges to a limit $q_\infty := \lim_{n\to\infty} q_n \in \overline{\mathbb{R}}$, it follows that*

$$q_\infty \in \big[\sup\{x \in \mathbb{R} \mid F_\infty(x) < p\}, \inf\{x \in \mathbb{R} \mid F_\infty(x) > p\}\big],$$

and thus if F_∞ is a cumulative distribution function, q_∞ is a p-quantile of F_∞.

Proof. It is practical to extend F_n and F_∞ to $\overline{\mathbb{R}}$ by $F_n(-\infty) = 0$ and $F_n(+\infty) = 1$, and analogously for F_∞. Let $w \in \overline{\mathbb{R}}$ be such that $F_\infty(w) < p$. By pointwise convergence, we have that there is some $n_0 \in \mathbb{N}$ such that $F_n(w) < p \leq F_n(q_n)$ for all $n > n_0$. As each F_n is non-decreasing, we have that $w < q_n$ for all $n > n_0$ and hence $q_\infty \geqslant w$. As this holds for every $w \in \{x \in \overline{\mathbb{R}} \mid F_\infty(x) < p\}$, we get that $q_\infty \geqslant \sup\{x \in \overline{\mathbb{R}} \mid F_\infty(x) < p\} = \sup\{x \in \mathbb{R} \mid F_\infty(x) < p\}$. In a similar way we may show that $q_\infty \leqslant \inf\{x \in \mathbb{R} \mid F_\infty(x) > p\}$, completing the proof.

Remark 2.7. As $\overline{\mathbb{R}}$ is compact, p-quantiles always have limit points (which in general can be infinite), and the above lemma shows that if F_∞ is also a distribution function, all the limit points are p-quantiles of F_∞ (and hence in this case they must be real). Moreover, in the case F_∞ is a cumulative distribution function, we may only need to assume that $F_n \to F_\infty$ almost everywhere (due to upper semi-continuity, excluding a null set would not change the supremum/infimum in the lemma). This gives a stronger version of direction (b) $\Rightarrow$ (c) in [6, Proposition 5.7, p. 112], where in our case we do not assume continuity, i.e.:

> Let F_n and F_∞ be cumulative distribution functions, with $F_n \to F_\infty$ pointwise almost everywhere, and q_n be a p-quantile of F_n for each n, $p \in (0, 1)$. Then, the sequence (q_n) has limit points that are all p-quantiles of F_∞. Thus if F_∞ has unique p-quantile, q_n converges to it.

Lemma 2.8. *Let $I, J \subset \mathbb{R}$ be open intervals, and $(F(a; x))_{a\in I}$ be a family of cumulative probability distribution functions of x on J, having positive densities $f(a; t)$ with respect to Lebesgue measure. Moreover assume that the densities are real analytic in both variables. Denote the respective p-quantiles by $q(a)$. Then, q is a real analytic function of a.*

Proof. As the densities are positive functions, the p-quantile is unique for each a. Hence, the function $q(a)$ is well defined implicitly as the solution $y = q(a)$ to the equation $F(a; y) - p = 0$. Let some $y_0 \in J$ and $a_0 \in I$ such that $F(a_0; y_0) - p = 0$. As F is real analytic and $\partial_2 F(a; y) = f(a; y) \neq 0$, by [14, Theorem 6.1.2] the equation $F(a; y) - p = 0$ has a real analytic solution $y = y(a)$ in a neighbourhood of a_0 such that $F(a_0; y(a_0)) - p = 0$. By uniqueness of the p-quantile this solution must be exactly $q(a)$, and hence q is real analytic.

