THE F. AND M. RIESZ THEOREM REVISITED

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1. Introduction.

The celebrated F. and M. Riesz theorem states: if \( \mu \) is a complex Borel measure on the unit circle \( T \) such that

\[
\hat{\mu}(n) = \int_T e^{-in\theta} d\mu(\theta) = 0 \quad \text{for } n = -1, -2, \ldots,
\]

then \( \mu \) is absolutely continuous with respect to Lebesque measure on \( T \). Some forty years later, Helson and Lowdenslager [4] generalized the F. and M. Riesz theorem to compact Abelian groups with ordered duals. deLeeuw and Glicksberg [1], Doss [2], [3], and Yamaguchi [11] shortly afterwards obtained a number of related results. In this note we present simple and perspicuous proofs of these theorems by using the Helson-Lowdenslager theorem and some other well-known facts. In particular, we will prove Yamaguchi's theorem without using the theory of disintegration.

We are very grateful to Professor S. Saeki for showing us the proof of Theorem C for \( G = \mathbb{R}^n \). His idea was also a guide for our proof for the case in which \( G \) contains a compact open subgroup.

2. Preliminaries and four theorems.

Let \( G \) be an Abelian group. We say that \( G \) is an ordered group if \( G \) contains a subsemigroup \( P \) such that \( P \cup (-P) = G \) and \( P \cap (-P) = \{0\} \). (We will refer to \( P \) as an order in \( G \).) It is well known that \( G \) is an ordered group if and only if \( G \) is torsion-free (see [5]).

Let \( G \) be a locally compact Abelian group and let \( \hat{G} \) be its dual group. (The term "locally compact Abelian group" means "locally compact Abelian group satisfying Hausdorff's separation axiom"). A fixed but arbitrary Haar measure on \( G \) will be denoted by \( m_G \). The symbol \( M(G) \) will denote the Banach algebra of all bounded regular complex Borel measures on \( G \) under con-

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volution multiplication and the total variation norm. For an element $x$ in $G$, $\delta_x$ denotes the Dirac measure at $x$. For $\mu$ in $M(G)$, let $\mu_a$ and $\mu_s$ be respectively the absolutely continuous and singular parts of $\mu$ with respect to $m_G$. We denote the Fourier-Stieltjes transform of a measure $\mu$ by $\hat{\mu}$ and convolution of measures $\mu$ and $\nu$ by $\mu \ast \nu$. For a subset $E$ of $\hat{G}$, $M_{E}(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish on $\hat{G}\setminus E$.

All notation and terminology not explained in the sequel is as in [5].

We now state the four Theorems mentioned in section 1.

**Theorem A** (Helson–Lowdenslager, cf. [9, Theorem 8.2.3]). Let $G$ be a compact Abelian group with ordered dual $\hat{G}$ and let $P$ be an order in $\hat{G}$. If $\mu$ is a measure in $M_{\mathcal{P}}(G)$, then $\mu_a$ and $\mu_s$ belong to $M_{\mathcal{P}}(G)$ and moreover $\hat{\mu}(0) = 0$.

**Theorem B** (deLeeuw–Glicksberg [1, Proposition 5.1]). Let $G$ be a compact Abelian group and let $\psi$ be a nontrivial homomorphism of $\hat{G}$ into the additive group $R$ of real numbers. If $\mu$ is a measure in $M(G)$ such that $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \hat{G}$ with $\psi(\gamma) \leq 0$, then $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ for all $\gamma \in \hat{G}$ with $\psi(\gamma) \leq 0$.

**Theorem C** (Doss [3, Lemma 1]). Let $G$ be a locally compact Abelian group with ordered dual $\hat{G}$ and let $P$ be an order in $\hat{G}$. If $\mu$ is a measure in $M_{\mathcal{P}}(G)$, then $\mu_a$ and $\mu_s$ also belong to $M_{\mathcal{P}}(G)$ and moreover $\hat{\mu}_s(0) = 0$.

