TWO-SHEETED COVERINGS OF THE DISC

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1. Introduction.

Let $X$ be a Riemann surface in which there is a holomorphic map $\varphi$ to the open unit disc $D$ with $\varphi$ of constant valence 2. One says that $X$, or the pair $(X, \varphi)$, is a two-sheeted covering of $D$. Let $N$ be the number of branch points of $\varphi$. Then $1 \leq N \leq \infty$. We will prove that if $N > 2$ and if $\varphi_1$, like $\varphi$, is a holomorphic map of $X$ to $D$ with $\varphi_1$ of valence 2, then $\varphi_1 = A(\varphi)$, where $A$ is an automorphism of $D$. This means in part that $\varphi_1$ and $\varphi$ have the same branch points, and thus one may speak without ambiguity of the branch points of $X$. (This is not possible if $1 \leq N \leq 2$. Then $X$ is, conformally, an annulus or the disc, and one finds that $\varphi_1 = A(\varphi(B))$ with $B \in \text{Aut } X$.) It is a corollary of "$\varphi_1 = A(\varphi)$" (or for that matter of "$\varphi_1 = A(\varphi(B))$") that if $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ are two-sheeted coverings of $D$, if

$$\Delta_k = \{\varphi_k(x) : x \text{ is a branch point of } \varphi_k\},$$

and if $X_1$ and $X_2$ are conformally equivalent, then $\Delta_2 = A(\Delta_1)$ with $A \in \text{Aut } D$. This should be new if $N$ is infinite. (The converse holds, i.e., $X_1$ and $X_2$ are conformally equivalent if $\Delta_2 = A(\Delta_1)$ with $A \in \text{Aut } D$, but I would guess this is not new.)

The theorem that gives $\varphi_1 = A(\varphi)$ if $N > 2$ is in part 2. If $N = \infty$, the theorem gives more. Then it identifies the holomorphic maps, of $X$ to $D$, of constant, finite valence, not just those of valence 2. Then in part 3 we use the theorem to study the proper holomorphic maps of $X$ to itself. E.g., we will prove that if $f \in \text{Prop } X$ but $\notin \text{Aut } X$, and if $f$ fixes a point, then the point is a branch point of $X$.

2. The main theorem.

A mapping $\varphi : X \to Y$ of topological spaces $X$ and $Y$ is said to be proper if inverses of bounded sets are bounded, i.e., if $\{\varphi \in E\}$ is bounded in $X$

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whenever $E$ is bounded in $Y$. A set is bounded if it is contained in a compact set.

Let $X$ be a Riemann surface in which there is a holomorphic map $\varphi$ to the open unit disc $D$ with $\varphi$ of constant, finite valence $m$. That is, $|\varphi| < 1$, while if $|\xi| < 1$, then the set where $\varphi = \xi$ consists of $m$ points counting multiplicities. In other words, if $\partial(F, w)$ is the order of vanishing of $F$ at $w$, then

\[
\sum_{\varphi(x) = \xi} (1 + \partial(\varphi', x)) = m.
\]

One says that $X$, or the pair $(X, \varphi)$, is an $m$-sheeted covering of $D$. Let $N$ be the number of points, in $X$, of branch order $m - 1$. Then $0 \leq N \leq \infty$. (The branch order of $x$ is $\partial(\varphi', x)$.)

Let $\varphi_1$, like $\varphi$, be a proper holomorphic map of $X$ to $D$. Put $m_1 = \varphi_1$ the number of times $\varphi_1$ vanishes counting multiplicities. Then $m_1 < \infty$.

**Theorem 1.** If $m_1 < N$, then

(i) $m$ divides $m_1$;
(ii) $\varphi_1 = g(\varphi)$, where $g$ is a proper holomorphic map of $D$ to itself of valence $m_1/m$, in other words, $g$ is a finite Blaschke product that vanishes $m_1/m$ times counting multiplicities.

Thus if the number of points of branch order $m - 1$, namely $N$, is infinite, then every proper holomorphic map of $X$ to $D$ is of valence zero mod $m$, and is obtained from $\varphi$ by composing with a finite Blaschke product.

2.1. **The proof of Theorem 1.** Our proof is elementary. Its main ingredient is an old lemma. If $f \in \mathcal{O}(D)$, let $N(f)$ be the number of times $f$ vanishes in $D$, counting multiplicities. Then the lemma is this:

**Lemma 1.** Let $f$ be a finite Blaschke product. Let $A$ be holomorphic in $D$ and bounded by 1 there. If $N(A - f) > N(f)$, then $A = f$.

**Proof.** (i) Let $N(f) = 0$. Then $f$ is a constant of modulus one, hence by the principle of maximum, $A = f$.

(ii) Let $N(f) > 0$. Because $N(A - f)$ is positive, there is, like in (i), a point $\xi$ in $D$ with $A(\xi) = f(\xi)$, but here $|f(\xi)| < 1$. Let $\zeta = f(\xi)$, and put

\[
A_1 = \frac{\zeta - A}{1 - \zeta A}, \quad f_1 = \frac{\zeta - f}{1 - \zeta f}, \quad \text{and} \quad w = \frac{\zeta - z}{1 - \zeta z}.
\]

Then $A_1 = wA_2$, $f_1 = wf_2$, and

\[
(1 - \zeta A)(1 - \zeta f)(A_2 - f_2)w = (|\xi|^2 - 1)(A - f).
\]
The identity gives
\[ 1 + N(A_2 - f_2) > 1 + N(f_2) \]
because
\[ N(A - f) > N(f) = N(f_1) = 1 + N(f_2). \]

By the induction hypothesis, \( A_2 = f_2 \), which means \( A = f \).

**The proof of Theorem 1.** If \( |ξ| < 1 \), put
\[ f(ξ) = \prod_{φ(x) = ξ} φ_1(x)^{1 + δ(φ', x)} \]
and
\[ A(ξ) = \left( \frac{1}{m} \sum_{φ(x) = ξ} (1 + δ(φ', x))φ_1(x) \right)^m. \]

It is plain that \( f \) and \( A \) are holomorphic in \( D \), with \( |f| < 1 \) and \( |A| < 1 \) there. (Less briefly: To each point \( x \) there is an open disc \( V \) of center \( x \) and a (cyclic) group \( G \), of \( 1 + δ(φ', x) \) automorphisms of \( V \) that fix \( x \), such that the orbits of points are the fibers of \( φ|V \). In other words, if \( y ∈ V \), then the set of points \( σ(y) \), \( σ ∈ G \), is the set in \( V \) where \( φ = φ(y) \). We may identify \( O(φ(V)) \) with \( \{ g(φ) : g ∈ O(φ(V)) \} \); then \( O(V) \) is an overring of \( O(φ(V)) \). The group \( G \) serves to tell who is in \( O(φ(V)) \). The test is this: let \( g ∈ O(V) \); then \( g ∈ O(φ(V)) \) iff \( g(σ) = g \) for every \( σ \) in \( G \). Put
\[ θ = \prod_{σ ∈ G} φ_1(σ); \]
then by the test, \( θ ∈ O(φ(V)) \), which means \( θ = g(φ) \) with \( g \) holomorphic in \( φ(V) \). Let \( |ξ| < 1 \). If \( f_ξ^* \) is the germ of \( f \) at \( ξ \), and \( g_x \) the germ of \( g \) at \( φ(x) \), then by (2.1),
\[ f_ξ^* = \prod_{φ(x) = ξ} g_x. \]