3. Monotonicity and limits

PROOF OF PROPOSITION 1.1. Fix $b > 0$. As the regularised incomplete beta function $I(x; a, b)$ is real analytic in x and a, Lemma 2.8 gives real analyticity of q. As the logarithmic derivative of $x^{a-1}(1-x)^{b-1}$ with respect to a is $\log x$, which is an increasing function of x, Lemma 2.4 gives us that q is also increasing. The limits at 0 and ∞ can be obtained by considering limits of the incomplete beta function and using Lemma 2.6. Let, for instance, some limit point $\lim_{n\to\infty} q(a_n) = q_\infty \in [0, 1]$ for a sequence $a_n \to \infty$. Then, the fact that $\lim_{a\to\infty} I(x; a, b)$ vanishes for $x \in [0, 1)$ and is a unit at $x = 1$ gives $q_\infty = 1$, hence $\lim_{a\to\infty} q(a) = 1$. A similar argument shows $\lim_{a\to 0} q(a) = 0$, as claimed.

PROOF OF PROPOSITION 1.2. By Proposition 1.1, ϕ is also a real analytic function. Regarding monotonicity, if $b = 1$ then $\phi(a) \equiv -\log p$. Assume $b > 1$. By using a change of variables in (1.2), we get

$$\int_{\phi(a)}^{\infty} e^{-s}(1 - e^{-s/a})^{b-1}\,\mathrm{d}s = p\int_0^{\infty} e^{-s}(1 - e^{-s/a})^{b-1}\,\mathrm{d}s \tag{3.1}$$

and hence the function ϕ is the $(1-p)$-quantile of the distribution with density

$$x \mapsto \frac{e^{-x}(1 - e^{-x/a})^{b-1}}{\int_0^{+\infty} e^{-s}(1 - e^{-s/a})^{b-1}\,\mathrm{d}s}. \tag{3.2}$$

We set $f(a; x) := e^{-x}(1 - e^{-x/a})^{b-1}$. The logarithmic derivative of f with respect to a is

$$\frac{\partial_1 f(a; x)}{f(a; x)} = -\frac{(b-1)xe^{-x/a}}{a^2(1 - e^{-x/a})}.$$

The derivative of this with respect to x is

$$\partial_x\left(\frac{\partial_1 f(a; x)}{f(a; x)}\right) = \frac{b-1}{a^3}\mathrm{e}^{-x/a}(a\mathrm{e}^{-x/a} - a + x)(-1 + \mathrm{e}^{-x/a})^{-2} \geq 0,$$

as the function $x \mapsto a\mathrm{e}^{-x/a} - a + x$ has positive derivative for $x > 0$ and vanishes at 0. Thus, by Lemma 2.4 we have that ϕ is increasing. The case $b < 1$ is similar.

For the asymptotic results, we notice that for $a \to 0$, we have that

$$\lim_{a\to 0} \frac{e^{-x}(1-e^{-x/a})^{b-1}}{\int_0^\infty e^{-s}(1-e^{-s/a})^{b-1}\,\mathrm{d}s} = \frac{e^{-x}}{\int_0^\infty e^{-s}\,\mathrm{d}s} = e^{-x}.$$

By Scheffe's lemma, the corresponding distributions, whose p-quantiles are equal to $\phi(a)$, converge to the exponential distribution, and hence by Lemma 2.6 $\lim_{a\to 0}\phi(a) = -\log p$. Similarly, for $a \to \infty$, as

$$\begin{aligned}\lim_{a\to\infty} \frac{e^{-x}(1-e^{-x/a})^{b-1}}{\int_0^\infty e^{-s}(1-e^{-s/a})^{b-1}\,\mathrm{d}s} &= \lim_{a\to\infty} \frac{e^{-x}}{\int_0^\infty e^{-s}\frac{(1-e^{-s/a})^{b-1}}{(1-e^{-x/a})^{b-1}}\,\mathrm{d}s} \\ &= \frac{e^{-x}x^{b-1}}{\int_0^\infty e^{-s}s^{b-1}\,\mathrm{d}s},\end{aligned}$$

it converges to the gamma distribution with parameter b and $\lim_{a\to\infty}\phi(a) = \gamma_b$, the $(1-p)$-quantile of the gamma distribution with parameter b.

REMARK 3.1. The distribution with density (3.2) is in fact the exponentiated McDonald generalised beta distribution with parameters $(a, b, 1/a)$, introduced by Nadarajah in [15, §4]. It can also be obtained by exponential tilting of the generalised exponential distribution from Gupta and Kundu [9].