**Theorem D** (Yamaguchi [11]). Let $G$ be a locally compact Abelian group and let $P$ be a subsemigroup of $\hat{G}$ such that $P \cup (-P) = \hat{G}$. If $\mu$ is a measure in $M_{\mathcal{P}}(G)$, then $\mu_a$ and $\mu_s$ also belong to $M_{\mathcal{P}}(G)$.

**Remark 2.1.** In his paper [11], Yamaguchi also showed the following. Let $G$, $\hat{G}$, and $P$ be as in Theorem D. If $\mu$ is a measure in $M_{\mathcal{P}}(G)$, then $\mu_a$ and $\mu_s$ belong to $M_{\mathcal{P}}(G)$. To prove Theorems C and D, it suffices to prove them with $M_{\mathcal{P}}(G)$ replaced by $M_{\mathcal{P}}(G)$: Yamaguchi proved this in [11, pp. 244–245]. We will prove Theorems C and D in this form.

The cited proof of Theorem A is unexceptionable, and Theorem A will be used in our work. We will generalize Theorem B. Doss’s proof of Theorem C is flawed, since he tacitly assumes that $P$ is Haar measurable (which as shown in [5] need not be the case). It seems worthwhile to present a short proof of Theorem C. Yamaguchi’s proof of Theorem D is in part impenetrable, and again our simple proof seems preferable.

3. Generalized Theorem B.

In this section we prove a generalization of Theorem B (see Theorem 3.6). We first prove the theorem for a compact metrizable Abelian group by using a result on measurable selections. We next prove it for all compact Abelian
groups by using a lemma due to Pigno and Saeki ([8, Lemma 4]).

We first prove a simple corollary of Theorem A.

**Lemma 3.1.** Let $G$ be a compact Abelian group with torsion-free dual $\hat{G}$ and let $P$ be a subsemigroup of $\hat{G}$ such that $P \cup (-P) = \hat{G}$. If $\mu$ is a measure in $M_P(G)$, then $\mu_a$ and $\mu_s$ also belong to $M_P(G)$.

**Proof.** We may suppose that $P \cap (-P) \neq \{0\}$; otherwise the lemma is Theorem A. Since $P \cap (-P)$ is a torsion-free Abelian group, there is a subsemigroup $Q$ of $P \cap (-P)$ such that $Q \cup (-Q) = P \cap (-P)$, and $Q \cap (-Q) = \{0\}$ (see [5, Remark (2.6)]). We write

$$P_1 = (P \setminus Q) \cup \{0\}$$

and

$$P_2 = (P \setminus (-Q)) \cup \{0\}.$$

A short argument, which we omit, shows that $P_1$ and $P_2$ are subsemigroups of $\hat{G}$, that $P_1 \cup (-P_1) = P_2 \cup (-P_2) = \hat{G}$, that $P_1 \cap (-P_1) = P_2 \cap (-P_2) = \{0\}$, and that $P_1 \cup P_2 = P$. Suppose that $\mu$ is a measure in $M_P(G)$. In particular we have $\hat{\mu}(\gamma) = 0$ for all $\gamma \in P_1$. Theorem A shows that $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ for all $\gamma \in P_1$; similarly $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ for all $\gamma \in P_2$. Since $P_1 \cup P_2 = P$, we have $\hat{\mu}_a(\gamma) = \hat{\mu}_s(\gamma) = 0$ for all $\gamma \in P$.

The following Lemma 3.2 is due to Ryll–Nardzewski [7], [10].

**Lemma 3.2.** Let $X$ be a metric space and let $Y$ be a separable and complete metric space. Let $\mathcal{F}$ be the family of all nonvoid closed subsets of $Y$. Let $\Sigma$ be a mapping from $X$ to $\mathcal{F}$ with the following property: $\{x \in X : \Sigma(x) \subset K\}$ is closed in $X$ for each closed subset $K$ of $Y$. Then there exists a mapping $\sigma$ from $X$ into $Y$ such that:

(i) $\sigma(x) \in \Sigma(x)$ for each $x \in X$;

and

(ii) $\sigma^{-1}(U)$ is a Borel subset of $X$ for each open subset $U$ of $Y$.

