PARACOMMUTATORS OF SCHATTEN – VON NEUMANN CLASS $S_p$, $0 < p < 1$

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1. Introduction.

In their paper [3], Janson and Peetre consider the paracommutator defined by

$$\langle T_b^* f \rangle (\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{b}(\xi - \eta) A(\xi, \eta) |\xi|^{d/2} |\eta|^{d/2} \hat{f}(\eta) d\eta$$

and obtain a series of results on $L^2$-boundedness and $S_p$-estimates for $1 \leq p \leq \infty$. In this paper we study corresponding $S_p$-estimates for $0 < p < 1$. For the notion of the Schatten – von Neumann class $S_p$, see McCarthy [4]. In the case $0 < p < 1$, $S_p$ is not a Banach space, only a quasi-Banach space. For it,

$$\|T_1 + T_2\|_{S_p} \leq \|T_1\|_{S_p} + \|T_2\|_{S_p}$$

holds. We shall repeatedly use this fact.

We shall give the assumptions on $A(\xi, \eta)$ in terms of $V_p(E \times F)$ defined below instead of $M(E \times F)$ in [3].

**Definition 1.** If $E, F \subset \mathbb{R}^d$, then we define

$$V_p(E \times F) = \{K(\xi, \eta) : K(\xi, \eta) = \sum \lambda_i f_i(\xi) g_i(\eta), f_i, g_i \text{ measurable,}$$

$$|f_i(\xi)| \leq 1 \text{ for } \xi \in E, |g_i(\eta)| \leq 1 \text{ for } \eta \in F, \sum |\lambda_i|^p < \infty \}$$

and

$$\|K\|_{V_p(E \times F)} = \inf(\sum |\lambda_i|^p)^{1/p}$$

the infimum being taken over all such decompositions in (3).

For $0 < p \leq 1$, $V_p(E \times F)$ is well defined, because $\sum |\lambda_i| \leq (\sum |\lambda_i|^p)^{1/p}$, so $\|K\|_{V_p(E \times F)} < \infty$ implies that the series $\sum \lambda_i f_i(\xi) g_i(\eta)$ converges absolutely and uniformly.

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Remark 1. In fact, we may assume that \(|f_i(\xi)| \leq 1\) and \(|g_i(\eta)| \leq 1\) hold only almost everywhere (a.e.) on \(E\) and \(F\) instead of \(|f_i(\xi)| \leq 1\) and \(|g_i(\eta)| \leq 1\) on \(E\) and \(F\) in Definition 1. The results of Theorems 1–3 below still hold, provided \(\varepsilon \to 0\) is along a sequence, and \(\varepsilon m\) are replaced by some points \(\eta_m^* \in Q_m^\varepsilon\). But for the sake of simplicity, we prefer Definition 1 in the above form.

It is easy to see that for \(0 < p_1 \leq p_2 \leq 1\), \(V_{p_1} \subset V_{p_2} \subset V_1 \subset M\), where \(V_1\) is the tensor product \(L^\alpha(E) \widehat{\otimes} L^\alpha(F)\) and \(M(E \times F)\) is the space of Schur multipliers, see Janson and Peetre [3].

Similarly, corresponding to Lemma 3.1 of [3], for \(V_p(E \times F)\) we have

**Proposition 1.** If \(\varphi(\xi, \eta) \in V_p(E \times F)\) and \(K(\xi, \eta) \in S_p(E \times F)\). Then \(\varphi K \in S_p(E \times F)\) and

\[
\|\varphi K\|_{S_p(E \times F)} \leq \|\varphi\|_{V_p(E \times F)} \|K\|_{S_p(E \times F)}, \quad 0 < p \leq 1.
\]

**Proof.** For any \(\varepsilon > 0\), let \(\varphi(\xi, \eta) = \sum \lambda_i f_i(\xi) g_i(\eta)\), where

\[
|f_i(\xi)| \leq 1, \quad |g_i(\eta)| \leq 1, \quad \sum |\lambda_i|^p \leq (\|\varphi\|_{V_p(E \times F)} + \varepsilon)^p.
\]

Then we have

\[
\|\varphi K\|_{S_p(E \times F)}^p \leq \sum |\lambda_i|^p \|f_i(\xi) K(\xi, \eta) g_i(\eta)\|_{S_p(E \times F)}^p \\
\leq \sum |\lambda_i|^p \|K\|_{S_p(E \times F)}^p \\
\leq (\|\varphi\|_{V_p(E \times F)} + \varepsilon)^p \|K\|_{S_p(E \times F)}^p.
\]

So (4) holds.

**Remark 2.** \(V_p(E \times F)\) is a quasi-Banach algebra but not a Banach algebra, as \(S_p(E \times F)\) is a quasi-Banach space but not a Banach space, for \(0 < p < 1\). \(\| \cdot \|_{V_p}\) induces a metric as \(\| \cdot \|_{S_p}^\varepsilon\) does. So the results analogous to Lemmas 3.3, 3.4, and 3.11 in [3] do not hold for \(V_p\) when \(0 < p < 1\).

As in [3], let \(A_k\) denote the set \(\{\xi \in \mathbb{R}^d : 2^k \leq |\xi| \leq 2^{k+1}\}\) and \(A_k = A_{k-1} \cup A_k \cup A_{k+1}\). Now we list some assumptions on \(A\) which will be used in the theorems below.

**A0:** There exists an \(r > 1\) such that \(A(r \xi, r \eta) = A(\xi, \eta)\).

**A_p^1:** \(\|A\|_{V_p(A_j \times A_k)} \leq C\), for all \(j, k \in \mathbb{Z}\).

**A_p^3(\alpha):** There exist \(\alpha > 0\) and \(0 < \delta < \frac{1}{2}\) such that

\[
\|A\|_{V_p(B \times B)} \leq C(r/|\xi_0|)\alpha,
\]

for every ball \(B = B(\xi_0, r)\) with the centre \(\xi_0\) and radius \(r < \delta|\xi_0|\).
Aₚ₄¹: For every ξ₀ ≠ 0, there exist η₀ ∈ Rᵈ and δ > 0 such that, with

\[ B₀ = B(ξ₀ + η₀, δ|ξ₀|) \text{ and } D₀ = B(η₀, δ|ξ₀|), \quad A(ξ, η)^{-1} ∈ Vₚ(B₀ × D₀). \]

Aₚ₉(α₀): A(ξ, η) satisfies Aₚ₁ and Aₚ₃(α). Furthermore, for every ε > 0 small enough, let \( \{Q'_m\}_{m ∈ Z^d} \) be a family of disjoint cubes with centres εm and sides ε, let \( \tilde{Q}_m^ε = 3Q'_m \) and let

\[ A_ε(ξ, η) = \sum_{m ∈ Z^d} A(ξ, εm)χ_{Q'_m}(η), \quad K_ε(ξ, η) = A(ξ, η) - A_ε(ξ, η). \]

Then

\[ \|K_ε\|_{Vₚ(Aₛ × Aₙ)} ≤ C(ε/2^k)^{α₀}, \] for every \( l ∈ Z, k > \log₂ ε, \)

and

\[ \|K_ε\|_{Vₚ(B × B)} ≤ C(ε/|ξ₀|)^{α₀}(r/|ξ₀|)^{α₀}, \] for every \( B = B(ξ₀, r) \)

with \( ε < r < δ|ξ₀|. \)

A₁₀(α): For any \( 0 ≠ θ ∈ Rᵈ, \) there exist a positive number \( δ < \frac{1}{2} \) and a subset \( V_θ \) of \( Rᵈ \) such that if \( N_r \) denotes the number of integer points contained in \( V_θ ∩ B_r, \) where \( B_r = B(0, r), \) then

\[ \lim_{r \to ∞} \frac{N_r}{r^d} > 0, \]

and for every \( n ∈ V_θ, \)

\[ \frac{1}{\|A(⋅ + n + θ, ⋅ + n)\|_{M(B × B)}} ≤ C|n|^α, \] where \( B = B(0, δ). \)

Remark 3. The assumption A₀ is about the homogeneity of \( A. \) The assumption Aₚ₁ is about the boundedness of \( A \) just like A₁ in [3]. Aₚ₁ implies A₁ in [3] and hence it implies that \( A ∈ L^∞(R^{2d}). \) The assumption Aₚ₃ is about the order of the zero at the diagonal \( \{ξ = η\} \) of \( A \) just like A₃ in [3]. The assumption Aₚ₄₁ is about non-degeneracy of \( A \) just like A₄₁ in [7]. It is stronger than A₄ in [3] but weaker than the one in Timotin [10] and [11]; for example, the kernel \( A(ξ, η) \) of commutator, see Example 2 below, satisfies Aₚ₄₁ but \( A ∉ C^∞(R^{2d} \setminus \{0\}). \) The assumption Aₚ₉ is about the smoothness of \( A \) on all of \( R^{2d}. \) It is not necessary for the \( S_p \)-estimates if \( 1 ≤ p ≤ ∞, \) but when \( 0 < p < 1, \) we need an assumption such as Aₚ₉. The assumption A₁₀(α) again is about the order of the zero at the diagonal \( \{ξ = η\} \) of \( A. \) Aₚ₃(α) says that the order is \( ≥ α, \) A₁₀(α) says that the order is \( ≥ α. \) A₁₀(α) will be used to characterize the "Janson-Wolff
phenomenon". It should be noticed that in the assumption A10(α), we use $M(B \times B)$ regardless of $p$.