4. Convexity of ϕ for $b < 1$

We rewrite (3.1) as

$$\int_0^{\phi(a)} e^{-s}(1-e^{-s/a})^{b-1}\,\mathrm{d}s = (1-p)\int_0^\infty e^{-s}(1-e^{-s/a})^{b-1}\,\mathrm{d}s. \tag{4.1}$$

We denote $f(a; s) = e^{-s}(1-e^{-s/a})^{b-1}$ and differentiating (4.1) we have

$$\phi'(a)f(a;\phi(a)) + \int_0^{\phi(a)} \partial_1 f(a;s)\,\mathrm{d}s = (1-p)\int_0^\infty \partial_1 f(a;s)\,\mathrm{d}s.$$

Differentiating again,

$$\begin{aligned}\phi''(a)f(a;\phi(a)) = (1-p)\int_{\phi(a)}^\infty \partial_1^2 f(a;s)\,\mathrm{d}s - p\int_0^{\phi(a)} \partial_1^2 f(a;s)\,\mathrm{d}s \\ - (\phi'(a))^2\partial_2 f(a;\phi(a)) - 2\phi'(a)\partial_1 f(a;\phi(a)).\end{aligned} \tag{4.2}$$

PROOF OF THEOREM 1.3. Let $b \in (0, 1)$. By Proposition 1.2, we have $\phi' < 0$, and as

$$\partial_2 f(a;s) = -e^{-s}(1-e^{-s/a})^{b-1} + \frac{b-1}{a}e^{-s}(1-e^{-s/a})^{b-2}e^{-s/a} < 0$$

and

$$\partial_1 f(a;s) = -s\frac{b-1}{a^2}e^{-s-s/a}(1-e^{-s/a})^{b-2} > 0,$$

we see that $\phi'(a)^2\partial_2 f(a;\phi(a)) < 0$ and $\phi'(a)\partial_1 f(a;\phi(a)) < 0$. In order to show that $\phi'' > 0$, using (4.2) what is left is to show that

$$(1-p)\int_{\phi(a)}^{\infty} \partial_1^2 f(a;t)\,\mathrm{d}t - p\int_0^{\phi(a)} \partial_1^2 f(a;t)\,\mathrm{d}t \geqslant 0. \tag{4.3}$$

By substituting $t/a = s$ in the above integrals, we get

$$\begin{aligned}(1-p)&\int_{\phi(a)}^{\infty} \partial_1^2 f(a;t)\,\mathrm{d}t - p\int_0^{\phi(a)} \partial_1^2 f(a;t)\,\mathrm{d}t\\ &= \frac{2(b-1)}{a}\left((1-p)\int_{\phi(a)/a}^{\infty} e^{-at}\eta(t)\,\mathrm{d}t - p\int_0^{\phi(a)/a} e^{-at}\eta(t)\,\mathrm{d}t\right),\end{aligned} \tag{4.4}$$

where

$$\eta(x) := xe^{-2x}(1-e^{-x})^{b-3}\left(e^x - 1 - \frac{x}{2}e^x + \frac{b-1}{2}x\right).$$

We now proceed to show (4.3). We see in Lemma 4.1 below that the function

$$w(x) := \left(1-\frac{x}{2}\right)e^x + \frac{b-1}{2}x - 1 \tag{4.5}$$

has a unique zero ρ on $(0,+\infty)$, and it is positive on $(0,\rho)$ and negative on (ρ,∞). Assume that $\phi(a) \geqslant \rho a$. As w and η have the same sign, we have that $\int_{\phi(a)/a}^{\infty} e^{-at}\eta(t)\,\mathrm{d}t < 0$. For the other integral, we have

