ON THE EMBEDDING AND DIAGONALIZATION
OF MATRICES OVER C(X)

KLAUS THOMSEN

1. Introduction.

In [2] Effros suggested the study of C*-algebras which arise as direct limits of C*-algebras of the form C(X) ⊗ F, where C(X) is the algebra of continuous functions on a nice compact topological space and F is a finite-dimensional C*-algebra. The idea is to generalize the almost complete theory of approximately finite-dimensional C*-algebras.

As a first step in the study of such direct limit C*-algebras one must study *-homomorphisms

$$\varphi : C(X) \otimes M_n \rightarrow C(Y) \otimes M_m.$$

The difficulties in imitating the theory of AF-algebras start already at this stage since two unital *-homomorphisms between C(X) ⊗ M_n and C(Y) ⊗ M_m need not be inner equivalent. Hence the question is if there is a canonical way of describing how C(X) ⊗ M_n can be embedded into C(Y) ⊗ M_m. The purpose of this note is to give such a description when Y satisfies certain topological conditions and the dimension of \( \varphi(C(X) \otimes M_n)(y) \subseteq M_m \) is constant over Y. As will become clear, the question is closely related to the question of which abelian C*-subalgebras of C(Y) ⊗ M_m can be diagonalized. Unless Y is a Stonean space, such a diagonalization is not automatically possible, see [4].

Although none of our results depend on the results of [4], the paper of Grove and Pedersen has been an indispensable source of inspiration.

2. Notation.

X, Y will denote compact Hausdorff spaces, M_n the n × n complex matrices, and U(n) the subset of M_n consisting of the unitary elements. We will identify C(X) ⊕ M_n with C(X, M_n), the continuous functions on X with values in M_n.

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If $B$ is a $C^*$-subalgebra of $C(X, M_n)$, we write $B(x)$ for the following $C^*$-subalgebra of $M_n$:

$$B(x) = \{ f(x) | f \in B \}, \quad x \in X.$$ 

$S_n$ will denote the symmetric group of order $n!$, and as in [4] the cohomology sets $H^1(X, S_n)$ and $H^1(X, U(n))$ will play an important role. For a definition of these sets, which will be sufficient, at least for our proofs, we refer to [5, p. 9–10].

3. Results.

**Lemma 1.** Let $\varphi: C(X) \otimes M_n \to C(Y) \otimes M_m$ be a unital $^*$-homomorphism. Then $n \mid m$, that is $n$ divides $m$, and

$$\dim[\varphi(C(X, M_n))](y) = n^2 \dim[\varphi(C(X) \otimes 1)](y), \quad y \in Y.$$ 

**Proof.** Let $y \in Y$. Then

$$C(X, M_n) \ni a \mapsto \varphi(a)(y)$$

defines a finite-dimensional representation of $C(X, M_n)$. Therefore there is a number $k(y) \in \mathbb{N}$ such that

$$\varphi(C(X, M_n))(y) \simeq M_n \oplus M_n \oplus \cdots \oplus M_n.$$

Since $\varphi(C(X) \otimes 1)(y)$ is the center of $\varphi(C(X, M_n))(y)$ by [6, Corollary 1], it is clear that

$$k(y) = \dim \varphi(C(X) \otimes 1)(y).$$

That $n \mid m$ follows from [1].

**Lemma 2.** Let $A$ be an abelian $C^*$-algebra in $C(X, M_n)$ containing the unit and such that $\dim A(x)$ is constant over $X$.

Assume that both $H^1(X, U(k))$ and $H^1(X, S_k)$ are trivial when $k \leq n$.

Then $A$ is diagonalizable.

**Proof.** Let $k = \dim A(x), x \in X$.

By [4, Lemma 5.2] we can find a finite open covering $\{ U_i | i = 1, 2, \ldots, N \}$ of $X$ and elements $Q^i_1, Q^i_2, \ldots, Q^i_k$ in $A$ such that $\{ Q^i_j(x) \}_{j=1}^k$ are the minimal projections in $A(x), x \in U_i, i = 1, 2, \ldots, N'$. 