Sometimes we write $T_b^u(A)$ to emphasize the kernel $A$.

The main results of this paper are the following four theorems.

**Theorem 1.** Suppose that $0 < p < 1$, $s,t > -d/2$, $\alpha > s + t + d/p$ and suppose further that $A(\xi, \eta)$ satisfies $A_p1$ and $A_p3(\alpha)$. Then $b \in B_p^{s+t+d/p}$ implies that $T_b^u \in S_p$ and

$$
\|T_b^u\|_{S_p} \leq C\|b\|_{B_p^{s+t+d/p}}.
$$

**Theorem 2.** Suppose that $0 < p < 1$, $s,t > -d/2$ and suppose further that $A(\xi, \eta)$ satisfies $A0$, $A_p1$ and $A_pA_2^1$. Then the a priori inequality

$$
\|b\|_{B_p^{s+t+d/p}} \leq C\|T_b^u\|_{S_p}
$$

holds for every $b \in B_p^{s+t+d/p}$.

**Theorem 3.** Suppose that $A(\xi, \eta)$ satisfies $A0$, $A_p1$, $A_p3(\alpha)$, $A_pA_2^1$, $A_p9(\alpha_0)$ and suppose further that $\alpha > \alpha_0 > 0, 0 < p < 1, s,t > -d/2$ and $s + t + d/p < \alpha$. Let

$$
A^{(s)}_b(\xi, \eta) = \sum_{m \in \mathbb{Z}^d, \theta \in \mathbb{Q}_*} A(\xi, e(m + \theta))\chi_{\mathcal{Q}_*}(\eta).
$$

Then for $b \in S'(\mathbb{R}^d)$ with $\mathcal{B}$ with compact support $\subset \mathbb{R}^d \setminus \{0\}$, $T_b^u \in S_p$ and $T_b^u(A^{(s)}_b) \in S_p$ uniformly in $\varepsilon \leq \varepsilon_0$ and $|\theta| \leq \sqrt{d}/3$ imply that $b \in B_p^{s+t+d/p}$ and that (6) holds.

**Theorem 4.** Suppose that $A(\xi, \eta)$ satisfies $A10(\alpha)$ and suppose further that $0 < p \leq d/(\alpha - s - t)$, $b \in S'(\mathbb{R}^d)$ with $\mathcal{B}$ with compact support $\subset \mathbb{R}^d$ such that $T_b^u \in S_p$. Then $b$ must be a polynomial.

**Remark 4.** The results of Theorems 3 and 4 are not as good as one would like. This is mainly because the analogue of Lemma 3.3 in [3] is false for $V_p$, when $p < 1$, so the restriction that $\mathcal{B}$ has compact support $\subset \mathbb{R}^d \setminus \{0\}$ or $\mathbb{R}^d$ cannot easily be removed. But from the proof of Theorem 1, see section 4 below, we see that under the hypotheses of Theorem 1, with $b_N = \sum_{k \in \mathbb{Z}} b \ast \psi_k$, we have $T_{b_N}^u \in S_p$ and $T_{b_N}^u \to T_b^u$ in the norm $S_p$. Let us define $T_b^u \in S_p$ in this way for $b \in S'(\mathbb{R}^d)$, of course, this is different from the natural definition of $T_b^u \in S_p$, and let us denote $T_b^u \in S_p$ strongly. Then, using Corollary 2 in section 4, we obtain formally good-looking results as follows.
Corollary 1. Suppose that $A(\xi, \eta)$ satisfies A0, A$_p$1, A$_p$3(\infty), A$_p$4$^\frac{1}{2}$, A$_p$9($\alpha_0$) and A10($\alpha$) and suppose further that $\alpha \geq \alpha_0 > 0$, $0 < p < 1$, $s, t > -d/2$, $\alpha > s + t$. Then

1) if $p > d/(\alpha - s - t)$, $T^a_b \in S_p$ strongly and $T^a_b(\nu^0) \in S_p$ strongly and uniformly in $\varepsilon \leq \varepsilon_0$ and $|\theta| \leq \sqrt{d}/3$ if and only if $b \in B^{s+t+d/p}$,

2) if $p \leq d/(\alpha - s - t)$, $T^a_b \in S_p$ strongly if and only if $b$ is a polynomial.

These theorems and corollary look somewhat complicated, but they cover at least paraproducts, higher order commutators of singular integral operators and some pseudo-differential operators, which are the cases of main interest. For Hankel operators, or equivalently for the one-dimensional commutators $[b, H]$, Peller [6] and Semmes [9] have obtained $S_p$-estimates, for $0 < p < 1$. So our results are generalizations of their results. In fact, our methods for proving Theorem 2 are close to those of Peller [6] and Semmes [9]. Their results can be obtained from Theorems 1 and 2. More generally, we consider $T^a_b = D'[\ldots, [b, H_1], \ldots, H_d]D'$, where $b$ is a function on $\mathbb{R}^d$, $H_i$ is defined by

$$H_i f(x) = \text{p.v.} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_d)}{x_i - y_i} \, dy_i, \quad 1 \leq i \leq d.$$  

It has the Fourier kernel

$$A(\xi, \eta) = C \prod_{i=1}^{d} \left[ I(\xi_i > 0 > \eta_i) - I(\xi_i < 0 < \eta_i) \right].$$

This kernel satisfies A0, A$_p$1, A$_p$3(\infty), A$_p$4$^\frac{1}{2}$. So the conclusions of Theorems 1 and 2 hold for it. Using an argument of Semmes ([9, pp. 261–265]), we get that if $0 < p < 1$, $s, t > -d/2$, then

$$D'[\ldots, [b, H_1], \ldots, H_d]D' \in S_p$$

if and only if $b \in B^{s+t+d/p}$ and furthermore, $\|D'[\ldots, [b, H_1], \ldots, H_d]D'\|_{S_p}$ is comparable to $\|b\|_{B^{s+t+d/p}}$. Here $D'[\ldots, [b, H_1], \ldots, H_d] \in S_p$ is defined in the natural way.

The phenomena of Theorems 3 and 4 do not appear for Hankel operators or one-dimensional commutators but they appear for general kernels $A(\xi, \eta)$. Thus we need other methods to deal with them. Our method to deal with the “Janson-Wolff phenomenon” in Theorem 4 is close to that of [2].

The proofs of Theorems 1, 2, and 4 are given in sections 4–6, respectively.
We omit the proof of Theorem 3, referring Peng [8]. In section 2 we examine some examples and in section 3 we present some lemmas which will be used in sections 4–6.

Results similar to Theorems 1 and 2 have also been given by Timotin [11].

2. Examples.

To show that a function belongs to $V_p(E \times F)$, the Fourier series expansion is often an efficient tool.

**Proposition 2.** Let $T^d = [-\pi, \pi]^d$, $u = [d(2/p - 1)] + 1$. Suppose that $A \in C^u(T^d \times T^d)$ and $supp \ A \subset Int(T^d \times T^d)$.

Then $A \in V_p(T^d \times T^d)$ and

$$
||A||_{V_p(T^d \times T^d)} \leq C \sum_{0 \leq |\alpha| + |\beta| \leq u} sup|D_\xi^\alpha D_\eta^\beta A|.
$$

**Proof.** Use the Fourier series expansion

$$
A(\xi, \eta) = \sum_{(m, n) \in \mathbb{Z}^{2d}} \hat{A}(m, n)e^{im \cdot \xi + in \cdot \eta}, \ \xi, \eta \in T^d.
$$

Thus

$$
||A||_{V_p(T^d \times T^d)} \leq \sum_{(m, n) \in \mathbb{Z}^{2d}} |\hat{A}(m, n)|^p \\
\leq \left\{ \sum_{(m, n) \in \mathbb{Z}^{2d}} (1 + |m|^2 + |n|^2)^u |\hat{A}(m, n)|^2 \right\}^{p/2} \\
\cdot \left\{ \sum_{(m, n) \in \mathbb{Z}^{2d}} (1 + |m|^2 + |n|^2)^{-m(p)/(2 - p)} \right\}^{(2 - p)/2} \\
= C \left\{ \sum_{0 \leq |\alpha| + |\beta| \leq u} C_{\alpha, \beta} ||D_\xi^\alpha D_\eta^\beta A||^2 \right\}^{p/2}.
$$

Therefore

$$
||A||_{V_p(T^d \times T^d)} \leq C \sum_{0 \leq |\alpha| + |\beta| \leq u} sup|D_\xi^\alpha D_\eta^\beta A|.
$$