This proves that \( f_ξ^* ∈ O_ξ \), which means \( f ∈ O(D) \). Likewise, \( A ∈ O(D) \). And it is plain that \( f \) is proper, because if \( m \) sequences in the disc \( D \) converge to the boundary, then their termwise product does too. Let \( l \) be the number of times \( f \) vanishes in \( D \). Then it is plain, once more, that \( l = m_1 \), which is to say that \( f \), like \( φ_1 \), vanishes \( m_1 \) times. (Less briefly: Let \( |ξ| < 1 \). Then by (2.2),
\[ \partial(f, \xi) = \sum_{\varphi(x) = \xi} \partial(g_x, \xi), \]

while by the corollary to Lemma 7 (infra),

\[
\partial(g, \varphi(x))(1 + \partial(\varphi', x)) = \partial(g(\varphi), x) = \partial(\theta, x) \\
= \sum_{\sigma \in \mathcal{G}} \partial(\varphi_1(\sigma), x) = \sum_{\sigma \in \mathcal{G}} \partial(\varphi_1(\sigma(x))(1 + \partial(\sigma', x)) \\
= \partial(\varphi_1, x)(1 + \partial(\varphi', x)),
\]

that is, \( \partial(g, \varphi(x)) = \partial(\varphi_1, x) \), hence by (2.3),

\[ \partial(f, \xi) = \sum_{\varphi(x) = \xi} \partial(\varphi_1, x). \]

Then

\[ l = \sum_{\xi \in D} \partial(f, \xi) = \sum_{\xi \in D} \sum_{\varphi(x) = \xi} \partial(\varphi_1, x) = \sum_{x \in X} \partial(\varphi_1, x) = m. \]

Let \( x \in X \). If the branch order of \( x \) is \( m - 1 \), then

\[ A(\varphi(x)) = \varphi_1(x)^m = f(\varphi(x)). \]

This means that

(2.4) \[ N(A - f) \geq N \]

if the left side is the number of times \( A - f \) vanishes in \( D \). The inequality (2.4), plus the hypothesis \( N > m_1 \), implies that \( N(A - f) > N(f) \). Then by the lemma, \( A = f \).

We may identify \( \mathcal{O}(D) \) with \( \{ g(\varphi) : g \in \mathcal{O}(D) \} \); then, in words used before, \( \mathcal{O}(X) \) is an overring of \( \mathcal{O}(D) \). It is to be proved that \( \varphi_1 \in \mathcal{O}(D) \). If \( x \in X \), let \( x_1, \ldots, x_m \) be the points where \( \varphi = \varphi(x) \). Because there are \( m \) points in the list, it is understood that \( x \) is in the list \( 1 + \partial(\varphi', x) \) times. Put \( w_k = \varphi_1(x_k) \).

Then we have proved that

(2.5) \[ \left( \frac{1}{m} \sum_{k=1}^{m} w_k \right)^m = \prod_{k=1}^{m} w_k. \]

If \( m = 2 \), the identity (2.5) implies that \( w_1 = w_2 \), which means \( \varphi_1 \in \mathcal{O}(D) \). If \( m > 2 \), the identity (2.5) does not imply that the \( w_k \) agree.

Let \(-1 < t < 1\). Then \((t - \varphi_1)/(1 - t\varphi_1)\), like \( \varphi_1 \), is a proper holomorphic
map of $X$ to $D$; it vanishes $m_1$ times because $\varphi_1$, being holomorphic and proper, is of constant valence. Hence by "$A = f$",

\begin{equation}
\left( \frac{1}{m} \sum_{k=1}^{m} \frac{t-w_k}{1-tw_k} \right)^m = \prod_{k=1}^{m} \frac{t-w_k}{1-tw_k}.
\end{equation}

Then (2.6) holds everywhere in $t$ because both sides are rational. Let $n$ be the number of $w_k$ that equal $w_1$. Then the right side of (2.6) has a pole of order $n$ at $1/w_1$, while the left side has a pole of order $m$ there. Thus $n = m$, which means, once more, that $\varphi_1 \in \mathcal{O}(D)$.

We have proved that $\varphi_1 = g(\varphi)$ with $g$ holomorphic in $D$; $|g| < 1$ there because $\varphi(X) = D$. Then $g$ is proper.

(Proof. Let $E$, contained in $D$, be bounded in $D$. Put $F = \{ g \in E \}$ and $G = \{ \varphi \in F \}$. Then $G = \{ \varphi_1 \in E \}$, hence $G$ is bounded in $X$, hence $\varphi(G)$, i.e. $F$, is bounded in $D$.)

Finally, if $k$ is the number of times $g$ vanishes in $D$, then by (2.1), $m_1 = km$. (Less briefly: By (2.1),

\[
km = \sum_{\xi \in D} \partial(g, \xi) \sum_{\varphi(x) = \xi} (1 + \partial(\varphi', x)) = \sum_{x \in X} \partial(g, \varphi(x))(1 + \partial(\varphi', x)),
\]

while by the corollary to Lemma 7 (infra),

\[
\text{the last sum} = \sum_{x \in X} \partial(g(\varphi), x) = m_1.
\]

2.2. How good is the theorem, in other words, can one say more if the number of points of branch order $m - 1$ is finite? I think it is fair to say no.

Let $m = 2$, let $N < \infty$, put

\[
A = \{ \varphi(x) : x \text{ is a branch point of } \varphi \},
\]

and let $\gamma$ be "the" finite Blaschke product that vanishes to order one everywhere in $A$. The number of points in $A$, like the number in $A_1$ ((2.7) infra), is $N$. By part 3 (infra), $\gamma(\varphi) = \theta^2$ with $\theta \in \mathcal{O}(X)$. Then $\theta$ is proper, but it is not a $g(\varphi)$. By (3.6'), it vanishes $N$ times.

Put

\begin{equation}
A_1 = \{ x \in X : x \text{ is a branch point of } \varphi \}.
\end{equation}

Because $m = 2$, the number of points in $A_1$ is $N$. Let $\gamma_1$ be holomorphic in $X$,
vanish to odd order infinitely often there, and vanish to odd order everywhere in $A_1$. (If we like, $\gamma_1 = B(\phi)\theta$ with $B$ a Blaschke product that vanishes to odd order infinitely often in $D$, but vanishes nowhere in $A$.) Let $(X_1, \varphi_1)$ be the Riemann surface of $\sqrt{\gamma_1}$, in other words, $(X_1, \varphi_1)$ is a 2-sheeted covering of $X$ with $\gamma_1(\varphi_1)$ a square in $\mathcal{O}(X_1)$. If $\varphi_2 = \varphi(\varphi_1)$, then the pair $(X_1, \varphi_2)$ is a 4-sheeted covering of the disc $D$ with $N$ points of branch order 3, no points of branch order 2, and infinitely many points of branch order 1, while $\theta(\varphi_1)$ is a proper holomorphic map of $X_1$ to $D$ that is not a $g(\varphi_2)$ because $\theta$ is not a $g(\varphi)$. Because $\theta$ vanishes $N$ times, $\theta(\varphi_1)$ vanishes $2N$ times.