$$\begin{aligned}\int_0^{\phi(a)/a} e^{-at}\eta(t)\,\mathrm{d}t &= \int_0^{\rho} e^{-at}\eta(t)\,\mathrm{d}t + \int_{\rho}^{\phi(a)/a} e^{-at}\eta(t)\,\mathrm{d}t\\ &\ge e^{-a\rho}\left(\int_0^{\rho}\eta(t)\,\mathrm{d}t + \int_{\rho}^{\phi(a)/a}\eta(t)\,\mathrm{d}t\right)\\ &\geqslant e^{-a\rho}\left(\int_0^{\rho}\eta(t)\,\mathrm{d}t + \int_{\rho}^{\infty}\eta(t)\,\mathrm{d}t\right)\\ &= e^{-a\rho}\int_0^{\infty}\eta(t)\,\mathrm{d}t = 0,\end{aligned}$$

by Lemma 4.2 below. Hence

$$\frac{2(b-1)}{a}\left((1-p)\int_{\phi(a)/a}^{\infty} e^{-at}\eta(t)\,\mathrm{d}t - p\int_0^{\phi(a)/a} e^{-at}\eta(t)\,\mathrm{d}t\right) \geqslant 0,$$

and by (4.4), (4.3) is proved for $\phi(a) \geqslant \rho a$.

Now, assume that $\phi(a) < \rho a$. We define

$$h(a;t) := \frac{\partial_1^2 f(a;t)}{(b-1)f(a;t)} = \frac{2t((a-t/2)e^{t/a} + (b-1)t/2 - a)}{a^4(e^{t/a}-1)^2}.$$

We further denote

$$h_0(s) := \frac{a^2}{2}h(a;as) = \frac{s((1-s/2)e^s + (b-1)s/2 - 1)}{(e^s-1)^2} = \frac{sw(s)}{(e^s-1)^2}. \tag{4.6}$$

By Lemma 4.3, h_0 is decreasing on $(0,\rho)$, hence $h(a;s)$ is also decreasing with respect to s on $(0,\rho a)$. Hence, for $t \in (0,\phi(a)) \subset (0,\rho a)$ we have $h(a;t) > h(a;\phi(a))$. For $t \in (\phi(a),\rho a)$, we analogously have $h(a;\phi(a)) > h(a;t)$, and if $t \in (\rho a,\infty)$, then $h(a;\phi(a)) > 0 > h(a;t)$. Hence,

$$\begin{aligned}
&(1-p)\int_{\phi(a)}^{\infty} \partial_1^2 f(a;t)\,\mathrm{d}t - p\int_0^{\phi(a)} \partial_1^2 f(a;t)\,\mathrm{d}t \\
&\quad = (b-1)\bigg((1-p)\int_{\phi(a)}^{\infty} h(a;t)f(a;t)\,\mathrm{d}t - p\int_0^{\phi(a)} h(a;t)f(a;t)\,\mathrm{d}t\bigg) \\
&\quad \geqslant (b-1)h(a;\phi(a))\bigg((1-p)\int_{\phi(a)}^{\infty} f(a;t)\,\mathrm{d}t - p\int_0^{\phi(a)} f(a;t)\,\mathrm{d}t\bigg) = 0
\end{aligned}$$

by (4.1). Thus (4.3) is proved. As the right-hand side of (4.2) is positive, then $\phi'' > 0$, implying that ϕ is strictly convex, as claimed.

LEMMA 4.1. *Fix $b > 0$. The function w in (4.5) has a unique zero ρ on $(0,\infty)$. We have that $w(x) > 0$ for $x < \rho$ and $w(x) < 0$ for $x > \rho$.*

PROOF. We have

$$w'(x) = \frac{1-x}{2}e^x + \frac{b-1}{2}$$

and

$$w''(x) = -\frac{x}{2}e^x < 0 \quad \text{for } x > 0.$$

Hence w' is strictly decreasing, and as $w'(0) = b/2$ and $\lim_{x\to+\infty} w'(x) = -\infty$, it changes its sign exactly once, and w is initially increasing and then decreasing, and is a concave function. As $w(0) = 0$ and $\lim_{x\to+\infty} w(x) = -\infty$, we get that w has unique zero $\rho \in (0,\infty)$, $w(x) > 0$ for $x < \rho$, and $w(x) < 0$ for $x > \rho$, as claimed.