Using Proposition 2, we can obtain the following three propositions.
Proposition 3. Let $A \in C^b(\tilde{A}_k \times \tilde{A}_1)$. Then

$$||A||_{V_p(\tilde{A}_k \times \tilde{A}_1)} \leq C \sup_{0 \leq |\alpha| + |\beta| \leq u} \sup_{\xi \in \tilde{A}_k, \eta \in \tilde{A}_1} |\xi|^a |\eta|^b |D^\alpha_x D^\beta_\eta A(\xi, \eta)|.$$

Proposition 4. Let $B = B(\xi_0, r)$. Then

$$||A||_{V_p(B \times B)} \leq C \sup_{0 \leq |\alpha| + |\beta| \leq u} \sup_{\xi, \eta \in B(\xi_0, 2r)} |D^\alpha_x D^\beta_\eta A(\xi, \eta)|.$$ 

Proposition 5. Suppose that $k \geq 1$ and $m \geq \max(u, k)$. Suppose further that $r < \frac{1}{4} |\xi_0|$ and $A \in C^m(B(\xi_0, 2r) \times B(\xi_0, 2r))$ with $D^a A(\xi_0, \xi_0) = 0$, when $|\alpha| \leq k - 1$. Then

$$||A||_{V_p(B \times B)} \leq C(r/|\xi_0|)^4 \sup_{|\alpha| \leq m} \sup_{\xi, \eta \in B(\xi_0, 2r)} |\xi_0|^a |D^a A(\xi, \eta)|.$$ 

Now we examine some examples.

Example 1. $N$th order commutators of singular integral operators. When $d = 1$, the singular integral operator $K$ is a scalar multiple of the Hilbert transform, so the $N$th order commutator has the kernel

$$A(\xi, \eta) = C(I(\xi > 0) - I(\xi < 0) < \eta).$$

($I(\ldots)$ denotes the indicator function, see [3].) It is clear that in this case $A(\xi, \eta)$ satisfies A0 for any $r > 0$, $A_{p1}$, $A_{p3}(\infty)$, $A_{p4^2}$, $A_{p9}(1)$, but not A10 for any $\alpha > 0$. So the conclusions of Theorems 1 and 2 hold for it, and the results of Semmes [9] can be easily obtained. The “Janson-Wolff phenomenon” described in Theorem 4 does not appear for it.

When $d \geq 2$, let $K_i$ denote a Calderón-Zygmund transform, i.e. the principal value convolution with a kernel $K_i$ whose Fourier transform $\hat{K}_i$ is homogeneous of degree 0, $C^\alpha(R^d \setminus \{0\})$ and has vanishing spherical mean values. The $N$th order commutator $[K_1, \ldots, [K_{N,5}], \ldots]$ has its kernel $A(\xi, \eta) = \prod_{i=1}^N [\hat{K}_i(\xi) - \hat{K}_i(\eta)]$. It is easy to check that in this case $A(\xi, \eta)$ satisfies A0 for any $r > 0$, $A_{p1}$, $A_{p3}(N)$, and $A_{p9}(1)$. If $A$ satisfies the non-degeneracy condition:

$$\text{(*) if } \prod_{i=1}^N (\hat{K}_i(\xi + \theta) - \hat{K}_i(\xi)) = 0 \text{ for all } \xi \text{ then } \theta = 0, \text{ then } A \text{ satisfies } A_{p4^2}.$$ 

If $A$ satisfies the non-degeneracy condition:

$$\text{(**) if } \prod_{i=1}^N D_\theta \hat{K}_i(\xi) = 0 \text{ for all } \xi \text{ then } \theta = 0, \text{ then } A \text{ satisfies } A_{p9}(N).$$

It is obvious that (**) $\Rightarrow$ (*), so if $A$ satisfies (**), then $A$ satisfies
A, A^1 and A(10(N). In this case, all of the conclusions of Theorems 1–4 and Corollary 1 hold.

**Example 2. Paraproducts.** The name "paraproduct" denotes an idea rather than a unique definition; several versions exist and can be used for the same purposes. For example, consider the paracommutator with the kernel

\[ A(\xi, \eta) = \varphi(|\xi|/|\xi - \eta|), \]

where \( \varphi \in C^\infty(0, \infty) \), \( \varphi = 1 \) on \((0, \delta)\) and \( \varphi = 0 \) on \((1 - \delta, \infty)\) for some \( \delta > 0 \). It is easy to check that in this case \( A(\xi, \eta) \) satisfies A0 for any \( r > 0 \), \( A_p1 \), \( A_p3(\infty) \), \( A_p4^1 \), \( A_p9(1) \), but does not satisfy A10 for any \( \alpha > 0 \). So the conclusions of Theorems 1–3 hold and the "Janson-Wolff phenomenon" described in Theorem 4 does not appear for it.

**Example 3. \( A(\xi, \eta) \) smooth.** Suppose that \( A \in C^\infty(\mathbb{R}^d \setminus \{0\}) \) and that, for each multi-index \( \alpha \), there exist a constant \( C_\alpha \) such that

\[ |D^\alpha A(\xi, \eta)| \leq C_\alpha (|\xi| + |\eta|)^{-|\alpha|} \]

and a positive integer \( N \) such that

\[ D^\alpha A(\xi, \eta) = 0 \quad \text{for} \quad |\alpha| \leq N - 1. \]

This is a kind of pseudo-differential operators studied by Coifman and Meyer [1]. Using Propositions 2–5, it is easy to check that \( A(\xi, \eta) \) satisfies \( A_p1 \), \( A_p3(N) \), and \( A_p9(1) \). If \( A(\xi, \eta) \) satisfies A0 for some \( r > 1 \) and A4 (see [3]), then it is not too hard to check that \( A(\xi, \eta) \) satisfies \( A_p4^1 \). If further \( A(\xi, \eta) \) satisfies

\[ (\triangle) \quad \text{for each} \quad \theta \neq 0 \quad \text{there exists} \quad \xi_1 \neq 0 \quad \text{with} \quad D^\theta_{\xi} A(\xi_1, \xi_1) \neq 0, \]

then \( A(\xi, \eta) \) satisfies A10(N).

So the conclusions of Theorems 1–4 and Corollary 1 hold for such paracommutators.

3. Some lemmas.

**Lemma 1.** If \( 0 < p < 1 \), \( T, S \in S_p \), then

\[ ||T + S||^p_{L^p} \leq ||T||^p_{L^p} + ||S||^p_{L^p}, \]

and equality holds if and only if \( T^* T S S = 0 \). (Cf. McCarthy [4].)

**Lemma 2.** If \( \{E_k\}_{k \in Z}, \{F_k\}_{k \in Z} \) are sets of disjoint subsets of \( \mathbb{R}^d \) such that \( E_k \cap F_l = \emptyset \) for \( k \neq l \), let \( Q_k, P_k \) denote the projections from \( L^2(\mathbb{R}^d) \) into
\[ L^2(E_k), L^2(F_k). \text{ Then} \]
\[ \left\| \sum_{k \in \mathbb{Z}} Q_k T P_k \right\|_{L^p}^p = \sum_{k \in \mathbb{Z}} \|Q_k T P_k\|_{L^p}^p \]
holds for \( T \in S_p(\mathbb{R}^d \times \mathbb{R}^d) \).

This is a consequence of Lemma 1.

**Lemma 3.** If \( F_1 \cap F_2 = \emptyset \), \( A \in V_p(E \times F_1) \), \( A \in V_p(E \times F_2) \). Then 
\( A \in V_p(E \times (F_1 \cup F_2)) \) and 
\[ \|A\|_{V_p(E \times (F_1 \cup F_2))} \leq C(\|A\|_{V_p(E \times F_1)} + \|A\|_{V_p(E \times F_2)}). \]
This is obvious.

**Lemma 4.** Let \( \chi \in C_0^\infty(\mathbb{R}^d) \), \( \text{supp} \chi \subseteq \Delta_0 \), \( \chi(\xi) = 1 \) on \( \Delta_0 \), and \( N \) be a fixed integer. Then
\[ \|\chi(\xi - \eta)\|_{V_p(\Delta_k \times \Delta_l)} \leq C(N) \text{ for } k, l \leq N \]
and
\[ \|\chi(\xi - \eta)\|_{V_p(B \times B)} \leq C(r) \text{ for all } B = B(\xi_0, r). \]
These are consequences of Propositions 3–4 in section 2.

**Lemma 5.** Let \( \Omega \) be a compact subset of \( \mathbb{R}^d \), \( 0 < p < 1 \). Then, for every \( r < p \), there exists a constant \( C \) such that
\[ \sup_{z \in \mathbb{R}^d} \frac{\| \varphi(x - z) \|}{1 + |z|^{d/r}} \leq C \left[ M[\varphi]^r(x) \right]^{1/r} \]
holds for all \( \varphi \in L^p_p = \{ \varphi \in L^p : \text{supp } \hat{\varphi} \subseteq \Omega \} \). (Cf. Triebel [12, p. 16 and p. 22].)