For \( x \in U_i \cap U_j \), we can define an element \( s_{ij}^x \in S_k \) by the requirement

\[
\text{Tr}(Q_{s_{ij}^x(m)}^l(x)Q_{s_{ij}^x}^l(x)) \neq 0 \quad \text{iff} \quad l = m
\]

\( m, l \in \{1, 2, \ldots, k\} \). Then \( Q_{s_{ij}^x(m)}^l(x) = Q_m^l(x) \) and the defining relation (1) shows that

\[
U_i \cap U_j \ni x \rightarrow s_{ij}^x \in S_k
\]
is continuous. Since \( s_{ij}^x s_{im}^x = s_{jm}^x \), \( x \in U_i \cap U_j \cap U_m \), we find that \((U_i, s_{ij})\) represent an element in \( H^1(X, S_k) \). This set is trivial by assumption. Therefore there are continuous functions

\[
s_i : U_i \rightarrow S_k
\]
such that

\[
s_{ij}^x = s_i^x (s_j^x)^{-1}, \quad x \in U_i \cap U_j.
\]

Then

\[
Q_{s_{ij}^x(m)}^i(x) = Q_{s_{ij}^x(m)}^j(x), \quad m \in \{1, 2, \ldots, k\}, \quad x \in U_i \cap U_j.
\]

For each \( m \in \{1, 2, \ldots, k\} \), we define \( Q_m \in C(X, M_n) \) by

\[
Q_m(x) = Q_{s_{ij}^x(m)}^i(x), \quad x \in U_i.
\]

Then \( \{Q_1(x), \ldots, Q_k(x)\} \) are the minimal projections in \( A(x) \) for all \( x \in X \).

For any sample \( d_1, d_2, \ldots, d_k \) of integers satisfying

\[
\sum_{i=1}^k d_i = n,
\]

the set \( \{x \in X | \text{Tr}(Q_i(x)) = d_i, \quad i = 1, 2, \ldots, m\} \) is open and closed. In order to show that span \( \{Q_1, Q_2, \ldots, Q_k\} \) can be diagonalized over \( X \), we can therefore restrict attention to such a set. Or, for simplicity of exposition, assume that

\[
\{x \in X | \text{Tr}(Q_i(x)) = d_i, \quad i = 1, 2, \ldots, m\} = X.
\]

Let \( p_1, p_2, \ldots, p_k \) be diagonal projections in \( M_n \) such that \( \text{Tr}(p_i) = d_i \), \( i = 1, 2, \ldots, k \). Fix \( x_0 \in X \). There is a unitary \( U \in C(X, M_n) \) such that

\[
U(x_0)Q_i(x_0)U(x_0)^* = p_i, \quad i = 1, 2, \ldots, k.
\]
Then

\[ \sup_i \| U(x)Q_i(x)U(x)^* - p_i \| < \frac{1}{2} \]

for all \( x \) in a neighbourhood of \( x_0 \).

Let \( g: [0, 1] \to [0, 2] \) be a continuous function which is zero in a neighbourhood of 0 and \( g(t) = 1/t, \quad t \geq \frac{1}{2} \).

Let \( W_i \in C(X, M_n) \) be given by

\[ W_i(x) = p_i \left[ g(p_i U(x)Q_i(x)U(x)^* p_i) \right]^{1/2} U(x)Q_i(x)U(x)^* \]

\( x \in X, \quad i = 1, 2, \ldots, k \). As shown by Glimm in the proof of [3, Lemma 1.8], we have that

\[ W_i(x)^* W_i(x) = U(x)Q_i(x)U(x)^* \]

\[ W_i(x)W_i(x)^* = p_i, \quad i = 1, 2, \ldots, k \]

for all \( x \) in the same neighbourhood of \( x_0 \) as above.

Let \( W = \sum_{i=1}^k W_i \). Then \( W(x) \) is a unitary such that

\[ WUQ_i(WU)^*(x) = p_i, \quad i = 1, 2, \ldots, k \]

for \( x \) in this neighbourhood.

We conclude that there is a finite covering \( \{ U_i | i = 1, 2, \ldots, N \} \) of \( X \) and continuous functions

\[ W_i: U_i \to U(n) \]

such that

\[ W_i(x)Q_j(x)W_i(x)^* = p_j, \quad j = 1, 2, \ldots, k, \quad x \in U_i. \]

Especially \( W_j(x)W_i(x)^* \in \{ p_1, p_2, \ldots, p_k \}' \), \( x \in U_i \cap U_j \). Let \( \mathcal{W} \) denote the unitary group of \( \{ p_1, p_2, \ldots, p_k \}' \). Then \( (U_i, W_jW_i^*) \) define an element in \( H^1(X, \mathcal{W}) \).