LEMMA 4.2. *For $b > 0$, it holds that*

$$\int_0^{\infty} se^{-2s}(1-e^{-s})^{b-3}\bigg(e^s - 1 - \frac{s}{2}e^s + \frac{b-1}{2}s\bigg)\,\mathrm{d}s = 0.$$

PROOF. In the course of the proof we assume that $\Re(b) > 2$, which may then be extended by analytic continuation. An elementary substitution yields

$$\mathrm{B}(a,b) = \int_0^\infty e^{-as}(1-e^{-s})^{b-1}\,\mathrm{d}s$$

and thus

$$\partial_1 \mathrm{B}(a,b) = \big(\psi(a) - \psi(a+b)\big)\mathrm{B}(a,b) = -\int_0^\infty s e^{-as}(1-e^{-s})^{b-1}\,\mathrm{d}s, \quad (4.7)$$

where $\psi := \Gamma'/\Gamma$ is the digamma function (see [3, Chapter 1]), and

$$\partial_1^2 \mathrm{B}(a,b) = \int_0^\infty s^2 e^{-as}(1-e^{-s})^{b-1}\,\mathrm{d}s. \quad (4.8)$$

We split the integral into three parts. The first one is

$$I_1 = \int_0^\infty s e^{-2s}(1-e^{-s})^{b-3}(e^s-1)\,\mathrm{d}s = \frac{\psi(b)+\gamma}{b-1},$$

by (4.7). The second is

$$\begin{aligned} I_2 &= \frac{b-1}{2}\int_0^\infty s^2 e^{-2s}(1-e^{-s})^{b-3}\,\mathrm{d}s \\ &= \frac{b-1}{2(b-2)}\left(\int_0^\infty s^2 e^{-s}(1-e^{-s})^{b-2}\,\mathrm{d}s - 2\int_0^\infty s e^{-s}(1-e^{-s})^{b-2}\,\mathrm{d}s\right) \\ &= \frac{b-1}{2(b-2)}\partial_1^2 \mathrm{B}(1,b-1) - \frac{\psi(b)+\gamma}{b-2}, \end{aligned}$$

using (4.7) and (4.8). The last one is

$$\begin{aligned} I_3 &= -\frac{1}{2}\int_0^\infty s^2 e^{-s}(1-e^{-s})^{b-3}\,\mathrm{d}s \\ &= -\frac{1}{2}\partial_1^2 \mathrm{B}(1,b-2) = -\frac{1}{2}\partial_a^2\left(\mathrm{B}(a,b-1)\frac{a+b-2}{b-2}\right)\bigg|_{a=1} \\ &= -\frac{b-1}{2(b-2)}\partial_1^2 \mathrm{B}(1,b-1) - \frac{\partial_1 \mathrm{B}(1,b-1)}{b-2} \\ &= -\frac{b-1}{2(b-2)}\partial_1^2 \mathrm{B}(1,b-1) + \frac{\gamma+\psi(b)}{(b-2)(b-1)}. \end{aligned}$$

We see that $I_1 + I_2 + I_3 = 0$, and the lemma is proved.

LEMMA 4.3. *Fix $b > 0$. The function h_0 in (4.6) is decreasing between 0 and its zero $\rho \in (0, \infty)$.*

PROOF. It is easy to see that $x/(e^x - 1)$ is decreasing. The rest is also decreasing as

$$\frac{(1 - x/2)e^x + ((b-1)/2)x - 1}{e^x - 1} = \frac{b}{2}\frac{x}{e^x - 1} + 1 - \frac{1}{2}\frac{x(e^x+1)}{e^x - 1}$$

and

$$\left(\frac{x(e^x+1)}{e^x-1}\right)' = \frac{e^{2x} - 2e^x x - 1}{(e^x - 1)^2} \geqslant 0,$$

as $(e^{2x} - 2e^x x - 1)' = 2e^x(e^x - x - 1) \geqslant 0$ and the numerator vanishes at 0. Thus, on $(0, \rho)$, h_0 is the product of decreasing, positive functions, hence decreasing, as claimed.