**Lemma 6.** Let \( 0 < p < \infty \), \( a > 0 \). For any \( a' > a \), there exist two positive constants \( C_1 \) and \( C_2 \) such that
\[ C_1 \left( \sum_{k \in \mathbb{Z}^d} \| \varphi(k/a') \|_p \right)^{1/p} \leq \|\varphi\|_p \leq C_2 \left( \sum_{k \in \mathbb{Z}^d} \| \varphi(k/a') \|_p \right)^{1/p} \]
holds for all \( \varphi \in \{ \varphi \in S' : \text{supp } \hat{\varphi} \subseteq B(0, a) \} \).

This is the theorem of Plancherel and Polya, see Triebel [12, p. 19–20].

**Lemma 7.** Let \( b \in L^{B_{00}(R/2)}_p \), \( 0 < p < 1 \), and let \( (\hat{b})_e(\xi) \) denote the periodic extension of \( \hat{b}(\xi) \) with the period \( 2\pi R \). Then
\[ \|(\hat{b})_e(\xi - \eta)\|_{V_p(\mathbb{R}^d \times \mathbb{R}^d)} \leq \mathbb{R}^{d/p - 4} \|b\|_p. \]
PROOF. By the homogeneity, it suffices to show (14) for \( R = 1 \). In that case, \( \text{supp} \hat{b} \subset B(0, \frac{1}{4}) \), and we extend \( \hat{b}(\xi) \) to a periodic function \( \hat{b}_c(\xi) \) with the period \( 2\pi \) and expand it into a Fourier series

\[
(\hat{b})_c(\xi) = \sum_{k \in \mathbb{Z}^d} b(k) e^{ik \cdot \xi}.
\]

Thus

\[
(\hat{b})_c(\xi - \eta) = \sum_{k \in \mathbb{Z}^d} b(k) e^{ik \cdot \xi} e^{-ik \cdot \eta}.
\]

Since \( |e^{ik \cdot \xi}| = |e^{-ik \cdot \eta}| = 1 \) for \( \xi, \eta \in \mathbb{R}^d \), by Lemma 6,

\[
\left( \sum_{k \in \mathbb{Z}^d} |b(k)|^p \right)^{1/p} \lesssim C\|b\|_p.
\]

Hence (14) holds.

4. Proof of Theorem 1.

Let \( \psi \in \mathcal{S}(\mathbb{R}^d) \) be such that \( \text{supp} \hat{\psi} \subset \mathcal{A}_0 \) and if \( \xi \neq 0 \), then \( \sum_{k = -\infty}^{\infty} \hat{\psi}_k(\xi) = 1 \) with \( \hat{\psi}_k(\xi) = \hat{\psi}(2^{-k}\xi) \). Thus we have

\[
b = \sum_{k = -\infty}^{\infty} b \ast \psi_k.
\]

Let \( \chi \in C_0^\infty(\mathbb{R}^d) \) be such that \( \chi(\xi) = 1 \) for \( \xi \in \mathcal{A}_0 \) and

\[
\text{supp} \chi \subset \left\{ \frac{1}{2} - \epsilon \leq |\xi| \leq 2 + \epsilon \right\} \text{ for some } 0 < \epsilon < \frac{1}{2}.
\]

Put \( \chi_k(\xi) = \chi(2^{-k}\xi) \). By Lemma 1, we have

\[
||T^p_b||_{\mathcal{S}^p} \lesssim \sum_{k = -\infty}^{\infty} ||T^p_{b \ast \psi_k}||_{\mathcal{S}^p}.
\]

Note that \( (b \ast \psi_k)^\wedge(\xi - \eta)A(\xi, \eta) = (b \ast \psi_k)^\wedge(\xi - \eta)\chi_k(\xi - \eta)A(\xi, \eta) \). By Proposition 1, we get

\[
||T^p_{b \ast \psi_k}||_{\mathcal{S}^p} = ||(b \ast \psi_k)^\wedge(\xi - \eta)\chi_k(\xi - \eta)A(\xi, \eta)||_{L^p(\mathbb{R}^d)} \lesssim C2^{kd(1 - p)}||b \ast \psi_k||_{\mathcal{S}^p}.
\]

By Lemma 7, we know that

\[
||(b \ast \psi_k)^\wedge(\xi - \eta)||_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim C2^{kd(1 - p)}||b \ast \psi_k||_{\mathcal{S}^p}.
\]
It suffices to show that
\[
\|\chi_k(\xi - \eta) A(\xi, \eta) \xi^l \eta^l \|_{S_p(\mathbb{R}^d)}^p \leq C 2^{dk_p + sk_p + tk_p}.
\]
In view of the homogeneity of the assumptions on $A$, it suffices to show (16) for $k = 0$, i.e.
\[
\|\chi(\xi - \eta) A(\xi, \eta) \xi^l \eta^l \|_{S_p(\mathbb{R}^d)}^p \leq C.
\]
To show (17), we use the analogue of the argument in [3] for $p = 1$. First of all, by Lemma 4, we have
\[
\|\chi(\xi - \eta) A(\xi, \eta) \xi^l \eta^l \|_{S_p(\mathbb{R}^d)}^p \\
\leq \|\chi(\xi - \eta) \xi^l \eta^l \|_{S_p(\mathbb{R}^d)}^p \\
\leq C \|\xi^l \eta^l \|_{L^2(\mathbb{R}^d)} \|\xi^l \eta^l \|_{L^2(\mathbb{R}^d)} \\
\leq C 2^{k_p s + d/2} 2^{l_p t + d/2}
\]
for $k, l \leq N$.

For $k \in \mathbb{Z}^d$, let $Q_k$ denote the cube with centre $4k$ and side $4$, and let $\bar{Q}_k$ be the concentric cube with side $9$. Note that if $\text{supp} \hat{f} \subseteq Q_k$, then
\[
\text{supp} \int \chi(\xi - \eta) A(\xi, \eta) \xi^l \eta^l \hat{f}(\eta) d\eta \subseteq \bar{Q}_k.
\]
Thus we have
\[
\|\chi(\xi - \eta) A(\xi, \eta) \xi^l \eta^l \chi_{Q_k}(\eta) \|_{S_p(\mathbb{R}^d \times \mathbb{R}^d)}^p = \|\chi(\xi - \eta) A(\xi, \eta) \xi^l \eta^l \|_{S_p(\mathbb{R}^d \times \mathbb{R}^d)}^p.
\]
When $|k| > 3\sqrt{d}/4\delta$, where $\delta$ is as in $A_p 3(\alpha)$, by Lemma 4 and $A_p 3(\alpha)$, we have
\[
\|\chi(\xi - \eta) A(\xi, \eta) \xi^l \eta^l \|_{S_p(\mathbb{R}^d \times \mathbb{R}^d)}^p \\
\leq \|\chi(\xi - \eta) \xi^l \eta^l \|_{S_p(\mathbb{R}^d \times \mathbb{R}^d)}^p \\
\leq C |k|^{-p_\alpha + ps + pt}.
\]
When $|k| \leq 3\sqrt{d}/4\delta$, note that $\bar{Q}_k \subseteq \bigcup_N^\infty A_k$, where $N$ is an integer depending only on $\delta$ and $d$. Thus we have
\[
\|\chi(\xi - \eta) A(\xi, \eta) \xi^l \eta^l \|_{S_p(\mathbb{R}^d \times \mathbb{R}^d)}^p \\
\leq \sum_{k = -\infty}^N \sum_{l = -\infty}^N \|\chi(\xi - \eta) A(\xi, \eta) \xi^l \eta^l \|_{S_p(\mathbb{R}^d \times \mathbb{R}^d)}^p.
\]
Using (18), we get
\[ \|\chi(\xi - \eta) A(\xi, \eta) \xi^i \eta^j \|_{S_p(\mathbb{Q}_1 \times \mathbb{Q}_2)}^p \]
\[ \leq \sum_{k = -\infty}^{N} \sum_{l = -\infty}^{N} 2^{k(p(s + d/2) + 1)} \]
\[ = C. \]

Since \( \alpha > s + t + d/p \), we have
\[ \sum_{|k| > 3\sqrt{d/4\delta}} |k|^{-p\alpha + ps + pt} < \infty. \]

Therefore (19) and (20) imply that
\[ \|\chi(\xi - \eta) A(\xi, \eta) \xi^i \eta^j \|_{S_p(\mathbb{R}^d \times \mathbb{R}^d)}^p \]
\[ \leq \sum_{k \in \mathbb{Z}^d} \|\chi(\xi - \eta) A(\xi, \eta) \xi^i \eta^j \|_{S_p(\mathbb{Q}_1 \times \mathbb{Q}_2)}^p \]
\[ \leq C \sum_{|k| > 3\sqrt{d/4\delta}} |k|^{-p\alpha + ps + pt} + C \sum_{|k| \leq 3\sqrt{d/4\delta}} \]
\[ = C. \]

This completes the proof of Theorem 1.