3. Proper holomorphic maps of $X$ to itself.

We return to part 1. Accordingly, the pair $(X, \varphi)$ is a two-sheeted covering of $D$ and $N$ is the number of branch points of $\varphi$. Let Prop $X$ be the semigroup of proper holomorphic maps of $X$ to itself. Then Prop $X$ contains Aut $X$. If $3 \leq N < \infty$, then by Riemann-Hurwitz, Prop $X = \text{Aut} X$. (If $N = 2$, then by an ad hoc proof, Prop $X = \text{Aut} X$.) What if $N$ is infinite? Then the inclusion can be proper, and it is this we study here. (If $N = 1$, $X \equiv D$, hence the inclusion is proper.)

Let $N = \infty$. Then by putting $\varphi_1 = \varphi(f)$ if $f \in \text{Prop} X$, we may use Theorem 1 to study Prop $X$. E.g., we find that if $H^\infty(X)$ separates points in $X$, if $f \in \text{Prop} X$, and if $f$ fixes a point, then $f \in \text{Aut} X$. (This is Theorem 4.) Put

$$
\Delta = \{\varphi(x) : x \text{ is a branch point of } \varphi\},
$$

and let $T = \Delta'$ (the derived set of $\Delta$). Then $\Delta$ is a discrete set in the disc $D$, while $T$ is a closed set in the circle $\partial D$. Both are nonempty ($\Delta$ is infinite, but $T$ may be finite).

Let $f \in \text{Prop} X$; then by Theorem 1, $\varphi(f) = \hat{f}(\varphi)$ with $\hat{f} \in \text{Prop} D$. Which finite Blaschke products are $\hat{f}$'s? The test is this: Let $g$ be a finite Blaschke product. Then the following imply one another:

(i) $g$ is an $\hat{f}$.

(ii) Let $\zeta \in D$. Then $\zeta \in \Delta$ iff $g(\zeta) \in \Delta$ and $g'$ vanishes to even order at $\zeta$. (If $g'(\zeta) \neq 0$, the order of vanishing is zero, which is even.)

We begin by proving the "$g = \hat{f}$ test". The proof is lengthy.

To $\varphi$ corresponds a period 2 automorphism of $X$, called $\sigma$. The proof and precise statement is this: Let $x \in X$. Then the set where $\varphi = \varphi(x)$ consists of $x$ plus one other point, say $y$. It is understood that $y = x$ if $x$ is a branch point of $\varphi$. Put $\sigma(x) = y$. Then $\varphi(\sigma) = \varphi$ and $\sigma(\varphi(x)) = x$. (Iota = the identity map.) To each point there is a disc in which $\sigma$ is either the identity or minus
the identity, which means $\sigma$ is holomorphic. (The first alternative holds if the point is not a branch point, the second if it is.) Then $\sigma \in \text{Aut } X$.

We have

$$\sigma(x) = x \text{ iff } x \text{ is a branch point of } \varphi.$$  \hspace{1cm} (3.1)

**Lemma 2.** Let $f \in \text{Prop } X$. Then $f(\sigma) = \sigma(f)$. In words, the proper maps of $X$ to itself commute with the period 2 automorphism that corresponds to $\varphi$.

**Proof.** By Theorem 1,

$$\varphi(f(\sigma)) = \hat{f}(\varphi(\sigma)) = \hat{f}(\varphi) = \varphi(f),$$

hence either $f(\sigma) = \sigma(f)$, which is to be proved, or $f(\sigma) = f$.

Let $f(\sigma) = f$. Then $f = g(\varphi)$ with $g : D \to X$. This gives $\hat{f} = \varphi(g)$ because $\hat{f}(\varphi) = \varphi(f)$, hence $N(\hat{f}')$ is infinite. $N(\hat{f}')$ is the number of times $\hat{f}'$ vanishes in $D$, counting multiplicities.) But $N(\hat{f}') = k - 1$ if $k$ is the valence of $f$.

We identify, once more, $\mathcal{O}(D)$ with $\{g(\varphi) : g \in \mathcal{O}(D)\}$; then, in words used twice before, $\mathcal{O}(X)$ is an overring of $\mathcal{O}(D)$. The automorphism $\sigma$ serves to tell who is in $\mathcal{O}(D)$. The test is this: let $f \in \mathcal{O}(X)$; then $f \in \mathcal{O}(D)$ iff $f(\sigma) = f$.

Let $\theta_1 \in \mathcal{O}(X)$ but $\not{\in} \mathcal{O}(D)$, put $\theta_2 = \theta_1 - \theta_1(\sigma)$, and put $\gamma_1 = \theta_2^2$. Then $\gamma_1$ is holomorphic in $D$. Because $\gamma_1$ does not vanish everywhere there, its order of vanishing, pointwise, is even or odd. The alternative is that the order of vanishing is everywhere infinite. Let $\gamma_2 \in \mathcal{O}(D)$ with $\partial(\gamma_2, z) = k$ if $\partial(\gamma_1, z) = 2k$ or $2k+1$. Then $\gamma_1 = \gamma_2^2 \gamma$ with $\gamma$ holomorphic in $D$ and all zeros of $\gamma$, if any, simple. Put

$$\theta = \theta_2/\gamma_2;$$  \hspace{1cm} (3.2)

then $\theta^2 = \gamma$, hence $\theta \in \mathcal{O}(X)$. By (3.2),

$$\theta(\sigma) = -\theta.$$  \hspace{1cm} (3.3)

**Lemma 3.** The pair $(1, \theta)$ is a basis of the $\mathcal{O}(D)$-module $\mathcal{O}(X)$.

**Proof.** Let $g \in \mathcal{O}(X)$, and put

$$A = (g + g(\sigma))/2, \quad B = (g - g(\sigma))/2\theta.$$ Then $g = A + B\theta$. It is to be proved that $B$, which is meromorphic in $D$, is, like $A$, holomorphic there. But $B^2\gamma$ is holomorphic, hence $B$ is too because the zeros of $\gamma$ are simple.
We have proved that the pair \((1, \theta)\) generates the \(\mathcal{O}(D)\)-module \(\mathcal{O}(X)\). On the other hand if \(A + B\theta = 0\), with \(A\) and \(B\) in \(\mathcal{O}(D)\), then by (3.3), \(A - B\theta = 0\), hence \(A = B = 0\). This means the pair \((1, \theta)\) is independent over \(\mathcal{O}(D)\).

**Corollary 1.** \(\theta\) separates the points of fibers of \(\varphi\).

**Proof.** \(\mathcal{O}(X)\) separates the points of \(X\).

**Lemma 4.** The set where \(\theta\) vanishes is the set of branch points of \(\varphi\), in symbols.

\[
\{ \theta = 0 \} = \{ x \in X : \sigma(x) = x \}
\]

by (3.1). Equivalently, \(\Delta\) is the set where \(\gamma\) vanishes.

**Proof.** By (3.3), the left side of (3.4) contains the right side.

Let \(\theta(x) = 0\). Then by (3.3) once more, \(\theta(\sigma(x)) = 0\), hence by Corollary 1, \(\sigma(x) = x\). This proves that the right side of (3.4) contains the left side. (Alternatively, let \(x \in X\). Then because \(\theta^2 = \gamma(\varphi)\),

\[
2\vartheta(\theta, x) = \vartheta(\gamma, \varphi(x))(1 + \vartheta(\varphi', x))
\]

by the corollary to Lemma 7 (infra). Thus if \(\theta(x)\) vanishes, then \(\vartheta(\varphi', x)\) is odd, which means \(x\) is a branch point of \(\varphi\).

**Lemma 5.** Let \(g\) be a finite Blaschke product. If \(g\) is an \(\tilde{f}\), then \(\gamma(g) = B^2\gamma\) with \(B \in \mathcal{O}(D)\).