**Lemma 3.3.** Let $G$ be a compact metrizable Abelian group, let $H$ be a closed subgroup of $G$, and let $\pi$ be the natural homomorphism of $G$ onto $G/H$. Then there exists a mapping $\sigma$ from $G/H$ into $G$ with the following properties:

(i) $\pi \circ \sigma(x) = \hat{x}$ for each $\hat{x} \in G/H$;

(ii) $\sigma^{-1}(U)$ is a Borel subset of $G/H$ for each open subset $U$ of $G$. 
PROOF. We use Lemma 3.2 with \( X = G/H, \ Y = G, \) and \( \Sigma(\hat{x}) = x + H, \) where \( \pi(x) = \hat{x}. \) We need only to verify that the set

\[ A_k = \{ \hat{x} \in G/H ; \Sigma(\hat{x}) = x + H \subset K \} \]

is closed in \( G/H \) for each closed subset \( K \) of \( G. \) This is simple. Indeed, let \( \{\hat{x}_n\} \) be a sequence in \( A_k \) such that \( \{\hat{x}_n\} \) converges to an element \( \hat{x} \) in \( G/H. \) Choose elements \( x_n \) and \( x \) in \( G \) such that \( \pi(x_n) = \hat{x}_n \) for \( n = 1, 2, \ldots \) and \( \pi(x) = \hat{x}. \) Since \( G \) is compact and metric, a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) converges to an element \( x_0 \) of \( G. \) Then \( \{\hat{x}_{n_j}\} \) converges to \( \hat{x}_0 = \pi(x_0), \) and so \( \hat{x}_0 = \hat{x}. \) For \( h \in H, \) we have \( x_0 + h \in K \) because \( \{x_{n_j} + h\} \) converges to \( x_0 + h \) and \( x_{n_j} + h \in K. \) That is, \( x_0 + H \subset K; \) and so \( x + H \subset K \) because \( \hat{x}_0 = \hat{x}. \)

LEMMA 3.4. Let \( G \) be a compact metrizable Abelian group and let \( P \) be a subsemigroup of \( \hat{G} \) such that \( P \cup (-P) = \hat{G}. \) If \( \mu \) is a measure in \( M_P(G), \) then \( \mu_a \) and \( \mu_s \) are also in \( M_P(G). \)

PROOF. Since \( G \) is a compact metrizable Abelian group, \( G \) may be regarded as a closed subgroup of the countably infinite dimensional torus \( G_0 \) (see \([9, \text{Theorem 2.2.6}]\)).

By Lemma 3.3, there exists a mapping \( \sigma \) from \( G_0/G \) to \( G_0 \) with the following properties:

(i) \( \pi \circ \sigma(\hat{x}) = \hat{x} \) for each \( \hat{x} \in G_0/G; \)

(ii) \( \sigma^{-1}(U) \) is a Borel subset of \( G_0/G \) for each open subset \( U \) of \( G_0, \)

where \( \pi \) denotes the natural homomorphism from \( G_0 \) onto \( G_0/G. \)

It suffices to show that \( \mu_s \) belongs to \( M_P(G) \) if \( \mu \) does. Assume the contrary: there is a measure \( \mu \) in \( M_P(G) \) such that \( \hat{\mu}_a(\gamma_0) \neq 0 \) for some \( \gamma_0 \in P. \) By considering \( \gamma_0 \mu, \) we may suppose that \( \hat{\mu}_a(0) \neq 0. \) We will also consider \( \mu \) as a measure in \( M(G_0). \)

Now we define a function on \( G_0/G \) for each bounded Borel function \( v \in M(G_0) \) as follows:

\[(1) \quad \hat{x} \mapsto v \circ \delta_{\sigma(\hat{x})}(f) \quad \text{for} \ \hat{x} \in G_0/G. \]

It is obvious that the mapping (1) is a bounded Borel function on \( G_0/G \) for each bounded Borel function \( f \) on \( G_0 \) and each \( v \in M(G_0). \) Thus we can define measures \( \lambda, \lambda_1, \) and \( \lambda_2 \) in \( M(G_0) \) as follows:
\[
\begin{align*}
\lambda(f) &= \int_{G_0/G} \mu * \delta_{\sigma(\hat{x})}(f) \, dm_{G_0/G}(\hat{x}); \\
\lambda_1(f) &= \int_{G_0/G} \mu_a * \delta_{\sigma(\hat{x})}(f) \, dm_{G_0/G}(\hat{x}); \\
\lambda_2(f) &= \int_{G_0/G} \mu_s * \delta_{\sigma(\hat{x})}(f) \, dm_{G_0/G}(\hat{x})
\end{align*}
\]

(1)

for \( f \in C(G_0) \).