**Corollary 2.** Under the same assumption as in Theorem 1.
If \( b \in B_p^{s + t + d/p} \) with supp \( \mathcal{B} \subset \mathbb{R}^d \setminus \{0\} \), then \( T_b^{\theta}(A_\theta^0) \in S_p \) and
\[ \| T_b^{\theta}(A_\theta^0) \|_{S_p} \leq C \|b\|_{B_p^{s + t + d/p}} \]
holds uniformly in \( \varepsilon \leq \varepsilon_0 \) and \( |\theta| \leq \sqrt{d}/3 \), for some \( \varepsilon_0 > 0 \).

**Proof.** We may assume that supp \( \mathcal{B} \subset \{|\xi| \geq 2^{-N_0}\} \), for some large number
\( N_0 > 0 \). Let \( \varepsilon < \varepsilon_0 = 2^{-N_0}/\sqrt{d} \). By the proof of Theorem 1, it suffices to show that \( A_\theta^\varepsilon \) satisfies \( A_p^1 \) and that for \( B = B(\xi_0, r) \) with \( 2^{-N_0} < r < \delta/2|\xi_0| \), holds
\[ \| A_\theta^\varepsilon \|_{V_p(B \times B)} \leq C(r/|\xi_0|)^\varepsilon. \]
In fact, if \( 2^{k+1} < \varepsilon/2 \),
\[ \| A_\theta^\varepsilon \|_{V_p(\Delta_k \times \Delta_k)} = 0; \]
if $2^{k+1} \geq \epsilon/2$,

$$\|A_\epsilon^a\|_{V_p(\delta_0 \times \delta_\epsilon)} = \left\| \sum_{\substack{m \in \mathbb{Z}^d \cap Q_{\epsilon,0}^* \neq \emptyset}} A(\xi, \epsilon(m + \theta)) \chi_{Q_{\epsilon}^*}(\eta) \right\|_{V_p(\delta_0 \times \delta_\epsilon)} \leq \left\| \sum_{\substack{m \in \mathbb{Z}^d \cap Q_{\epsilon,0}^* \neq \emptyset}} A(\xi, \epsilon(m + \theta)) \chi_{Q_{\epsilon}^*}(\eta) \right\|_{V_p(\delta_0 \times \delta_\epsilon)} \leq \|A\|_{V_p(\delta_0 \times \delta_\epsilon)} \leq C \quad \text{(by Lemma 3)}.$$ 

So $A_\epsilon^a$ satisfies $A_p 1$. If $2^{-N_0} < r < d/2|\xi_0|$, then

$$\|A_\epsilon^a\|_{V_p(B \times B)} = \left\| \sum_{\substack{m \in \mathbb{Z}^d \cap Q_{\epsilon,0}^* \neq \emptyset}} A(\xi, \epsilon(m + \theta)) \chi_{Q_{\epsilon}^*}(\eta) \right\|_{V_p(B \times B)} \leq \left\| \sum_{\substack{m \in \mathbb{Z}^d \cap Q_{\epsilon,0}^* \neq \emptyset}} A(\xi, \epsilon(m + \theta)) \chi_{Q_{\epsilon}^*}(\eta) \right\|_{V_p(B \times B)} \leq \|A\|_{V_p(B \times B)} \leq C \left( \frac{r + 2^{-N}}{|\xi_0|} \right)^\alpha \leq C \left( \frac{r}{|\xi_0|} \right)^\alpha \quad \text{(because } r + 2^{-N_0} \leq 2r < \delta|\xi_0|.\text{)}$$

5. Proof of Theorem 2.

For the sake of simplicity, we assume that $r = 2$ in $A_0$. It is easy to show that $A_p A_\frac{1}{2}$ is equivalent to the following statement.

For every $\xi_0 \neq 0$, there exist $\eta_0 \in \mathbb{R}^d$ and $\delta > 0$ with $\eta_0 \notin \{0, -\xi_0\}$ and $\delta < \frac{1}{4}\min(|\xi_0 + \eta_0|, |\eta_0|, 1)$ such that, with $B_0 = B(\xi_0 + \eta_0, \delta|\xi_0|)$ and $D_0 = B(\eta_0, \delta|\xi_0|), A(\xi, \eta)^{-1} \in V_p(B_0 \times D_0)$.

By the compactness of $A_0$, there exist finite sets of points $\{\xi_0^j\}_{j=1}^J$ in $A_0$ and $\{\eta_0^j\}_{j=1}^J$, with corresponding open balls $B(\xi_0^j, \delta^j)\cap B(\eta_0^j, \delta^j)$, such that $\eta_0^j \neq 0, \delta_0^j \neq -\xi_0^j$,

$$\bigcup_{j=1}^J B(\xi_0^j, \delta^j) \ni A_0, \quad \delta^j < \frac{1}{4}\min(|\xi_0^j + \eta_0^j|, |\eta_0^j|, 1)$$

and, with $B_j = B(\xi_0^j + \eta_0^j, \delta^j)$ and $D_j = B(\eta_0^j, \delta^j)$,

$$A^{-1} \in V_p(B_j \times D_j).$$
We choose the positive functions $h'_j(\xi)$ and $h_j(\eta)$ such that $h'_j, h_j \in C^\infty_0(\mathbb{R}^d)$, supp $h'_j = \bar{B}_j$, $h'_j(\xi) > 0$ on $B_j$, supp $h_j = \bar{D}_j$ and $h_j(\eta) > 0$ on $D_j$. Let

$$
\hat{\psi}(\xi) = \sum_{j=1}^{J} \int |\xi + \eta|^m |\eta|^l h'_j(\xi + \eta)h_j(\eta)d\eta.
$$

(21)

Then $\hat{\psi} \in C^\infty_0(\mathbb{R}^d)$, supp $\hat{\psi} \subset \{ \frac{1}{2} \leq |\xi| \leq 2 + \frac{1}{2} \}$ and $\hat{\psi}(\xi) \geq C > 0$ on $\Delta_0$. Thus $\psi$ can be used to define the norm of $B^{s+t+d/p}_p$.

Let $\hat{\psi}' \in C^\infty_0(\mathbb{R}^d)$ with support $\subset \{ \frac{1}{8} \leq |\xi| \leq 4 \}$ and $\hat{\psi}'(\xi) = 1$ on $\{ \frac{1}{4} \leq |\xi| \leq 3 \}$. Thus $\psi'$ can be used to define the norm of $B^{s+t+d/p}_p$ also, and

$$
\| \cdot \|_{B^{s+t+d/p}_p(\psi)} \approx \| \cdot \|_{B^{s+t+d/p}_p(\psi')}.
$$

Let $\hat{\psi}_k(\xi), \hat{\psi}'_k(\xi)$ denote $\hat{\psi}(2^{-k}\xi), \hat{\psi}'(2^{-k}\xi)$ respectively.

For $\eta^0 \neq 0, \xi^0 + \eta^0 \neq 0$, there exist $r_1$ and $r_2$ with $0 < r_1 < r_2 < r_2 < \infty$ such that

$$
\begin{equation}
    r_1 \leq |\eta^0|, \quad |\xi^0 + \eta^0| \leq r_2, \quad j = 1, ..., J.
\end{equation}
$$

For the sake of simplicity, we assume that $3/4 \leq |\eta^0|, |\xi^0 + \eta^0| \leq 2\frac{1}{2}$, thus

$$
\text{supp } h'_j, \text{ supp } h_j \subset \{ \frac{1}{2} \leq |\xi| \leq 1\frac{3}{2} \} = \bar{\Delta}_0, \quad \text{for } j = 1, ..., J.
$$

The proof of the general case is similar.