**Proof.** By Lemma 2, \(\theta(f(\sigma)) = \theta(\sigma(f)) = -\theta(f)\), hence by Lemma 3, \(\theta(f) = B\theta\). Then

\[
\gamma(g(\varphi)) = (B^2 \gamma)(\varphi)
\]

because the left side \(= \gamma(\varphi(f)) = \theta^2(f)\).

The converse is true too: \(g\) is an \(\tilde{f}\) if \(\gamma(g) = B^2\gamma\). This is Lemma 6 (infra).

Let \(x \in X\). Then by (3.5),

\[
(3.6') \quad \theta'(x) \neq 0 \quad \text{if } x \text{ is a branch point of } \varphi,
\]

while of course

\[
(3.6'') \quad \varphi'(x) \neq 0 \quad \text{if } x \text{ is not a branch point of } \varphi.
\]

**Lemma 6.** Let \(g \in \text{Prop } D\), and let

\[
\gamma(g) = B^2\gamma \quad \text{with } B \in \mathcal{O}(D).
\]

Then \(g(\varphi) = \varphi(f)\) with \(f \in \text{Prop } X\), that is, \(g\) is an \(\tilde{f}\).
Proof. Let $x \in X$. Then, counting multiplicities, the set where $\varphi = g(\varphi(x))$ consists of two points, $y_1$ and $y_2$ say. By (3.7), $\theta(y_1) = B(x)\theta(x)$ or $\theta(y_1) = -B(x)\theta(x)$. If the first alternative holds, put $f(x) = y_1$, if the second holds, put $f(x) = y_2$. (If both hold, $y_1 = y_2$.) Then

$$
\varphi(f) = g(\varphi) \quad \text{and} \quad \theta(f) = B\theta.
$$

By (3.6) and (3.8), $f$ is holomorphic because it is continuous, while by the first identity in (3.8), $f$ is proper.

**Lemma 7.** Let $f$ and $g$ be formal power series with $f(0) = 0$ but $f \neq 0$. If $\partial G$ is the order of vanishing of the formal power series $G$, then

$$
\partial(g(f)) = (\partial g)(1 + \partial f').
$$

**Proof.** Without loss of generality $g \neq 0$. If

$$
g = g_1 t^l + \ldots, \quad g_i \neq 0,
$$

and

$$
f = f_k s^k + \ldots, \quad f_k \neq 0, \quad k > 0,
$$

then

$$
g(f) = g_l(f_k s^k + \ldots)^l + \ldots = g_l f_k s^{kl} + \ldots
$$

and

$$
f' = k f_k s^{k-1} + \ldots,
$$

hence

$$
\partial(g(f)) = kl = (1 + k - 1)l = (1 + \partial f') \partial g.
$$

**Corollary.** Let $X_1$ and $X_2$ be Riemann surfaces, let $f$ be holomorphic in $X_1$, with values in $X_2$, but not a constant, and let $g$ be holomorphic in $X_2$. If $x \in X_1$, and if $\partial(F, w)$ is the order of vanishing of $F$ at $w$, then

$$
\partial(g(f), x) = \partial(g, f(x))(1 + \partial(f', x)).
$$

We now come to the proof of the "$g = \tilde{f}$ test". Accordingly, $g$ is a finite Blaschke product.

Let (i) hold. By Lemma 5,

$$
\gamma(g) = B^2 \gamma, \quad B \in \mathcal{O}(D).
$$
Then

(3.10) \[ \gamma'(g)g' = 2BB'\gamma + B^2\gamma'. \]

A. Let \( \xi \in \Delta \). Then by (3.9) and Lemma 4, \( g(\xi) \in \Delta \). Because \( \delta(\gamma, \xi) = 1 \),

\[ \gamma(\xi + t) = \gamma_1 t + \ldots, \quad \gamma_1 \neq 0, \]

while

\[ B(\xi + t) = B_p t^p + \ldots, \quad B_p \neq 0. \]

These give

\[ 2BB'\gamma + B^2\gamma' = (2p + 1)B_p^2t^2p + \ldots, \]

hence by (3.10),

\[ 2p = \delta(\gamma'(g)g', \xi) = \delta(\gamma'(g), \xi) + \delta(g', \xi) = \delta(g', \xi) \]

because \( \gamma'(g(\xi)) \neq 0 \). In words, \( g' \) vanishes to even order at \( \xi \).

B. Let \( g(\xi) \in \Delta \) and let \( \delta(g', \xi) = 2l \). Then

\[ \gamma'(g(\xi + t)) = A_0 + \ldots, \quad A_0 \neq 0, \]

and

\[ g'(\xi + t) = G_{2l}t^{2l} + \ldots, \quad G_{2l} \neq 0, \]

while

\[ B(\xi + t) = B_p t^p + \ldots, \quad B_p \neq 0, \]

and

\[ \gamma(\xi + t) = \gamma_k t^k + \ldots, \quad \gamma_k \neq 0. \]

These plus (3.10) give

\[ A_0 G_{2l}t^{2l} + \ldots = (2p + k)B_p^2 \gamma_k t^{2p+k-1} + \ldots. \]
If \( k = 0 \), that is, if \( \xi \notin \Delta \), then by (3.9), \( p > 0 \), hence \( 2l = 2p - 1 \). This proves \( \xi \in \Delta \).

We have proved that (i) gives (ii).

Let (ii) hold. Then \( \gamma(g) = A\gamma \) with \( A \in \mathcal{O}(D) \). It is to be proved that \( A \) is a square.

Let \( A(\xi) = 0 \). Then \( g(\xi) \in \Delta \), hence

\[
1 + \partial(g', \xi) = \partial(\gamma, g(\xi))(1 + \partial(g', \xi)) = \partial(\gamma(g), \xi)
\]

by the corollary to Lemma 7.

A. Let \( \xi \in \Delta \). Then \( \partial(g', \xi) = 2l \). By (3.11),

\[
1 + 2l = \partial(\gamma(g), \xi) = \partial(A, \xi) + \partial(\gamma, \xi) = \partial(A, \xi) + 1
\]

or \( 2l = \partial(A, \xi) \).

B. Let \( \xi \notin \Delta \). Then \( \partial(g', \xi) = 1 + 2l \). By (3.11) once more,

\[
2 + 2l = \partial(\gamma(g), \xi) = \partial(A, \xi) + \partial(\gamma, \xi) = \partial(A, \xi).
\]

We have proved that \( A \) vanishes to odd order nowhere in \( D \). This means \( A \) is a square, in other words, \( \gamma(g) = B^2\gamma \) with \( B \in \mathcal{O}(D) \). Thus by Lemma 6, (ii) gives (i).

**Corollary 2.** Let \( g \) be a finite Blaschke product. If \( g \) is an \( \hat{f} \), then \( T \), the derived set of \( \Delta \), is completely \( g \)-invariant. In symbols,

\[
\text{(3.12)} \quad T = \{ g \in T \}.
\]

**Proof.** If \( k \) is the valence of \( g \), then by the \( g = \hat{f} \) test:

\[
\text{(3.13)} \quad \text{there are at most } k - 1 \text{ points in } \{ g \in \Delta \} \setminus \Delta.
\]

Let \( g(\xi) \in T \) with \( \xi \notin T \). Then there are infinitely many points in \( \{ g \in \Delta \} \setminus \Delta \), but this contradicts (3.13). This proves that the left side of (3.12) contains the right.