Since \( \{ p_1, p_2, \ldots, p_k \}' \simeq M_{d_1} \oplus M_{d_2} \oplus \cdots \oplus M_{d_k} \), we have that

\[ \mathcal{W} \simeq U(d_1) \times U(d_2) \times \cdots \times U(d_k). \]
Since $H^1(X, U(d_i)) = 0$ for all $i$ by assumption, we conclude that $H^1(X, \mathcal{W}) = 0$. Hence there are continuous functions

$$V_i: U_i \to \mathcal{W}$$

such that $W_j(x)W_i^*(x) = V_j^*(x)V_i(x), \ x \in U_i \cap U_j$. Define $T \in C(X, M_n)$ by

$$T(x) = V_j(x)W_j(x), \quad x \in U_j.$$

Then $T$ diagonalizes $A$.

**Remark.** If dim $A(x)$ is constantly equal to $n$, the above proof works with the assumptions that

$$H^1(X, S_n) = H^1(X, U(1)) = 0.$$

Since $H^1(X, U(1)) \simeq H^2(X, Z)$, the lemma that results follows from some of the arguments used to prove Theorem 5.3 in [4]. In this case the lemma is very close to Theorem 1.4 of [4]. The morale of the lemma is that the worst or most effective obstruction to diagonalization of matrices over $C(X)$ arises from the fact that the number of eigenvalues of a normal element in $C(X, M_n)$ can vary over $X$. If this number is constant the multiplicites are irrelevant as long as $X$ satisfies the assumptions of the lemma.

**Theorem 3.** Let $X, Y$ be compact Hausdorff spaces, and

$$\varphi: C(X, M_n) \to C(Y, M_m)$$

a unital $*$-homomorphism (such that especially $n|m$). Assume that

$$\dim \varphi(C(X, M_n))(y)$$

is constant over $Y$ and that $H^1(Y, U(k)) = H^1(Y, S_k) = 0, \ k \leq m/n$. Then there is a unitary $U \in C(Y, M_m)$ and $m/n$ continuous functions $\psi_i: Y \to X, \ i = 1, 2, \ldots, m/n$, such that

$$\varphi(f)(y) = U(y)$$

where

$$\begin{bmatrix}
    f \circ \psi_1(y) \\
    f \circ \psi_2(y) \\
    \vdots \\
    0 \\
    0 \\
    \vdots \\
    f \circ \psi_{m/n}(y)
\end{bmatrix}$$

$$y \in Y, \ f \in C(X, M_n).$$
PROOF. Define elements \( \tilde{e}_{ij} \) of \( C(X, M_n) \) by
\[
\tilde{e}_{ij}(x) = e_{ij}, \quad x \in X
\]
where \( \{e_{ij}\} \) is the standard system of matrix units in \( M_n \). Since \( \varphi \) is unital,
\[
\{ \varphi(\tilde{e}_{ij})(y) \}
\]
is a system of matrix units in \( M_m \) for all \( y \in Y \). Let \( \{f_{ij}\} \) be the standard system of matrix units in \( M_m \), and define
\[
c_{ij} = \sum_{d=1}^{m/n} f_{i+(d-1)m, j+(d-1)n}, \quad i, j = 1, 2, \ldots, n.
\]
For each \( y_0 \in Y \) there is a unitary \( U \in C(Y, M_m) \) such that
\[
U(y_0)\varphi(\tilde{e}_{ij})(y_0)U(y_0)^* = c_{ij}, \quad i, j = 1, 2, \ldots, n.
\]
But then
\[
\sup_{ij} ||U(y)\varphi(\tilde{e}_{ij})(y)U(y)^* - c_{ij}|| < \frac{1}{2}
\]
in a neighbourhood of \( y_0 \). Take a function \( g \) as in the proof of Lemma 2, and define an element \( W \in C(Y, M_m) \) by
\[
W(y) = \sum_{i=1}^{n} c_{i1} [g(c_{11} U(y)\varphi(\tilde{e}_{11})(y)U(y)^* c_{11})]^{1/2} U(y)\varphi(\tilde{e}_{i1})(y)U(y)^*, \quad y \in Y.
\]
Then \( W(y) \) is a unitary such that
\[
W(y)U(y)\varphi(\tilde{e}_{ij})(y)U(y)^*W(y)^* = c_{ij}, \quad i, j = 1, 2, \ldots, n
\]
for all \( y \) in the above neighbourhood of \( y_0 \). The details needed to verify this can be found in [3, proof of lemma 1.8] and [1, proof of lemma 2.3].