Note that the equalities (1) hold for each bounded Borel function \( g \) on \( G_0 \). This can be easily verified by approximating a bounded Borel function on \( G_0 \) by continuous functions on \( G_0 \).

We will show that \( \lambda_1 \) and \( \lambda_2 \) are respectively the absolutely continuous and singular parts of \( \lambda \) with respect to \( m_{G_0} \). Once we have proved this fact, the Lemma can be established as follows. Define

\[
\bar{P} = \{ \gamma \in \hat{G}_0 : \gamma|G \in P \},
\]

where \( \gamma|G \) denotes the restriction of \( \gamma \) to \( G \). It is obvious that \( \bar{P} \) is a subsemigroup of \( \hat{G}_0 \) and that \( \bar{P} \cup (-\bar{P}) = \hat{G}_0 \). If \( \gamma \) is an element of \( \bar{P} \), then \( \gamma|G \in P \) and therefore we have

\[
\mu * \delta_{\sigma(\hat{x})}(\bar{\gamma}) = \hat{\mu}(\gamma)(-\sigma(\hat{x}), \gamma) \\
= \hat{\mu}(\gamma|G)(-\sigma(\hat{x}) \gamma) \\
= 0
\]

for each \( \hat{x} \in G_0/G \). We infer that

\[
\hat{\lambda}(\gamma) = \lambda(\bar{\gamma})
\]

\[
= \int_{G_0/G} \mu * \delta_{\sigma(\hat{x})}(\bar{\gamma}) \, dm_{G_0/G}(\hat{x})
\]

\[
= 0
\]
for each $\gamma \in \tilde{P}$. On the other hand, we have

$$\hat{\lambda}_s(0) = \hat{\lambda}_2(0)$$

$$= \int_{\tilde{G}_0/G} \mu_s * \delta_{\sigma(\check{x})}(1)dm_{\tilde{G}_0/G}(\check{x})$$

$$= \hat{\mu}_s(0) \neq 0.$$ 

The group \( \tilde{G}_0 \) is the weak direct sum of countably many copies of the integers and so is torsion-free. This contradicts Lemma 3.1.

To complete the present proof, we need to prove that $\lambda_1$ and $\lambda_2$ are respectively the absolutely continuous and singular parts of $\lambda$ with respect to $m_{\tilde{G}_0}$. Since $\lambda = \lambda_1 + \lambda_2$, it is sufficient to show the following:

(I) $\lambda_1$ is absolutely continuous with respect to $m_{\tilde{G}_0}$;

(II) $\lambda_2$ is singular with respect to $m_{\tilde{G}_0}$.

To prove (I), let $E$ be a Borel subset of $G_0$ such that $m_{\tilde{G}_0}(E) = 0$. Since

$$0 = m_{\tilde{G}_0}(E)$$

$$= \int_{\tilde{G}_0/G} \int_{G} 1_E(x + y)dm_G(y)dm_{\tilde{G}_0/G}(\check{x}) \quad (\check{x} = \pi(x)),$$

there exists a Borel subset $A$ of $G_0/G$ such that $m_{\tilde{G}_0/G}(A) = 0$ and

$$\int_{G} 1_E(x + y)dm_G(y) = 0$$

for all $x$ in $G_0$ such that $\pi(x) \in A^c \subset G_0/G$. For $x \in A^c$, we have

$$\mu_a * \delta_{\sigma(\check{x})}(1_E) = \int_{G} 1_E(\sigma(\check{x}) + y) \frac{d\mu_a}{dm_G}(y)dm_G(y).$$