Let \( \xi \in \Delta \). Then by the \( g = \hat{f} \) test, \( g(\xi) \in \Delta \). Thus \( T \) contains \( g(T) \), in other words, the right side of (3.12) contains the left.

3.1. **Four theorems on Prop X.** Let \( g \in \text{Prop} \ D \). This, i.e., \( g \) is a finite Blaschke product that is not a constant, is a standing hypothesis. Let \( E \subset T \).
Then by $E \cong T$ we mean that $g$, if restricted to $E$, is a homeomorphism of $E$ with $T$. This amounts to saying $E$ is closed, $g$ is univalent in $E$, and $g(E) = T$. Let $k$ be the valence of $g$.

**Corollary 3.** Let $g$ be an $\hat{f}$, and let $T \neq \partial D$. Then

$$T = \bigcup_{l=1}^{k} T_l$$

with the $T_l$ disjoint and with $T_l \cong T$. This means in part that $T$ is the union of $k$ disjoint copies of itself.

**Proof.** Let $\xi \in \partial D \setminus T$. Here we use $T \neq \partial D$. The circle $\partial D$ is the union of $k$ nonoverlapping arcs whose endpoints, $z_1, \ldots, z_k$, satisfy $g(z) = \xi$. If $A_l$ is the $l$th arc, let $T_l = A_l \cap T$. Then $T_l$ is closed, $g$ is univalent in $T_l$ because, by Corollary 2, the endpoints of $A_l$ are not in $T$, and, by Corollary 2 once more, $g(T_l) = T$. In brief, $T_l \cong T$, while the $T_l$, whose union is $T$, are disjoint because, once more, the endpoints of the $A_l$ are not in $T$.

If $l$ is a positive integer, let $g_l$ be the $l$th iterate of $g$. In symbols,

$$g_1 = g, \ g_2 = g_1(g), \ldots, g_l = g_{l-1}(g).$$

If $l = 0$, $g = i$. Then $g_l$, like $g$, is a proper holomorphic map of $D$, but its valence is $k^l$.

**Corollary 4.** Let $g$ be an $\hat{f}$, let $T \neq \partial D$, and let $l$ be a positive integer. Then $T$ is the union of $k^l$ disjoint copies of itself.

**Proof.** $g_l$ is an $\hat{f}$.

Here is the first theorem on Prop $X$.

**Theorem 2.** Let $T \neq \partial D$. If $T$ is not the union of large numbers of disjoint copies of itself, e.g., if the number of components of $T$ is finite, or if the number of isolated points of $T$ is finite, then Prop $X = \text{Aut } X$.

**Proof.** Let $f \in \text{Prop } X$. By Corollary 4, $\hat{f}$ is univalent, which means $f$ is too.

Put $\theta(t) = e^{it}g'(e^{it})/g(e^{it})$ if $-\infty < t < \infty$, and let $i\theta(0) = \log g(1)$. Then

$$g(e^{it}) = e^{i\theta(0)}.$$  \hspace{1cm} (3.14)

If $\zeta_1, \ldots, \zeta_k$ are the zeros of $g$, i.e., if

$$g(z) = e^{it} \prod_{m=1}^{k} \frac{\zeta_m - z}{1 - \overline{\zeta_m}z}.$$
then
\[(3.15) \quad \theta'(t) = \sum_{m=1}^{k} \frac{(1 - |\zeta_m|^2)}{|1 - \zeta_m e^{i\mu}|^2} \]

which means in part
\[(3.16) \quad \theta' > 0. \]

Put \( \mu = \min \theta' \). Then \( \mu > 0 \). Let \( |\xi| = 1 \). Then there are \( k \) points, \( t_1 < t_2 < \ldots < t_k \), in the interval \([0, 2\pi)\), that satisfy \( g(e^{it}) = \xi \). Put \( t_{k+1} = t_1 + 2\pi \). Because \( \mu \leq \theta' \),
\[(t_{l+1} - t_l) \mu < \theta(t_{l+1}) - \theta(t_l), \]
while by (3.14) and (3.16), the right side is \( 2\pi \). Thus
\[(3.17) \quad t_{l+1} - t_l < 2\pi / \mu \quad \text{if} \quad 1 \leq l \leq k. \]

**Lemma 8** (Fatou [1]). Let \( |\xi| = 1 \). If \( g \) is not univalent and if \( g \) fixes the origin, then the union, over positive integers \( l \), of the sets where \( g_l = \xi \) is dense in \( \partial D \).

**Proof.** We have \( g'_1 = g', \ g'_2 = g'_1(g)g', \ldots, g'_I = g'_{I-1}(g)g' \). Thus in \( \partial D \),
\[|g'_I| \geq \mu^I \quad \text{because} \quad |g'(e^{i\mu})| = \theta'(t). \]
Then by (3.17), the circle \( \partial D \) is the union of \( k^I \) (nonoverlapping) arcs of length \( < 2\pi / \mu^I \) whose endpoints, \( z_1, \ldots, z_n \), satisfy \( g_l(z) = \xi \), while by (3.15), \( \mu > 1 \) because \( k \geq 2 \) and \( g(0) = 0 \). This proves that the union in the lemma is dense in the circle.

**Corollary 5.** Let \( g \in \text{Prop } D \) but \( \not\in \text{Aut } D \). Let \( \xi \in \partial D \). If \( g \) fixes a point in \( D \), then
\[\bigcup_{l=1}^{\infty} \{g_l = \xi\} \]
is dense in \( \partial D \).

**Proof.** The \( g \) in the corollary is conjugate to a \( g \) that fixes the origin, namely, \( A(g(A)) \) if \( A \) is the period 2 automorphism of \( D \) that takes the origin to the point fixed by \( g \).

Here is our second theorem on \( \text{Prop } X \).

**Theorem 3.** Let \( T \neq \partial D \). Let \( f \in \text{Prop } X \). If \( f \) fixes a point, then \( f \in \text{Aut } X \).

**Proof.** If \( f \) fixes \( x \), then \( \hat{f} \) fixes \( \varphi(x) \). This implies, by Corollaries 2 and 5, that \( \hat{f} \) is univalent. Then \( f \) is too.
3.1.1. Our standing hypothesis is that $g \in \text{Prop } D$. Let $g \notin \text{Aut } D$, and let $g$ fix $p$, $p \in D$. Let $\zeta \in D$, and put

$$\Delta_\zeta = \bigcup_{l=0}^{\infty} \{ g_l = \zeta \}.$$ 

**Lemma 9.** If $g(z)$ is conjugate to a power of $z$, let $\zeta \neq p$. Then $\Delta_\zeta$ is infinite.

**Proof.** Let $k$ be a positive integer. If $\zeta \neq p$, and if $x$ is a point with $g_k(x) = \zeta$, then $x, g(x), g_2(x), \ldots, g_{k-1}(x), \zeta$ are distinct. E.g., if $g_3(x) = g_7(x) = g_4(g_3(x))$, then $g_3(x) = p$ because $g \notin \text{Aut } D$, hence $\zeta = g_{k-3}(p) = p$.

If $\zeta = p$, let $y$ be such that $y \neq \zeta$ while $g(y) = \zeta$. (There is such a point because otherwise $g(z)$ would be conjugate to a power of $z$.) Let $g_k(x) = y$. Then $x, g(x), g_2(x), \ldots, g_{k-1}(x), y$ are distinct.