We fix a positive integer $M$, which is large enough and whose choice will be specified later. We define operators $T_i, i = 0, ..., M - 1$, by

$$
(T_i f)(\xi) = (2\pi)^{-d} \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \int \hat{b}(\xi - \eta)A(\xi, \eta)|\xi|^m |\eta|^l \chi_{\Delta_{M+i}}(\xi) \chi_{\bar{\Delta}_{M+i}}(\eta)\hat{f}(\eta)d\eta,
$$

(22)

where $\bar{\Delta}_k = 2^{k}\bar{\Delta}_0$. Note that $\|T_i\|_{S_p} \leq \|T^{o}_b\|_{S_p}$, so we have

$$
\sum_{i=0}^{M-1} \|T_i\|_{S_p}^p \leq M \|T^{o}_b\|_{S_p}^p.
$$

(23)

We put

$$
T_i = T^{(1)}_i + T^{(2)}_i,
$$

(24)

where $T^{(1)}_i$ is defined by
\[(T_i^{(1)}) \hat{f}(\xi)\]
\[(25) \quad = (2\pi)^{-d} \sum_{k = -\infty}^{\infty} \sum_{l = -\infty}^{\infty} \int b(\xi - \eta) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\xi) \chi_{\bar{A}_{km+i}}(\eta) \hat{f}(\eta) d\eta \]
and \(T_i^{(2)}\) by
\[(T_i^{(2)}) \hat{f}(\xi)\]
\[(26) \quad = (2\pi)^{-d} \sum_{k = -\infty}^{\infty} \sum_{l = -\infty}^{\infty} \int b(\xi - \eta) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\xi) \chi_{\bar{A}_{km+i}}(\eta) \hat{f}(\eta) d\eta \]
\[+ (2\pi)^{-d} \sum_{l = -\infty}^{\infty} \sum_{k = -\infty}^{\infty} \int b(\xi - \eta) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\xi) \chi_{\bar{A}_{km+i}}(\eta) \hat{f}(\eta) d\eta.\]

We estimate the "\(S_p\)-norm" of \(T_i^{(1)}\) from below and the "\(S_p\)-norm" of \(T_i^{(2)}\) from above separately. First of all we have
\[
\left\| \left\{ \sum_{l = -\infty}^{k-1} \chi_{\bar{A}_{km+i}}(\xi) \right\} b(\xi - \eta) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\eta) \right\|_{L^p} \]
\[\leq \|\chi_{(0, 2^{a-1})^{M+i+2}}(\xi) b(\xi - \eta) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\eta) \|_{L^p},\]
where \(\chi_{(0, b]}(\xi)\) is the characteristic function of \(\{ |\xi| \leq b \}\). If \(M\) is large enough, then
\[\hat{\psi}_{km+i}(\xi - \eta) = 1 \quad \text{on} \quad \{ |\xi| \leq 2^{-(k-1)M+i+2} \} \times \bar{A}_{km+i}.\]

Thus we have
\[
\|\chi_{(0, 2^{a-1})^{M+i+2}}(\xi) b(\xi - \eta) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\eta) \|_{L^p} \]
\[= \|\chi_{(0, 2^{a-1})^{M+i+2}}(\xi) b(\xi - \eta) \hat{\psi}_{km+i}(\xi - \eta) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\eta) \|_{L^p} \]
\[= \|\chi_{(0, 2^{a-1})^{M+i+2}}(\xi) (b * \psi_{km+i}^\ast) \hat{\psi}(\xi - \eta) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\eta) \|_{L^p} \]
\[\leq C \|b * \psi_{km+i}^\ast\| L_p 2^{(kM+i+1)(d+pd)} \|\chi_{(0, 2^{a-1})^{M+i+2}}(\xi) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\eta) \|_{L^p} \]
(by Proposition 1 and Lemma 7).

We claim that
\[(27) \quad \|\chi_{(0, 2^{a-1})^{M+i+2}}(\xi) A(\xi, \eta)|\xi|^{\alpha} |\eta|^{\alpha} \chi_{\bar{A}_{km+i}}(\eta) \|_{L^p} \]
\[\leq C 2^{(kM+i)(sp+ip+dp)} 2^{-MP(s+d/2)}.\]
In fact, to show (27), by homogeneity of $A(\xi, \eta)$, it is sufficient to show it for $k = 0, i = 0$. In that case we have

$$
\|\chi_{(0, 2^{-M+1})}(\xi) A(\xi, \eta) |\xi|^{n}|\eta|^l \chi_d(\eta)\|_{S_p}
\leq \sum_{k = -1}^{1} \sum_{l = -\infty}^{-M+1} \|A(\xi, \eta) |\xi|^{n}|\eta|^l\|_{S_p(d_1 \times d_k)}
\leq C \sum_{k = -1}^{1} \sum_{l = -\infty}^{-M+1} 2^{lp(s+d/2)} 2^{k(p(t + d/2))} = C 2^{-Mp(s+d/2)},
$$

i.e. (27) holds.

Thus we obtain

$$
\left\| \left\{ \sum_{i = -\infty}^{i-1} \chi_{d_{i+1}}(\xi) \right\} \tilde{\delta}(\xi - \eta) A(\xi, \eta) |\xi|^{n}|\eta|^l \chi_{d_{i+1}}(\eta) \right\|_{S_p}^p
\leq C 2^{-Mp(s+d/2)} 2^{(kM+i)(sp+tp+d)} \|b * \psi_{k'M+i}^r\|_{p}^p.
$$

Similarly, we obtain

$$
\left\| \chi_{d_{i+1}}(\xi) \tilde{\delta}(\xi - \eta) A(\xi, \eta) |\xi|^{n}|\eta|^l \left\{ \sum_{i = -\infty}^{i-1} \chi_{d_{i+1}}(\eta) \right\} \right\|_{S_p}^p
\leq C 2^{-Mp(t+d/2)} 2^{(lM+i)(sp+tp+d)} \|b * \psi_{lM+i}^r\|_{p}^p.
$$

Hence we get the estimate of the "$S_p$-norm" of $T_i^{(2)}$ from above

$$
\|T_i^{(2)}\|_{S_p}^p \leq C \left[ 2^{-Mp(s+d/2)} + 2^{-Mp(t+d/2)} \right] \sum_{k = -\infty}^{\infty} 2^{kM+i}(sp+tp+d) \|b * \psi_{k'M+i}^r\|_{p}^p.
$$

Consequently,

$$
(28) \sum_{i = 0}^{M-1} \|T_i^{(2)}\|_{S_p}^p \leq C \left[ 2^{-Mp(s+d/2)} + 2^{-Mp(t+d/2)} \right] \|b\|_{B^{p+\delta}(\psi)}^p.
$$

Now we are going to estimate the "$S_p$-norm" of $T_i^{(1)}$ from below. By Lemma 2

$$
\|T_i^{(1)}\|_{S_p}^p = (2\pi)^{-dp} \sum_{k = -\infty}^{\infty} \|\chi_{d_{i+1}}(\xi) \tilde{\delta}(\xi - \eta) A(\xi, \eta) |\xi|^{n}|\eta|^l \chi_{d_{i+1}}(\eta)\|_{S_p}^p.
$$
We claim that when $M_1$ is large enough

$$\|x_{\mathcal{A}_{M_1}}(\xi)\hat{A}(\xi, \eta)A(\xi, \eta)\|_{L^p_x}^p \geq CM_1^{-d_2(1+\delta)}\|b*\psi_{\mathcal{A}_{M_1}} + \|_{L^p_x}^p - M_1^{-N_2}\|b*\psi_{\mathcal{A}_{M_1}} + \|_{L^p_x}^p.$$  

(29)

In fact, by the homogeneity of $A(\xi, \eta)$ it is sufficient to show (29) for $k = i = 0$.

Since $\text{supp} h'_j = \bar{B}_j$ and $\text{supp} h_j = \bar{D}_j$, $A_pA_{\frac{1}{2}}$ gives that

$$\|\hat{A}(\xi, \eta)\sum_{j=1}^J |\xi|^l |\eta|^l h_j'(\xi)h_j(\eta)\|_{L^p_x}^p \leq \left( \max_{1 \leq j \leq J} \|A^{-1}\|_{V_{\bar{B}_j, \bar{D}_j}}^p \right) \sum_{j=1}^J \|\hat{A}(\xi, \eta)A(\xi, \eta)|\xi|^l |\eta|^l h'_j(\xi)h_j(\eta)\|_{L^p_x}^p \leq C \sum_{j=1}^J \|\hat{A}(\xi, \eta)A(\xi, \eta)|\xi|^l |\eta|^l x_{\mathcal{A}_0}(\xi)\|_{L^p_x}^p.$$  

Note that $\text{supp} h'_j, \text{supp} h_j \subset \mathcal{A}_0$. We therefore get

$$\|\hat{A}(\xi, \eta)\sum_{j=1}^J |\xi|^l |\eta|^l h'_j(\xi)h_j(\eta)\|_{L^p_x}^p \leq C \|\hat{A}(\xi, \eta)A(\xi, \eta)|\xi|^l |\eta|^l x_{\mathcal{A}_0}(\xi)\|_{L^p_x}^p.$$  

We consider the operator defined by

$$(Sf')(\xi) = \int \hat{A}(\xi, \eta)\sum_{j=1}^J |\xi|^l |\eta|^l h'_j(\xi)h_j(\eta) f(\eta)d\eta$$  

as an operator from $L^2((3\mathbb{T})^d)$ to $L^2((3\mathbb{T})^d)$. It is clear that the family

$$\{e_n(\eta)\}_{n \in \mathbb{Z}^d} = \{(6\pi)^{-d/2}e^{in\cdot \eta/3}\}_{n \in \mathbb{Z}^d}$$  

forms a complete basis of $L^2((3\mathbb{T})^d)$.