Since $\sigma(\check{x}) \in \pi^{-1}(\check{x})$, it follows that $\mu_a * \delta_{\sigma(\check{x})}(1_E) = 0$. Accordingly we have
\[ \lambda_1(E) = \int_{G_0/G} \mu_s * \delta_{\sigma(\hat{x})}(1_E) dm_{G_0/G}(\hat{x}) \]

\[ = \int_A \mu_s * \delta_{\sigma(\hat{x})}(1_E) dm_{G_0/G}(\hat{x}) + \]

\[ + \int_{A^c} \mu_s * \delta_{\sigma(\hat{x})}(1_E) dm_{G_0/G}(\hat{x}) \]

\[ = 0. \]

This proves (I).

We now prove (II). Employing the canonical decomposition of \( \mu_s \) as a linear combination of nonnegative (singular!) measures, we may suppose that \( \mu_s \) is nonnegative and that \( \|\mu_s\| = 1. \)

Since \( G_0 \) is compact and metrizable, \( C(G_0) \) contains a countable dense subset \( \{f_n\} \). Let \( \varepsilon \) be a positive real number. For \( n = 1, 2, \ldots, \), Luzin's theorem shows that there exists a compact subset \( E_n \) of \( G_0/G \) such that \( m_{G_0/G}(E_n^c) < \varepsilon/2^n \) and \( \hat{x} \to \mu_s * \delta_{\sigma(\hat{x})}(f_n) \) is continuous on \( E_n \). We write \( E = \bigcap_{n=1}^{\infty} E_n \). Then \( E \) is compact and \( m_G(E^c) < \varepsilon \) and \( \hat{x} \to \mu_s * \delta_{\sigma(\hat{x})}(f_n) \) is continuous on \( E \) for each \( h \in C(G_0) \). The measure \( \mu_s * \delta_{\sigma(\hat{x})} \) is singular with respect to \( m_G \) for each \( \hat{x} \in G_0/G \). A short argument, which we omit, shows that for each \( \hat{x} \in E \), there is an \( f \in C(G_0) \) with \( 0 \leq f \leq 1 \) such that:

\[ \begin{cases} 1 = \|\mu_s * \delta_{\sigma(\hat{x})}\| < \mu_s * \delta_{\sigma(\hat{x})}(f) + \varepsilon; \\ \delta_x * m_G(f) < \varepsilon \quad \text{for } x \in \pi^{-1}(\{\hat{x}\}), \end{cases} \]

since \( G + \sigma(\hat{x}) = \pi^{-1}(\{\hat{x}\}) \).

Since \( \hat{x} \to \mu_s * \delta_{\sigma(\hat{x})}(f) \) is continuous on \( E \) and \( x \to \delta_x * m_G(f) \) is continuous on \( G \), the inequalities (11) hold on some neighborhood \( U_{\hat{x}} \) of \( \hat{x} \) in \( E \). Since \( E \) is compact, there exist \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k \) in \( E \) such that \( \bigcup_{j=1}^k U_{\hat{x}_j} = E \). We denote the \( f^s \) that correspond to \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k \) by \( f_1, f_2, \ldots, f_k \), respectively. Now we define a function \( g \) on \( G_0 \) as follows:
\[
g = \begin{cases} 
  f_1 & \text{on } \pi^{-1}(U_{\hat{x}_1}); \\
  f_2 & \text{on } \pi^{-1}(U_{\hat{x}_2} \setminus U_{\hat{x}_1}); \\
  \vdots & \vdots \\
  f_k & \text{on } \pi^{-1}(U_{\hat{x}_k} \setminus \bigcup_{j=1}^{k-1} U_{\hat{x}_j}); \\
  0 & \text{on } \pi^{-1}(E'). 
\end{cases}
\]