**Lemma 10.** If $g(z)$ is conjugate to a power of $z$, let $\zeta \neq p$. Then $\Delta_\zeta$ does not satisfy the Blaschke condition, i.e.,

$$\sum_{z \in \Delta_\zeta} (1 - |z|) = \infty.$$ 

**Proof.** If $g'$ vanishes nowhere in $\Delta_\zeta$, let $\xi = \zeta$. Otherwise, let $x_1, \ldots, x_t$ be the points, in $\Delta_\zeta$, where $g'$ vanishes, let $m(1), \ldots, m(t)$ be integers such that $g_{m(1)}(x_1) = \ldots = g_{m(t)}(x_t) = \zeta$, and let $\xi \in \Delta_\zeta$ with $\xi \neq g_l(x_\xi)$ if $0 \leq l \leq m(s)$ and $1 \leq s \leq t$. There is such a point $\xi$ because, by Lemma 9, $\Delta_\zeta$ is infinite. Then

(i) $\Delta_\zeta$ contains $\Delta_\zeta$ because $\xi \in \Delta_\zeta$;

(ii) $g'$ vanishes nowhere in $\Delta_\zeta$.

To prove (ii), let $x_1 \in \Delta_\zeta$. Then $\xi = g_l(x_1)$ say, which means $\xi = g_{l-m(1)}(\xi)$ because $l > m(1)$. On the other hand, because $\xi \in \Delta_\zeta$, $\xi = g_k(\xi)$ say, hence

$$\xi = g_{l-m(1)+k}(\xi) \quad \text{and} \quad \xi = g_{l-m(1)+k}(\xi).$$

But $\xi \neq \xi$ while $l-m(1)+k > 0$. This proves (ii) because $g \notin \text{Aut } D$.

Let $\Delta_\zeta$ satisfy the Blaschke condition. Then there is a Blaschke product $B$ that vanishes to order one everywhere in $\Delta_\zeta$ while vanishing to order zero elsewhere. Put $\theta = B(g)$. Then $\theta$, like $B$, is a Blaschke product. Let $\theta(x) = 0$. Then $g(x) \in \Delta_\zeta$, hence $x \in \Delta_\zeta$, hence $\theta'(x) \neq 0$ because $\theta'(x) = B'(g(x))g'(x)$. In words, the zeros of $\theta$ are simple and each is a zero of $B$. Then $B/\theta \in \mathcal{O}(D)$, hence

$$|B/\theta| \leq 1$$

because $|B| < 1$. The inequality (3.18) means $|B| \leq |B(g)|$, hence $|B| \leq |B(g)|$. 
Then $|B| \leq |B(p)|$ because the iterates of $g$ converge in $D$ to the point fixed by $g$. This proves that $\Lambda_\xi$ does not satisfy the Blaschke condition. Then by (i), neither does $\Lambda_\zeta$.

**Corollary 6.** If $g$ is an $f$, then $\Lambda$ does not satisfy the Blaschke condition.

**Proof.** If $x \in \Lambda$, then by the $g = f$ test, each point of the sequence

$$g(x), g_2(x), \ldots, g_l(x), \ldots$$

is in $\Lambda$. This implies that $g_l(x) = p$ for some $l$ because $\Lambda$ is discrete in the disc while, once more, the iterates of $g$ converge there to the point fixed by $g$. In other words,

$$(3.19) \quad p \in \Lambda \quad \text{and} \quad \Lambda \subset \Lambda_p.$$  

The previous lemma means in part this: to prove the corollary it is enough to prove there is a point $\zeta$ such that $\Lambda$ contains $\Lambda_\zeta$. There are two possibilities. The first is that $g'$ vanishes to odd order nowhere in $\Lambda_p$, the second is $g'$ vanishes to odd order somewhere in $\Lambda_p$.

(i) $g'$ vanishes to odd order nowhere in $\Lambda_p$. Then $\Lambda = \Lambda_p$.

To prove this, let $x \in \Lambda_p$. Let $k$ be the first integer with $g_k(x) \in \Lambda$. If $k \geq 1$, then $g(g_{k-1}(x)) \in \Lambda$. But $g'$ vanishes to even order at the point $g_{k-1}(x)$ because this point, like $x$, is in $\Lambda_p$, hence by the "$g = f$ test", $g_{k-1}(x) \in \Lambda$. Thus $k = 0$, that is, $x \in \Lambda$.

(ii) $g'$ vanishes to odd order somewhere in $\Lambda_p$. Let $x_1, \ldots, x_t$ be the points, in $\Lambda_p$, where $g'$ vanishes to odd order, let $m$ be an integer such that $g_m(x_1) = \ldots = g_m(x_t) = p$, and let $\zeta \in \Lambda$ with $\zeta \neq g_l(x_s)$ if $0 \leq l \leq m$ and $1 \leq s \leq t$. Then $\Lambda$ contains $\Lambda_\zeta$.

To prove this, we first prove $\zeta \neq g_l(x_s)$ if $l \geq 0$ and $1 \leq s \leq t$. If $g_l(x_s) = \zeta$, then $l > m$, hence $\zeta = g_{l-m}(p) = p$. But $\zeta \neq p$.

Let $x \in \Lambda_\zeta$. Then $\zeta = g_l(x)$ say. Once more, let $k$ be the first integer with $g_k(x) \in \Lambda$. If $k \geq 1$, then by the "$g = f$ test", $g'$ vanishes to odd order at $g_{k-1}(x)$. Then $g_{k-1}(x) = x_1$ say, hence $\zeta = g_{l-k+1}(x_1)$ because $l \geq k - 1$. This proves $k = 0$.

One has, in the symbols of the proof of the corollary,

$$(3.20) \quad \Lambda = \Lambda_p \setminus \left( \bigcup_{s=1}^{t} \Lambda_{x_s} \right).$$

The identity means in part that $g$ determines $\Lambda$ if $g$ is an $f$. How can one exploit this, or for that matter, how can one exploit (3.20)?

We may paraphrase the corollary in this way.
Theorem 4. Let \( \Delta \) satisfy the Blaschke condition. Equivalently, let \( H^\infty(X) \) separate points in \( X \). Let \( f \in \text{Prop } X \). If \( f \) fixes a point, then \( f \in \text{Aut } X \).

Proof. By the hypothesis on \( f \), \( \hat{f} \) fixes a point in \( D \), hence by the hypothesis on \( \Delta \), \( f \) is univalent.

Here is the fourth theorem on \( \text{Prop } X \). We might have proved this at the outset.

Theorem 5. Let \( f \in \text{Prop } X \) but \( \notin \text{Aut } X \). Then \( f \) fixes at most one point. If \( f \) fixes a point, then the point is a branch point. (Neither of these need hold if \( f \in \text{Aut } X \).

Proof. Let \( f \) fix \( x \) and put \( p = \varphi(x) \). Then \( \hat{f} \) fixes \( p \), hence by (3.19), \( p \in \Delta \) because \( \hat{f} \notin \text{Aut } D \). This proves the second assertion of the theorem. For the first assertion, let \( f \) fix \( y \) and put \( q = \varphi(y) \). Then \( \hat{f} \) fixes \( q \) as well as \( p \), hence \( q = p \) because otherwise \( \hat{f}(z) \) would be \( z \) everywhere, but \( f \) is neither \( \iota \) nor \( \sigma \). That is, \( \varphi(y) = \varphi(x) \), hence by the second assertion, \( y = x \).

3.2. The theorems are pretty good. What we mean is this:

A. There is an \( X \), whose \( N \) is infinite, that has a proper holomorphic map of itself of valence 2 that fixes a point.