5. Logarithmic concavity of *q*

In this section, we shall prove Theorem 1.4. In order to have a more concise notation, we shall often omit the argument a from the notation of functions q, ϕ and ξ. The following lemma will be the key to this proof.

LEMMA 5.1. *Let $\xi := -\log q$. We have that*

$$\xi' = \sum_{n=0}^{\infty} \frac{1}{a+b+n} Y_{n+b}(\xi) - \sum_{n=0}^{\infty} \frac{1}{a+n} Y_n(\xi), \tag{5.1}$$

where

$$Y_c(\xi) := \frac{\int_0^{\xi} e^{ct}(1 - e^{-t})^{b-1}\,\mathrm{d}t}{e^{c\xi}(1 - e^{-\xi})^{b-1}}.$$

PROOF. Taking logarithms of

$$\frac{\mathrm{B}(e^{-\xi}; a, b)}{\mathrm{B}(a, b)} = p,$$

we get

$$\log(\mathrm{B}(e^{-\xi}; a, b)) - \log\Gamma(a) + \log\Gamma(a+b) = \log(p\Gamma(b)).$$

Differentiating by a we get then

$$\xi' e^{-a\xi}(1 - e^{-\xi})^{b-1} = (\psi(a+b) - \psi(a))\mathrm{B}(e^{-\xi}; a, b) + \partial_2 \mathrm{B}(e^{-\xi}; a, b). \tag{5.2}$$

Here, the incomplete beta function is considered as a function of three variables, and ∂_2 denotes differentiation with respect to the second. We shall prove the identity

$$\begin{aligned}&(\psi(a+b)-\psi(a))\mathrm{B}(e^{-x};a,b)+\partial_2\mathrm{B}(e^{-x};a,b)\\&=\sum_{n=0}^{\infty}\left(\frac{e^{-(n+a+b)x}}{a+b+n}\mathrm{B}(1-e^{-x};b,-n-b)-\frac{e^{-(n+a)x}}{a+n}\mathrm{B}(1-e^{-x};b,-n)\right),\end{aligned}\tag{5.3}$$

valid for all $x > 0$. As

$$\mathrm{B}(1-e^{-x};b,-c)=\int_0^x e^{ct}(1-e^{-t})^{b-1}\,\mathrm{d}t,$$

(5.3) combined with (5.2) implies (5.1).

We shall prove (5.3) using Laplace transforms. In particular, we shall prove that the Laplace transforms of the left- and the right-hand sides of (5.3) are equal, which shall imply (5.3) by uniqueness of the Laplace transform. We denote the Laplace transform of a function f by

$$\mathscr{L}_t[f(t)](s)=\int_0^\infty e^{-st}f(t)\,\mathrm{d}t,$$

for $\Re s > 0$. To get the Laplace transform of the left-hand side, we start with

$$\mathrm{B}(s,b)=\mathscr{L}_x\left[(1-e^{-x})^{b-1}\right](s)$$

for $\Re s > 0$. Hence

$$\mathscr{L}_x\left[e^{-(a+c)x}\int_0^x e^{ct}(1-e^{-t})^{b-1}\,\mathrm{d}t\right](s)=\frac{\mathrm{B}(s+a,b)}{s+a+c}$$

using elementary properties of the Laplace transform related to multiplication and integration. Thus,

$$\begin{aligned}&\mathscr{L}_x\left[\sum_{n=0}^{\infty}\left(\frac{e^{-(n+a+b)x}}{a+b+n}\mathrm{B}(1-e^{-x};b,-n-b)-\frac{e^{-(n+a)x}}{a+n}\mathrm{B}(1-e^{-x};b,-n)\right)\right](s)\\&=\mathrm{B}(s+a,b)\sum_{n=0}^{\infty}\left(\frac{1}{(s+a+b+n)(a+b+n)}-\frac{1}{(s+a+n)(a+n)}\right)\\&=\frac{\mathrm{B}(s+a,b)}{s}(\psi(a)-\psi(a+b)-\psi(a+s)+\psi(a+b+s)),\end{aligned}\tag{5.4}$$

using the non-negativity of individual terms in the summands to interchange integration and summation in the first equality, and the well-known infinite series involving the digamma function,

$$\psi(a+b) - \psi(a) = \sum_{k=0}^{\infty} \left(\frac{1}{k+a} + \frac{1}{k+a+b} \right) \tag{5.5}$$

(which can be obtained, for example, by using [16, 5.7.6]), in the second.