Then \( g \) is a Borel function on \( G_0 \) such that \( 0 \leq g \leq 1, 1 - \varepsilon < \mu_s * \delta_{\sigma(\hat{x})}(g) \), and \( \delta_s * m_G(g) < \varepsilon \) for each \( \hat{x} \in E \) and \( x \in \pi^{-1}(\{\hat{x}\}) \). Hence we have

\[
\lambda_2(g) = \int_{G_0 / G} \mu_s * \delta_{\sigma(\hat{x})}(g) \, dm_{G_0 / G}(\hat{x}) > 1 - 2\varepsilon
\]

and

\[
m_{G_0}(g) = \int_{G_0} g \, dm_{G_0}
\]

\[
= \int_{G_0 / G} \int_{G} g(x + y) \, dm_{G}(y) \, dm_{G_0 / G}(\hat{x})
\]

\[
= \int_{E} \int_{G} g(x + y) \, dm_{G}(y) \, dm_{G_0 / G}(\hat{x})
\]

\[
< \varepsilon m_{G_0 / G}(E) \leq \varepsilon.
\]

Since this holds for each \( \varepsilon > 0 \), \( \lambda_2 \) is singular with respect to \( m_{G_0} \).

We quote the following lemma from Pigno and Saeki [8, Lemma 4].

**Lemma 3.5.** Let \( G \) be a nonmetrizable locally compact Abelian group, and let \( D \) be a \( \sigma \)-compact subset of \( G \) with \( m_G(D) = 0 \). Then, given a \( \sigma \)-compact subset \( \Delta \) of \( \hat{G} \), we can find a \( \sigma \)-compact, non-compact, open subgroup \( \Gamma \) of \( \hat{G} \) which contains \( \Delta \) and satisfies \( m_G(D + \Gamma^\perp) = 0 \).

**Theorem 3.6.** Let \( G \) be a compact Abelian group and let \( P \) be a subsemigroup of \( \hat{G} \) such that \( P \cup (-P) = \hat{G} \). If \( \mu \) is a measure in \( M_P(G) \), then \( \mu_s \) and \( \mu_s \) also belong to \( M_P(G) \).
PROOF. It suffices to show that \( \hat{\mu}_s(\gamma) = 0 \) for all \( \gamma \in P \). Since \( \mu_s \) is a singular measure, we can choose a \( \sigma \)-compact subset \( E \) of \( G \) such that \( m_\sigma(E) = 0 \) and \( |\mu_s|(E^c) = 0 \). Let \( \gamma_0 \) be any element of \( P \). By Lemma 3.5, there is a countable subgroup \( \Gamma \) of \( \hat{G} \) containing \( \gamma_0 \) such that

\[
m_\sigma(E + \Gamma^\perp) = 0.
\]

Let \( \pi \) be the natural homomorphism from \( G \) onto \( G/\Gamma^\perp \). By (1), we have

\[
(\pi(\mu))_s = \pi(\mu_s),
\]

where \( \pi(\mu) \) denotes the image of \( \mu \) under \( \pi : \pi(\mu)(A) = \mu(\pi^{-1}(A)) \) for Borel subsets \( A \) of \( G/\Gamma^\perp \). Write \( P' = P \cap \Gamma \). Then \( P' \) is a subsemigroup of \( \Gamma \) such that \( P' \cup (-P') = \Gamma \), and \( (\pi(\mu))'(\gamma') = 0 \) for all \( \gamma' \in P' \). (Recall that the dual group of \( G/\Gamma^\perp \) is \( \Gamma \) and therefore \( \hat{\mu}(\gamma) = (\pi(\mu))'(\gamma) \) for all \( \gamma \in \Gamma \).) Since \( \Gamma = (G/\Gamma^\perp)' \) is countable \( G/\Gamma^\perp \) is metrizable and hence (2) and Lemma 3.4 imply that \( (\pi(\mu_s))'(\gamma') = (\pi(\mu)_s)'(\gamma') = 0 \) for all \( \gamma' \in P' \). It follows that \( \hat{\mu}(\gamma_0) = (\pi(\mu))'(\gamma_0) = 0 \). Since \( \gamma_0 \) is an arbitrary element of \( P \), we have \( \hat{\mu}_s(\gamma) = 0 \) for all \( \gamma \in P \).

REMARK 3.7. Let \( \psi \) be as in Theorem B. If we put

\[
P = \{ \gamma \in \hat{G} ; \psi(\gamma) \leq 0 \}
\]

and apply Theorem 3.6, we obtain Theorem B.