Thus we can find a finite covering \( \{U_i, i = 1, 2, \ldots, N\} \) of \( Y \) and continuous functions
\[
W_i: U_i \to U(m)
\]
such that
\[ W_k(y)\varphi(\tilde{e}_{ij})(y)W_k(y)^* = c_{ij}, \quad i, j = 1, 2, \ldots, n, \quad y \in U_k. \]

Let \( \mathcal{W} \) denote the unitary group of \( \{c_{ij}\}' \subseteq M_m \). Then \( (U_i, W_iW_j^*) \) defines an element in
\[ H_1(Y, \mathcal{W}). \]

Since \( \{c_{ij}\}' \simeq M_{m/n} \), our assumption on \( Y \) assures that there are continuous functions
\[ V_i : U_i \to \mathcal{W} \]
such that
\[ W_iW_j^* = V_i^*V_j \quad \text{over} \quad U_i \cap U_j. \]

Define \( S \in C(Y, M_m) \) by
\[ S(y) = V_i(y)W_i(y), \quad y \in U_i, \quad i = 1, 2, \ldots, N. \]

Then \( S \) is a unitary in \( C(Y, M_m) \) such that
\[ S(y)\varphi(\tilde{e}_{ij})(y)S(y)^* = c_{ij}, \quad y \in Y, \quad i, j = 1, 2, \ldots, n. \]

Let \( f \in C(X) \), \( i, j \in \{1, 2, \ldots, n\} \). Then
\[ S\varphi(f \otimes e_{ij})S^*(y) = S\varphi(f \otimes 1)\varphi(\tilde{e}_{ij})S^*(y) \]
\[ = S\varphi(f \otimes 1)S^*(y)c_{ij} = c_{ij}S\varphi(f \otimes 1)S^*(y), \quad y \in Y. \]

Hence \( S\varphi(C(X) \otimes 1)S^*(y) \subseteq \{c_{ij}\}' \) for \( y \in Y \).

Since \( \{c_{ij}\}' \simeq M_{m/n} \) and \( \dim \varphi(C(X) \otimes 1)(y) \) is constant over \( X \) by Lemma 1, we conclude from Lemma 2 that there is a unitary \( T \in C(Y, M_m) \) such that \( T(y) \in \{c_{ij}\}' \) and
\[ TS\varphi(f \otimes 1)S^*T^*(y) \]
is diagonal for all \( f \in C(X), \ y \in Y \).
Let
\[ p_1 = \sum_{i=1}^{n} f_{ii}, \quad p_2 = \sum_{i=n+1}^{2n} f_{ii}, \ldots, \quad p_{m/n} = \sum_{i=m-n+1}^{m} f_{ii}. \]

Since \( TS\varphi(C(X) \otimes 1)S^*T^*(y) \subseteq \{c_{ij}\} \),
\[ TS\varphi(C(X) \otimes 1)S^*T^*(y) \subseteq \text{span}\{p_1, \ldots, p_{m/n}\} \]
for all \( y \in Y \).

For each \( y \in Y \) there are then elements
\[ \psi_1(y), \psi_2(y), \ldots, \psi_{m/n}(y) \in X \]
determined by
\[ TS\varphi(f \otimes 1)S^*T^*(y)p_i = f(\psi_i(y))p_i, \quad i = 1, 2, \ldots, \frac{m}{n} \]
\( f \in C(X) \). Clearly, \( \psi_i : Y \to X \), are continuous functions.

The desired unitary, \( U \), is \( TS \) and it is a routine matter to check that \( U, \psi_1, \psi_2, \ldots, \psi_{m/n} \) have the right property.

It is clear that there is a great freedom in the choice of the unitary \( U \) of Theorem 3. But the question is how much freedom there is in the choice of the functions \( \psi_1, \psi_2, \ldots, \psi_{m/n} \). This is answered by the following

**Proposition 4.** Let \( X, Y \) be compact Hausdorff spaces and let
\[ \varphi : C(X, M_n) \to C(Y, M_m) \]
be a unital *-homomorphism such that \( \text{dim} \varphi(C(X, M_n))(y) \) is constant over \( Y \).

Assume \( \psi_1, \psi_2, \ldots, \psi_{m/n} \) are continuous functions from \( Y \) to \( X \) such that (2) holds for some unitary \( U \). Let \( \varphi_1, \varphi_2, \ldots, \varphi_{m/n} \) be continuous functions from \( Y \) to \( X \).