Thus we have

$$Sf = \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} (f, e_n)(Se_m, e_m)e_m.$$  

(30)

Let $M_1$ be a positive integer, large enough. Let $P_k$ denote the orthogonal
projection from $L^2((3T)^d)$ onto
\[ \text{span}\{e_{M_1n+k}\}_{n \in \mathbb{Z}^d}, \quad k \in \{0, \ldots, M_1 - 1\}^d. \]

Thus we have
\[
I = \bigoplus_{k \in \{0, \ldots, M_1 - 1\}^d} P_k
\]
and
\[
\sum_{k \in \{0, \ldots, M_1 - 1\}^d} \|P_kSP_k\|_{S_p}^p \leq M_1^d\|S\|_{S_p}^p.
\]

We put
\[
P_kSP_k = S_k^{(1)} + S_k^{(2)}
\]
where
\[
S_k^{(1)}f = \sum_{n \in \mathbb{Z}^d} (f, e_{M_1n+k})(Se_{M_1n+k}, e_{M_1n+k})e_{M_1n+k}
\]
\[
S_k^{(2)}f = \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d, m \neq n} (f, e_{M_1n+k})(Se_{M_1n+k}, e_{M_1m+k})e_{M_1m+k}.
\]

For $S_k^{(1)}$, since
\[
(Se_{M_1n+k}, e_{M_1n+k})
\]
\[
= \int \int \hat{b}(\xi - \eta) \sum_{j=1}^J |\xi|^m|\eta|^m h_j(\xi) h_j(\eta) e_{M_1n+k}(\eta) e_{M_1n+k}(-\xi) d\eta d\xi
\]
(by changing variables $\xi \rightarrow \xi' + \eta$)
\[
= \int \int \hat{b}(\xi) \sum_{j=1}^J |\xi + \eta|^m|\eta|^m h_j(\xi + \eta) h_j(\eta) e_{M_1n+k}(-\xi) d\eta d\xi
\]
\[
= C \int \hat{b}(\xi) \tilde{\psi}(\xi)e^{-i(M_1n+k) \cdot \xi/3} d\xi
\]
\[
= Cb \ast \psi(-(M_1n+k)/3),
\]
we have
\[ \| S_k^{(1)} \|_S^p = \sum_{n \in \mathbb{Z}^d} \| (S e_{M,n+k}, e_{M,n+k}) \|_p = C \sum_{n \in \mathbb{Z}^d} |b \ast \psi(-(M_1n+k)/3)|^p \]
and
\[ \sum_{k \in \{0, \ldots, M_1-1\}^d} \| S_k^{(1)} \|_S^p = C \sum_{k \in \{0, \ldots, M_1-1\}^d} \sum_{n \in \mathbb{Z}^d} |b \ast \psi(-(M_1n+k)/3)|^p \]
(34)
\[ = C \sum_{n \in \mathbb{Z}^d} |b \ast \psi(n/3)|^p \geq C |b \ast \psi|^p \] (by Lemma 6).

For \( S_k^{(2)} \), we estimate its "\( S_p \) norm" from above,
\[ \| S_k^{(2)} \|_S^p \]
\[ \leq \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d \atop m \neq n} \| (Se_{M,n+k}, e_{M,m+k}) \|_p \]
\[ = \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d \atop m \neq n} \left| \int \int b(\xi - \eta) \sum_{j=1}^J |\xi|^\alpha |\eta|^\alpha h_j(\xi) h_j(\eta) e_{M,n+k}(\eta) e_{M,m+k}(-\xi) d\xi d\eta \right|^p \]
\[ = \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d \atop m \neq n} \left| \int b(\xi) e_{M,m+k}(-\xi) \int \sum_{j=1}^J |\xi + \eta|^\alpha |\eta|^\alpha h_j(\xi + \eta) h_j(\eta) e_{M,(n-m)}(\eta) d\eta d\xi \right|^p \]

Let \( I(\xi, \eta) \) denote \( \sum_{j=1}^J |\xi + \eta|^\alpha |\eta|^\alpha h_j(\xi + \eta) h_j(\eta) \), and write
\[ I^2(\xi, z) = \int I(\xi, \eta) e^{-iz \cdot \eta} d\eta, \quad I^{12}(y, z) = \int I^2(\xi, z) e^{-iz \cdot \xi} d\xi. \]

Then
\[ \| S_k^{(2)} \|_S^p \]
\[ \leq C \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d \atop m \neq n} \left| \int b(\xi) \tilde{\psi}'(\xi) e_{M,m+k}(-\xi) I^2(\xi, M_1(m-n)/3) d\xi \right|^p \]
\[ = C \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d \atop m \neq n} \| b \ast \psi' \ast I^{12}(-(M_1n+k)/3, M_1(m-n)/3) \|_p \]
\[ = C \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d \atop m \neq n} \left| \int b \ast \psi'(-(M_1n+k)/3-y) I^{12}(y, M_1(m-n)/3) dy \right|^p. \]
Since \( I(\xi, \eta) \in C_0^\infty \), for every fixed \( N > 0 \),

\[
|I^{12}(y, z)| \leq C \frac{1}{1 + |y|^N} |z|^{-N}.
\]

Let \( Q_n' \) denote the cube with centre \(-1/3n\) and side 1/3. We choose \( r < p \) and \( N \) sufficiently large. For \( x \in Q'_{M_n + k} \), by Lemma 5,

\[
\left| \int b \ast \psi'(- (M_1 n + k)/3 - y)I^{12}(y, M_1 (m - n)/3) dy \right|
\leq C \int \frac{|b \ast \psi'[x - (y + x + (M_1 n + k)/3)]|}{1 + |y + x + (M_1 n + k)/3|^{d/r}} \times
\frac{1 + |y + x + (M_1 n + k)/3|^{d/r}}{1 + |y|^N} dy M_1^{-N} |m - n|^{-N}
\leq C |M| b \ast \psi'|r(x)|^{1/r} M_1^{-N} |m - n|^{-N}.
\]

Integrating over \( x \in Q'_{M_1 n + k} \), we get

\[
\left| \int b \ast \psi'(- (M_1 n + k)/3 - y)I^{12}(y, M_1 (m - n)/2^k) dy \right|^p
\leq C M_1^{-Np} |m - n|^{-Np} \int_{Q'_{M_1 n + k}} (M|b \ast \psi'|r(x))^{p/r} dx.
\]

Finally, we obtain

\[
\|S^{(2)}\|_{L_p} \leq C M_1^{-Np} \sum_{n \in \mathbb{Z}^d} \int_{Q'_{M_1 n + k}} (M|b \ast \psi'|r(x))^{p/r} dx.
\]

and

\[
\sum_{k \in \{0, \ldots, M_1 - 1\}^d} \|S^{(2)}\|_{L_p} \leq C M_1^{-Np} \int_{\mathbb{R}^d} (M|b \ast \psi'|r(x))^{p/r} dx
\leq C M_1^{-Np} \|b \ast \psi'|_{L_p}^p.
\]
Combining (32), (33), (34), and (35), we obtain

\[ M_{d}^{*} ||S||_{S_{p}}^{p} \geq (C ||b*\psi||_{p}^{p} - CM_{1}^{-N}||b*\psi'||_{p}^{p}) \]

i.e. (29) holds.

Combining (28) and (29), we obtain

\[ M ||T_{b}^{*}\|_{S_{p}}^{p} \]

\[ \geq CM_{1}^{-d}[||b||_{B^{s+d}_{t,\psi}}^{p} - M_{1}^{-N}||b||_{B^{s+d}_{t,\psi}}^{p}] - 
\]

\[ - C[2^{-M_{p}(s+d/2)} + 2^{-M_{p}(t+d/2)}]||b||_{B^{s+d}_{t,\psi}}^{p} \]

\[ \geq CM_{1}^{-d}[||b||_{B^{s+d}_{t,\psi}}^{p} - M_{1}^{-N}||b||_{B^{s+d}_{t,\psi}}^{p}] - 
\]

\[ - C[2^{-M_{p}(s+d/2)} + 2^{-M_{p}(t+d/2)}]||b||_{B^{s+d}_{t,\psi}}^{p}. \]

We now choose \( M_{1} \) and \( M \) large enough. Thus we finally obtain

\[ C ||T_{b}^{*}\|_{S_{p}}^{p} \geq ||b||_{B^{s+d}_{t,\psi}}^{p}. \]

Theorem 2 has been proved.


We give the proof of Theorem 4 only in the case \( p < 1 \). For the case \( p \geq 1 \), Theorem 4 can be improved, see Corollary 3 below.

If \( b \) is not a polynomial, there exists \( 0 \neq \theta \in \text{supp} \; \delta \). Without loss of generality, we assume that \( |\theta| = 1 \). By A10(x), we find \( \delta > 0 \) and a subset \( V_{\delta} \) of \( \mathbb{R}^{d} \) such that

\[ \lim_{r \to \infty} \frac{N_{r}}{r^{d}} > 0 \]

and for every \( n \in V_{\theta}, \)

\[ \left| \int_{A(n + \theta, \cdot + n)} \right|_{M(B_{x} \times B_{y})} \leq C |n|^{a}, \quad \text{where} \quad B_{\delta} = B(0, \delta). \]

Let \( C_{a} \) denote \( \sup\{\langle T_{b}^{*}\psi, \sigma \varphi \rangle\}, \) where \( \varphi \) and \( \psi \) range over all functions with \( ||\varphi||_{2}, ||\psi||_{2} \leq 1 \), \( \text{supp} \; \hat{\sigma} \in B(n + \theta, \delta) \) and \( \text{supp} \; \hat{\psi} \in B(n, \delta) \).