Now, we shall compute the Laplace transform of the right-hand side of (5.3) and show it are equal to that of the left-hand side. By elementary computations we have

$$\mathrm{B}(e^{-x}; a, b) = \mathrm{B}(a, b) - \int_0^x e^{-at}(1 - e^{-t})^{b-1} \,\mathrm{d}t,$$

hence

$$\mathscr{L}_x\big[\mathrm{B}(e^{-x}; a, b)\big](s) = \frac{1}{s}\mathrm{B}(a, b) - \frac{1}{s}\mathrm{B}(s + a, b).$$

Differentiating, we get

$$\mathscr{L}_x\big[\partial_2 \mathrm{B}(e^{-x}; a, b)\big](s) \\ = \frac{\psi(a) - \psi(a+b)}{s}\mathrm{B}(a, b) - \frac{\psi(a+s) - \psi(a+b+s)}{s}\mathrm{B}(s + a, b).$$

Combining the last two inequalities with (5.4), by uniqueness of the Laplace transform, (5.3) holds, hence the lemma is proven.

REMARK 5.2. The proof presented above was suggested by the anonymous referee. We would like here to give a short outline of the original, and considerably longer and more technical, proof of Lemma 5.1, as it demonstrates the construction of the right hand side of (5.1). Binomial expansions in (1.2) give

$$\sum_{n=0}^{\infty} \frac{\Gamma(a+b)}{\Gamma(a)(a+n)} \binom{b-1}{n} (-1)^n e^{-(n+a)\xi} = p\Gamma(b).$$

By differentiation and lengthy manipulations on the infinite sums that arise, key tools being the infinite series (5.5) involving the digamma function, and expanding binomial sums, we arrive to

$$
\begin{aligned}
&\xi'(1-e^{-\xi})^{b-1} \\
&\quad= \sum_{n=0}^{\infty} \frac{1}{a+n}\binom{b-1}{n}(-1)^n e^{-n\xi}\Bigg(\sum_{k\neq n}\Big(\frac{1}{k-n}-\frac{1}{k+b-n}\Big)-\frac{1}{b}\Bigg) \\
&\qquad+\sum_{n=0}^{\infty}\frac{1}{a+n}\Bigg(\sum_{k\neq n}\binom{b-1}{k}\frac{(-1)^k e^{-k\xi}}{k-n}-\binom{b-1}{n}(-1)^n e^{-n\xi}\xi\Bigg) \\
&\qquad+\sum_{n=0}^{\infty}\frac{1}{a+n+b}\sum_{k=0}^{\infty}\binom{b-1}{k}\frac{(-1)^k e^{-k\xi}}{n+b-k},
\end{aligned}
$$

and the infinite sums inside divided by the factor in the left-hand side, essentially by summing back binomial terms, are shown to represent the Y_c functions in (5.1).

LEMMA 5.3. *Let $b > 1$ and $c > 0$. Then, Y_c is increasing on $(0,\infty)$. Moreover, $Y_c(x)$, $Y_c'(x)$ are decreasing with respect to c for fixed x.*

PROOF. We rewrite

$$
\begin{aligned}
\frac{\int_0^x e^{ct}(1-e^{-t})^{b-1}\,\mathrm{d}t}{e^{cx}(1-e^{-x})^{b-1}} &= \int_0^x e^{c(t-x)}\Big(\frac{1-e^{-t}}{1-e^{-x}}\Big)^{b-1}\,\mathrm{d}t \\
&= \int_0^x e^{c(t-x)}\Big(\frac{e^x-e^{x-t}}{e^x-1}\Big)^{b-1}\,\mathrm{d}t \\
&= \int_0^x e^{-cv}\Big(\frac{e^x-e^v}{e^x-1}\Big)^{b-1}\,\mathrm{d}v.
\end{aligned}
$$