B. There is an \( X \), whose \( N \) is infinite, whose \( T \) is not the circle, and whose \( \Delta \) satisfies the Blaschke condition, that has a proper holomorphic map of itself of valence 2.

C. There is an \( X \), whose \( T \) is the circle, and whose \( \Delta \) does not satisfy the Blaschke condition, that has a proper holomorphic map of itself of valence 2 that fixes no point.

All three have a period 2 automorphism that fixes two points, neither point being a branch point, and a period 2 automorphism that fixes no point.

We will omit the proofs of the first two. Here is the proof of the third. We will work not in the disc \( D \), but in the right half plane \( H \). This is all right, because if \((X, \varphi)\) is a two-sheeted covering of \( H \), and if \( \varphi_1 \) is the Cayley transform, \((\varphi - 1)/(\varphi + 1)\), of \( \varphi \), then \((X, \varphi_1)\) is a two-sheeted covering of \( D \), and vice versa. Let

\[
g(z) = z + \frac{1}{z}.
\]

Then \( g \) maps \( H \) bivalently onto itself.
Lemma 11 (Fatou [1]). There is a \( \theta \) that is holomorphic in \( H \), that is not a constant, and that is \( g \)-invariant. In symbols,

\[
\theta \in \mathcal{C}(H), \quad \theta \notin \mathbb{C}, \quad \text{and} \quad \theta(g) = \theta.
\]

Proof. The key to \( \theta \) is this: iterate the square of \( g \). We begin though by iterating \( g \). Let

\[
z_0 = z, \quad z_{l+1} = z_l + \frac{1}{z_l}, \quad A_l = \frac{1}{z_l z_l}, \quad \text{and} \quad x_l + iy_l = z_l.
\]

Then

\[
(3.21) \quad x_{l+1} = x_l (1 + A_l),
\]

hence \( x_l \uparrow s \). It is understood that \( x_0 > 0 \). If \( s < \infty \), then by (3.21) once more, \( A_l \to 0 \), hence \( y_l \to t \) with \( -\infty < t < \infty \) because

\[
(3.21A) \quad y_{l+1} = y_l (1 - A_l).
\]

Then \( A_l \to |s + it|^{-2} \), but this is not 0. This proves \( s = \infty \). Thus

\[
(3.22) \quad \sum_{k=0}^{\infty} A_k = \infty
\]

because

\[
x_{l+1} = x_0 \prod_{k=0}^{l} (1 + A_k),
\]

while

\[
(3.23) \quad A_k \to 0
\]

because \( A_k \leq 1/x_k^2 \). By (3.22) and (3.23), \( y_l \to 0 \) because

\[
y_{l+1} = y_0 \prod_{k=0}^{l} (1 - A_k).
\]

Then

\[
(3.24) \quad y_l/x_l \to 0.
\]
By (3.21) and (3.21A),

(3.25) \[ |y_{l+1}/x_{l+1}| < |y_{l}/x_{l}|. \]

Let \( f(w) = w + 1/w + 2 \). Then \( f \) maps the slit plane \( \{z^2 : z \in H\} \) bivalently onto itself. One has \( g_l(z)^2 = f_l(z^2) \), but we do not use this if \( l \geq 2 \). Anyway, passing to \( f \) and its iterates amounts to squaring the iterates of \( g \).

Let

\[ w_0 = w, \quad w_{l+1} = w_l + \frac{1}{w_l} + 2, \quad B_l = \frac{1}{w_l \bar{w}_l}, \quad \text{and} \quad u_l + iv_l = w_l. \]

Then, like \( y_{l+1} \),

(3.26) \[ v_{l+1} = v_0 \prod_{k=0}^{l} (1 - B_k), \]

while, unlike \( x_{l+1} \),

(3.27) \[ u_{l+1} = u_l (1 + B_l) + 2. \]

Let \( u_0 > 0 \). Then by (3.27), \( u_l > 2l \), hence

(3.28) \[ B_l < 1/4l^2. \]

By (3.26) and (3.28), the harmonic functions \( v_l \) converge uniformly in the half plane \( u > 0 \), to \( V \) say. Then

\[ V = v_1 \prod_{k=1}^{\infty} (1 - B_k). \]

The infinite product is positive, hence the sign of \( V \) changes with that of \( v_1 \). This proves \( V \) is not a constant. We have \( v_{l+1} = v_l(f) \) because \( w_{l+1} = w_l(f) \), hence \( V = V(f) \). Thus if \( F \) is holomorphic in \( u > 0 \) with \( \text{im} F = V \) there, then \( F(f) = F + \gamma \) with \( \gamma \in \mathbb{R} \). In other symbols,

\[ F(w + 1/w + 2) = F(w) + \gamma. \]

Let \( H^{1/2} \) be the sector of opening \( \pi/2 \), i.e.,

\[ H^{1/2} = \{ x + iy : |y| < x \}, \]

and put \( G(z) = F(z^2) \) if \( z \in H^{1/2} \). By (3.25), \( z + 1/z \) is in \( H^{1/2} \) if \( z \) is. Then

\[ G(z + 1/z) = F(z^2 + 1/z^2 + 2) = F(z^2) + \gamma = G(z) + \gamma. \]
Thus in $H^{1/2}$,

\[(3.29) \quad G(g_l) = G + l\gamma \quad \text{if} \quad l = 0, 1, 2, \ldots .\]

We may combine (3.29) with (3.24) to continue $G$ to $H$. Put $X_l = \{g_l \in H^{1/2}\}$. Then in $X_k \cap X_l$, $G(g_k) - k\gamma = G(g_l) - l\gamma$, while by (3.24), $H$ is the union of the $X_l$. This continues $G$ to $H$ if in $X_l$ we let $G = G(g_l) - l\gamma$. Then $G \in \mathcal{O}(H)$ with $G(g) = G + \gamma$.

We now come to $\theta$. If $\gamma = 0$, put $\theta = G$, while if $\gamma \neq 0$, put $\theta = e^{2\pi i G/\gamma}$. Then $\theta \in \mathcal{O}(H)$ with $\theta(g) = \theta$. Finally, $\theta$ is not a constant since $V$ is not.

Fatou's proof, which is of greater subtlety, gives more. Namely, the value of $\gamma$, which is 2.

To $\xi$ in $H$, or for that matter in $P$, corresponds its $g$-orbit $A_\xi$. This is to say,

\[
A_\xi = \bigcup_{k,l=0}^{\infty} \{g_k = g_l(\xi)\}.
\]

(The $A_\xi$ in 3.1.1 is just a piece of the $A_\xi$ here.) Then

\[(3.30) \quad A_\xi = \{g \in A_\xi\},\]

in words used before, $A_\xi$ is completely $g$-invariant. It is the least set containing $\xi$ that is.

**Corollary 7.** Let $\xi \in H$. Then $A_\xi$ is discrete in $H$.

**Proof.** The set where $\theta = \theta(\xi)$, which is discrete in $H$, contains $A_\xi$.

**Lemma 12.** $A_\xi$ does not satisfy the Blaschke condition.

**Proof.** By (3.22),

\[
\sum_{k=0}^{\infty} \frac{4x_k}{|z_k + 1|^2} = \infty
\]

because $x_k \geq 1$ if $k$ is large.