If \( g \) and \( h \) are \( C_{a}^{\infty} \) functions with \( ||g||_{2} = 1/C_{a}, \; ||h||_{2} = 1/C_{t} \) \( (C_{a} > 0 \) depends on \( S, \; C_{t} > 0 \) depends on \( t \); see below), \( \text{supp} \; g, \; \text{supp} \; h \subset B(0, \delta) \), then we have,
for any fixed \( n \in V_\theta \) with \(|n| > 6\),
\[
\left| \int \int \hat{b}(\xi + \theta - \eta)g(\xi)h(\eta)\,d\xi d\eta \right|
\]
\[
= \left| \int \int \hat{b}(\xi + \theta - \eta)A(\xi + n + \theta, \eta + n)|\xi + n + \theta|^{\nu} g(\xi)h(\eta)\,d\xi d\eta \right|
\]
\[
\times |\xi + n + \theta|^{-s}|\eta + n|^{-t}g(\xi)h(\eta)\,d\xi d\eta.
\]
Since \( A(\xi + n + \theta, \eta + n)^{-1} \in M(B_\delta \times B_\delta) \), it has the representation
\[
A(\xi + n + \theta, \eta + n)^{-1}\chi_{B_\delta}(\xi)\chi_{B_\delta}(\eta) = \int_{\Omega} \beta(\xi, \omega)\gamma(\eta, \omega)d\mu(\omega)
\]
where
\[
\|\beta\|_{L^\infty(B_\delta \times \Omega)}, \|\gamma\|_{L^\infty(B_\delta \times \Omega)} \leq 1, \quad \mu(\Omega) \leq C|n|^s.
\]
Let
\[
\beta'(\xi, \omega) = \beta(\xi, \omega)|\xi + n + \theta|^{-s}(|n| - 2)^s,
\]
\[
\gamma'(\eta, \omega) = \gamma(\eta, \omega)|\eta + n|^{-t}(|n| - 2)^t.
\]
and
\[
\mu'(\omega) = \mu(\omega)(|n| - 2)^{-s-1}.
\]
Then
\[
\|\beta'\|_{L^\infty(B_\delta \times \Omega)} \leq C_s, \|\gamma'\|_{L^\infty(B_\delta \times \Omega)} \leq C_t, \quad \mu'(\Omega) \leq C_s|n|^{-s-t}.
\]
Thus we obtain
\[
\left| \int \int \hat{b}(\xi + \theta - \eta)g(\xi)h(\eta)\,d\xi d\eta \right|
\]
\[
= \left| \int_{\Omega} \int \hat{b}(\xi + \theta - \eta)A(\xi + n + \theta, \eta + n)|\xi + n + \theta|^{\nu} g(\xi)h(\eta)\times
\]
\[
\beta'(\xi, \omega)h(\eta)\,d\xi d\eta d\mu'(\omega) + \gamma'(\eta, \omega)d\xi d\eta d\mu'(\omega)
\]
\[ 
\begin{align*}
\leq & \int_{\Omega} d\mu'(\omega) \left| \int \int B(\xi - \eta) A(\xi, \eta) |\xi|^s |\eta|^{t} (\eta - \mathbf{n}, \omega)(g\beta')(\eta - \mathbf{n} - \theta, \omega) \, d\xi d\eta \right| \\
\leq & \mu'(\Omega) \sup_{\|\phi\|_2,\|\psi\|_2 \leq 1} |T_{b}^{u}\psi, \phi\rangle = \mu'(\Omega)a_{n} \leq C_{sr}|n|^{-s-t}a_{n}.
\end{align*}
\]

Since \( \theta \in \text{supp} \, B \) we can find \( g \) and \( h \) such that

\[ 
\left| \int \int B(\xi + \theta - \eta) g(\xi) h(\eta) d\xi d\eta \right| > 0,
\]

thus we get

\[ 
a_{n} \geq C|n|^{-a-s+t} \quad \text{for} \quad n \in V_{0} \quad \text{and} \quad |n| > 6.
\]

We claim that

\[ 
(38) \quad \|T_{b}^{u}\|_{S_{p}}^{p} \geq C \sum_{n \in V_{0}} a_{n}^{p}.
\]

Then, by (36),

\[ 
\|T_{b}^{u}\|_{S_{p}}^{p} \geq C \sum_{n \in V_{0}} |n|^{(-a+s+t)p} = \infty
\]

\[ 
\text{this contradicts } T_{b}^{u} \in S_{p}. \text{ This contradiction shows that } b \text{ must be a polynomial.}
\]

To show (38), we assume that \( \text{supp} \, B \subset \{ |\xi| \leq M - 2 \} \), where \( M > 2 \) is a positive integer.

Let \( V_{r} = \{ n \in V_{0} : n = Mk + r \} \) for \( r \in \{ 0, 1, \ldots, M - 1 \} \), let \( P_{r}^{(k)} \) denote the projection from \( L^{2}(\mathbb{R}^{d}) \) to \( L^{2}(B(Mk + r, \delta)) \), let \( Q_{r}^{(k)} \) denote the projection from \( L^{2}(\mathbb{R}^{d}) \) to \( L^{2}(B(Mk + r + \theta, \delta)) \), and write

\[ 
P_{r} = \sum_{Mk + r \in V_{0}} P_{r}^{(k)},
\]

\[ 
Q_{r} = \sum_{Mk + r \in V_{0}} Q_{r}^{(k)}.
\]

Then we have

\[ 
\sum_{r \in \{ 0, \ldots, M - 1 \}^{d}} \|Q_{r} T_{b}^{u} P_{r}\|_{S_{p}}^{p} \leq M^{d} \|T_{b}^{u}\|_{S_{p}}^{p}.
\]
We note that

\[(39) \quad Q_r T_b^{st} P_r = \sum_{Mk + r \in V_0} Q_r^{(k)} T_b^{st} P_r^{(k)},\]

because when \( k \neq j \), \( Mk + r \in V_0 \) and \( Mj + r \in V_0 \),

\[
(Q_r^{(k)} T_b^{st} P_r^{(j)} f) \hat{\xi} = \int \chi_{B(Mk + r + \theta, \delta)}(\xi - \eta) A(\xi, \eta) |\xi|^p |\eta|^q \chi_{B(Mj + r, \delta)}(\eta) \hat{f}(\eta) d\eta.
\]

Also, if \( \xi \in B(Mk + r + \theta, \delta) \), \( \eta \in B(Mj + r, \delta) \), then

\[
|\xi - \eta| = |\xi - (Mk + r + \theta) - \eta + (Mj + r) + \theta + M(k - j)| \\
\leq M|k - j| - |\theta| - 2\delta \\
> M - 2.
\]

Thus, since \( \text{supp} \delta \subset \{|\xi| \leq M - 2\} \), we have

\[
Q_r^{(k)} T_b^{st} P_r^{(j)} = 0.
\]

Therefore, by Lemma 2, we have

\[
||Q_r T_b^{st} P_r||_{S_p}^p = \sum_{Mk + r \in V_0} ||Q_r^{(k)} T_b^{st} P_r^{(k)}||_{S_p}^p \\
= \sum_{Mk + r \in V_0} ||\delta(\xi - \eta) A(\xi, \eta) |\xi|^p |\eta|^q \chi_{B(Mk + r + \theta, \delta)}(\xi) \chi_{B(Mk + r, \delta)}(\eta)||_{S_p}^p \\
\geq \sum_{Mk + r \in V_0} ||\delta(\xi - \eta) A(\xi, \eta) |\xi|^p |\eta|^q||_{S_n(B(Mk + r + \theta, \delta) \times B(Mk + r, \delta))}^p \\
= \sum_{Mk + r \in V_0} a_{MK + R}^p.
\]

Finally, we get
\[ \| T^w_b \|_{S_p}^c \geq N^{-d} \sum_{r \in \{0, \ldots, M-1\}^d} \sum_{M^k + r \in V_\alpha} a^w_{M^k + r} = M^{-d} \sum_{n \in V_\alpha} a^w_n, \]
i.e. (38) holds.

This completes the proof of Theorem 4.

**Corollary 3.** Suppose \( A(\xi, \eta) \) satisfies A10(\( \alpha \)), \( 1 \leq p \leq d/(\alpha - s - t) \) and \( b \in S'(\mathbb{R}^d) \) such that \( T^w_b \in S_p \). Then \( b \) must be a polynomial.

**Proof.** When \( p \geq 1 \), (38) always holds. Noting that the argument in the proof of Theorem 4 up to (38) does not need the assumption that \( b \) is such that \( \hat{b} \) has compact support, it follows that Corollary 3 holds.

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**References**