Then there is a unitary \( W \) in \( C(Y, M_m) \) such that (2) holds with \( \varphi_i \) substituted for \( \psi_i \), \( i = 1, 2, \ldots, m/n \), and \( W \) substituted for \( U \) if and only if
\[ \{\psi_1(y), \psi_2(y), \ldots, \psi_{m/n}(y)\} = \{\varphi_1(y), \varphi_2(y), \ldots, \varphi_{m/n}(y)\}, \quad y \in Y. \]

If \( Y \) is connected this condition is equivalent to
\[ \{\psi_1, \psi_2, \ldots, \psi_{m/n}\} = \{\varphi_1, \varphi_2, \ldots, \varphi_{m/n}\}. \]
PROOF. Assume first that (2) holds for \( \varphi_1, \ldots, \varphi_{m/n} \) and \( W \). Taking the trace, it follows that

\[
\sum_{i=1}^{m/n} f(\varphi_i(y)) = \sum_{i=1}^{m/n} f(\psi_i(y))
\]

for all \( y \in Y, \ f \in C(X) \).

Since \( X \) is a compact Hausdorff space, this is only possible if

\[
(3) \quad \{ \varphi_1(y), \varphi_2(y), \ldots, \varphi_{m/n}(y) \} = \{ \psi_1(y), \psi_2(y), \ldots, \psi_{m/n}(y) \}, \quad y \in Y.
\]

Conversely, assume that (3) holds.

Let \( k = \dim \varphi(C(X) \otimes 1)(y), \quad y \in Y \).

Let \( N_1, N_2, \ldots, N_k \) and \( M_1, M_2, \ldots, M_k \) be two partitions of \( \{1, 2, \ldots, m/n\} \) into \( k \) disjoint non-empty subsets such that \( \#N_i = \#M_i, \quad i = 1, 2, \ldots, k \).

Consider

\[
\cap_{i=1}^{k} \{ y \in Y | \varphi_i(y) = \psi_j(y) = \psi_n(y) = \psi_m(y), \quad i, j \in N_i, \quad n, m \in M_i \}.
\]

Such a subset of \( Y \) is called a configuration.

From (3) and the assumption that \( \dim(C(X) \otimes 1)(y) = k \), it follows that the configurations form a finite covering of \( Y \) by mutually disjoint subsets. Since each configuration is obviously closed, we see that they are in fact both closed and open.

Let \( \{ F_1, F_2, \ldots, F_N \} \) denote the non-empty configurations. For each \( i \in \{1, 2, \ldots, N\} \), it is clear that we can find a unitary \( \tilde{W}_i \in M_m \) such that

\[
\tilde{W}_i \begin{bmatrix} f \circ \psi_1(y) & f \circ \psi_2(y) & 0 \\ f \circ \psi_2(y) & \ddots & \vdots \\ 0 & \ddots & f \circ \psi_{m/n}(y) \end{bmatrix} \tilde{W}_i^* = \begin{bmatrix} f \circ \varphi_1(y) & f \circ \varphi_2(y) & 0 \\ f \circ \varphi_2(y) & \ddots & \vdots \\ 0 & \ddots & f \circ \varphi_{m/n}(y) \end{bmatrix}
\]

\( y \in F_i, \quad f \in C(X, M_n) \).

Define

\[
W_i(y) = \begin{cases} \tilde{W}_i, & y \in F_i \\ 0, & y \notin F_i \end{cases}
\]
and

\[ V = \sum_{i=1}^{N} W_i. \]

Then \( V \) is a unitary in \( C(Y, M_m) \) such that

\[
V(y) = \begin{bmatrix}
    f \circ \psi_1(y) & 0 \\
    f \circ \psi_2(y) & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & f \circ \psi_{m/n}(y)
\end{bmatrix} = \begin{bmatrix}
    f \circ \varphi_1(y) \\
    f \circ \varphi_2(y) \\
    \vdots \\
    0 \\
    f \circ \varphi_{m/n}(y)
\end{bmatrix}
\]

\( y \in Y \). Let \( W = UV^* \). Then (2) holds for \( \varphi_1, \varphi_2, \ldots, \varphi_{m/n} \) and \( W \).

If \( Y \) is connected, there is only one non-empty configuration. Hence (3) implies that

\[ \{\psi_1, \psi_2, \ldots, \psi_{m/n}\} = \{\varphi_1, \varphi_2, \ldots, \varphi_{m/n}\} \]

in this case.

Remark. The conclusion of Theorem 3 is not true if \( H^1(Y, U(k)) \neq 0 \) or \( H^1(Y, S_k) \neq 0 \) for some \( k \leq m/n \). This follows from [4, Theorems 4.1 and 4.2], or rather their proofs, since the non-diagonalizable elements constructed there generate abelian C*-subalgebras with constant dimension over \( X_0 \).

REFERENCES