**Lemma 13** (Fatou [1], Julia [2]). $\Delta_\xi$ is dense in the imaginary line.

**Proof** (in brief). Let $z_0 \in i\mathbb{R} \setminus A_\infty$, and let $D$ be an open disc of center $z_0$. By (3.21A), $A_l \not\to 0$, hence by Montel,

\[
\bigcup_{l=0}^{\infty} g_l(D)
\]

meets $A_\xi$. Then $D$ does too.
We now come to the $X$ in the statement $C$. Let $\xi \in H$ with $Z \notin \Delta \xi$. Then
\begin{equation}
(3.31) \quad g' \text{ vanishes nowhere in } \Delta \xi.
\end{equation}

By Corollary 7, there is a $\gamma$ in $G(H)$ that vanishes to order one everywhere in $\Delta \xi$ while vanishing to order zero elsewhere. Let $X$ be the Riemann surface of $\sqrt{\gamma}$. Then $X$ is a two-sheeted covering, of $H$, whose $\Delta$ is $\Delta \xi$. By (3.30) and (3.31), there is an $f$ in Prop $X$ with $\hat{f} = g$. Then $f$ is of valence 2 and fixes no point. Put $G(z) = 1/z$. Then $g(G) = g$, hence $\Delta$ is completely $G$-invariant, hence by the "$g = \hat{f}$ test" once more, there is an $F$ in Prop $X$ with $\hat{F} = G$. Then $F$ and $F(\sigma)$ are period 2 automorphisms. One of these fixes two points, neither point being a branch point, the other fixes no point. Finally, $\Delta$ does not satisfy the Blaschke condition, this is Lemma 12, while in the disc $D$, $T = \partial D$ by Lemma 13.

3.3. Three problems. We state each in terms of the two-sheeted covering $X$, but by Theorem 1, plus the "$g = \hat{f}$ test", all are problems on finite Blaschke products.

PROBLEM 1. Can we have $f, g \in \text{Prop } X$ but $\notin \text{Aut } X$ with $g$ fixing a point and $f$ fixing no point?

PROBLEM 2. Let $\chi$ be the number of points fixed by the proper maps that are not automorphisms, i.e., $\chi$ is the cardinality of

$$\{x \in X : \exists f \in \text{Prop } X \text{ but } \notin \text{Aut } X \text{ with } f(x) = x\}.$$  

By Theorem 5, $\chi$ is at most countably infinite. What are the possible values of $\chi$? One can have $\chi \geq 2$, but we do not know, for example, if 1, 2, or $\omega$ is possible.

The underlying problem is one of size. Namely, how large can the semigroup $\text{Prop } X \setminus \text{Aut } X$ be? I think the answer is "not very". This would mean that in Problem 1 there is no such pair $(f, g)$, while in Problem 2, $\chi$ is finite.

Let $f$ and $g$ be holomorphic maps, of $D$ to itself, such that $g$ is proper while $f$ fixes no point. Suppose that to each positive integer $l$ there is a positive integer $m$ such that the $l$th iterate of $f$ followed by the $m$th iterate of $g$ fixes the origin, in symbols, $g_m(f_l(0)) = 0$. Is this possible? It seems unlikely, especially if $g$ fixes the origin and $f$ is proper. This, i.e., "not possible if $g(0) = 0$ and $f \in \text{Prop } D$", would imply that in the first problem the answer is yes.

This is a good place to prove that Prop $X$ is countable, which means that Prop $X$ is countably infinite if it is not equal to Aut $X$. In other words, in
terms of cardinality, Prop $X$ is as small as it can be provided it is not equal to $\text{Aut } X$.

Let $g \in \text{Prop } D$ and put $k$ the valence of $g$. If $\xi \in D$, let $(g = \xi)$ be the unordered $k$-tuple of points where $g = \xi$. Thus if the point $x$ is such that $g(x) = \xi$ with multiplicity $l$, then $x$ is listed $l$ times in the tuple $(g = \xi)$.

**Lemma 14.** Let $f, f_1 \in \text{Prop } X$ and let $\xi \in D$. If the tuples $(\hat{f} = \xi)$ and $(\hat{f}_1 = \xi)$ are equal, then

$$\frac{\xi - \hat{f}}{1 - \xi \hat{f}} = \mu \frac{\xi - \hat{f}_1}{1 - \xi \hat{f}_1}$$

where $\mu$ is a root of unity.

**Proof.** Let $A$ be the period 2 automorphism of $D$ that takes the origin to the point $\xi$, in symbols, $A(w) = (\xi - w)/(1 - \xi w)$. The hypothesis implies that $A(\hat{f}) = \mu A(\hat{f}_1)$ where $|\mu| = 1$. It is to be proved that $\mu$ is a root of unity.

Put $g = A(\hat{f}(A))$, $g_1 = A(\hat{f}_1(A))$, $\phi_1 = A(\phi)$, and $\Delta_1 = A(\Delta)$. Then

(3.32) \[ g = \mu g_1, \]

while $g(\phi_1) = A(\hat{f}(\phi)) = A(\phi(f)) = \phi_1(f)$, which means that $g$ is an $\hat{f}$. Likewise, $g_1$ is an $\hat{f}$. By the "$g = \hat{f}$ test" there is at most a finite number of points $\zeta$, in $\Delta_1$, such that $\Delta_1$ does not meet $\{g_1 = \zeta\}$. Call these points, if any, $\zeta_1, \ldots, \zeta_t$. Let $\theta \in \Delta_1$ with $|\theta| \neq |\zeta_k|$ if $1 \leq k \leq t$. Because $\theta$ is not a $\zeta_k$, there is a $y$ in $\Delta_1$ with $g_1(y) = \theta$, but then by (3.32), $\mu \theta = g(y)$, hence $\mu \theta \in \Delta_1$. Then $\mu^2 \theta \in \Delta_1$, etc. This proves, if $\theta \neq 0$, that $\mu$ is a root of unity because $\Delta_1$ is discrete in the disc.

**Theorem 6.** Prop $X$ is countable.

**Proof.** If $\xi \in \Delta$, let $\Delta^\xi$ be the set of those unordered tuples, $(\hat{f} = \xi)$, whose components belong to $\Delta$. Then $\Delta^\xi$ is countable because $\Delta$ is, in symbols,

(3.33) \[ \Delta^\xi = \{(\hat{f}_1 = \xi), (\hat{f}_2 = \xi), \ldots\}. \]

Let $A$, once more, be the period 2 automorphism of $D$ that takes the origin to the point $\xi$.

Let $f \in \text{Prop } X$. By the "$g = \hat{f}$ test", there is a $\xi$ in $\Delta$ with $(\hat{f} = \xi)$ in $\Delta^\xi$, but then the tuple $(\hat{f} = \xi)$ is equal to one of the tuples in the right side of (3.33), to $(\hat{f}_1 = \xi)$ say. By the lemma, $\hat{f} = A(\mu A(\hat{f}_1))$ where $\mu$ is a root of unity.

**Problem 3.** If $0 \leq t < 1$, let $\mu(t)$ be the number of points in $\Delta$ bounded by $t$. 


Then
\[ \int_0^1 \mu(t) \, dt = \sum_{z \in A} (1 - |z|). \]

Because of Theorem 4, we ask if

\[ (3.34) \quad \text{Prop } X = \text{Aut } X \]

if \( \mu \) is not too large? E.g., is there a \( p, \ 1 < p < \infty \), such that (3.34) holds if

\[ \int_0^1 \mu(t)^p \, dt < \infty? \]

Failing this, what if

\[ \int_0^1 2^{\mu(t)} \, dt < \infty? \]

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