Differentiating, we get

$$
\begin{aligned}
&\partial_x\Bigg(\int_0^x e^{-cv}\Big(\frac{e^x-e^v}{e^x-1}\Big)^{b-1}\,\mathrm{d}t\Bigg) \\
&\quad= \int_0^x e^{-cv}\partial_x\Big(\frac{e^x-e^v}{e^x-1}\Big)^{b-1}\,\mathrm{d}v \\
&\quad= \int_0^x e^{-cv+x}(b-1)\Big(\frac{e^x-e^v}{e^x-1}\Big)^{b-2}\frac{e^v-1}{(e^x-1)^2}\,\mathrm{d}v
\end{aligned}
$$

and this completes the proof.

PROOF OF THEOREM 1.4. We shall show the convexity of $\xi = -\log q$, which is equivalent to logarithmic concavity of q. The case $b < 1$ is given by Theorem 1.3, as $a\xi'' = \phi'' - 2\xi' > 0$. For $b = 1$, we have $\xi = (\log(1/p))/a$ hence $\xi'' = 0$. For $b > 1$, differentiating (5.1) we get

$$\xi'' = \sum_{n=0}^{\infty} \frac{1}{(a+n)^2} Y_n(\xi) - \sum_{n=0}^{\infty} \frac{1}{(a+b+n)^2} Y_{n+b}(\xi) + \left(\sum_{n=0}^{\infty} \frac{1}{a+b+n} Y'_{n+b}(\xi) - \sum_{n=0}^{\infty} \frac{1}{a+n} Y'_n(\xi)\right)\xi' > 0$$

using that $\xi' < 0$ and Lemma 5.3, completing the proof.

REMARK 5.4. We notice that (5.1) also gives

$$q' = \sum_{n=0}^{\infty} \frac{1}{a+b+n} \frac{\int_q^1 t^{-n-b-1}(1-t)^{b-1}\,\mathrm{d}t}{q^{-n-b-1}(1-q)^{b-1}} - \sum_{n=0}^{\infty} \frac{1}{a+n} \frac{\int_q^1 t^{-n-1}(1-t)^{b-1}\,\mathrm{d}t}{q^{-n-1}(1-q)^{b-1}}.$$

Appendix

Finally, we want to see how the p-quantile depends on the second parameter of the beta distribution. For clarity, from now on we denote the p-quantile of the beta distribution with parameters a and b by $q_p(a, b)$. We shall consider a constant, and try to relate q as a function of b with the previous results.

A simple change of variables $s = 1 - t$ in (1.1) gives the functional relation

$$I(x; a, b) = 1 - I(1 - x; b, a),$$

which implies

$$p = I(q_p(a, b); a, b) = 1 - I(q_p(a, b); b, a) \implies I(q_p(a, b); b, a) = 1 - p = I(q_{1-p}(a, b); b, a)$$

and, using the uniqueness of the p-quantile, we get

$$q_p(a, b) = 1 - q_{1-p}(b, a).$$

Hence, by Proposition 1.1, we get that q_p is decreasing in b and

$$\lim_{b\to 0} q_p(a, b) = 1, \quad \lim_{b\to\infty} q_p(a, b) = 0.$$

Moreover, we have

$$(1 - q_p(a, b))^b = q_{1-p}(b, a)^b = e^{-\varphi_{1-p}(b)},$$

where $\varphi_{1-p}(b) = -b \log q_{1-p}(b, a)$, hence the behaviour of $q_p(a, b)$ as a function of b can again be studied similarly through the function φ_p. We also easily see that $b \mapsto 1 - q_p(a, b)$ is log-concave. We remark that numerical evidence shows that $b \mapsto q_p(a, b)$ itself is not (log-)concave/convex. However, the function $b \mapsto \varphi_p(b)$ seems to be convex.

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