Theorem 3.6 is strictly stronger than Theorem B. To see this, consider the compact group \( T^3 \) and its dual group \( Z^3 \). Let \( P \) be

\[
\{(x, y, z) \in Z^3 ; z > 0 \} \cup \{(x, y, 0) \in Z^3 ; x \geq 0 \}.
\]

Plainly \( P \) is a subsemigroup of \( Z^3 \) such that \( P \cup (-P) = Z^3 \). If \( \psi \) is a nonzero homomorphism of \( Z^3 \) into \( R \) nonnegative on \( P \), we have \( \psi((x, y, z)) = ax \) with \( a > 0 \). Thus \( \psi \) vanishes for all \( (x, y, 0) \) and

\[
P \subsetneq \psi^{-1}(\{ x \in R ; x \geq 0 \}).
\]

Thus Theorem B cannot prove Theorem 3.6.

4. Proof of Theorem C.

As we noted in Remark 2.1, we will prove Theorem C with \( M_p(G) \) replaced by \( M_{p^*}(G) \). We make use of the structure theorem for locally
compact Abelian groups (see [6, Theorem (24.30)]) and examine two cases.

We may suppose that \( \hat{G} \) is nondiscrete: otherwise the Theorem reduces to Theorem A. It suffices to show that if \( \mu \) is a measure in \( M_p(G) \), then \( \mu_s \) belongs to \( M_p(G) \). By the structure theorem, \( \hat{G} \) has the form \( \mathbb{R}^n \oplus X \), where \( n \) is a nonnegative integer and \( X \) is a locally compact Abelian group containing a compact open subgroup \( A \). We examine two cases.

**Case I: \( n = 0 \).** Since \( X = \hat{G} \) is nondiscrete, \( A \) is infinite, and \( \hat{G}/A \) is discrete. If we put \( H = A^\perp \), then \( H \) is a compact open subgroup of \( G \), and \( G/H \) is discrete. The dual group of \( G/H \) is of course \( H^\perp = A \). Let \( \mu \) be a measure in \( M_p(G) \). Since \( \mu \) has \( \sigma \)-compact support, there exists a sequence \( \{x_n\} \) of elements in \( G \) such that \( \mu = \sum_{n=1}^{\infty} \mu_{x_n + H} \) and \( x_i + H \neq x_j + H \) if \( i \neq j \), where \( \mu_{x_n + H} \) denotes the restriction of \( \mu \) to \( x_n + H \). Observe that

\[
||\mu|| = \sum_{n=1}^{\infty} ||\mu_{x_n + H}||.
\]

Put

\[
\lambda_n = \mu_{x_n + H} * \delta_{-x_n} \quad \text{for} \ n = 1, 2, \ldots
\]

so that \( \lambda_n \in M(H) \) and \( \mu = \sum_{n=1}^{\infty} \lambda_n * \delta_{x_n} \). We obtain

\[
\hat{\mu}(\gamma) = \sum_{n=1}^{\infty} \hat{\lambda}_n(\gamma)(-x_n, \gamma) \quad \text{for} \ \gamma \in \hat{G}.
\]

Since \( H \) is open in \( G \), it is obvious that \( \mu_s = \sum_{n=1}^{\infty} (\lambda_n)_s * \delta_{x_n} \), and so

\[
\hat{\mu}_s(\gamma) = \sum_{n=1}^{\infty} ((\lambda_n)_s)(\gamma)(-x_n, \gamma) \quad \text{for} \ \gamma \in \hat{G}.
\]

We will now show that if \( \gamma \in P \), then \( \hat{\lambda}_n(\gamma + A) = 0 \) for \( n = 1, 2, \ldots \). (Recall that the dual group of \( H \) is \( \hat{G}/A \).) As a measure in \( M(H) \), \( \lambda_n \) has a Fourier-Stieltjes transform constant on cosets of \( H^\perp = A \). Thus we may write \( \hat{\lambda}_n(\gamma + A) \) for \( \gamma \in \hat{G} \). For a fixed \( \gamma \) in \( P \), define

\[
v = \sum_{n=1}^{\infty} \hat{\lambda}_n(\gamma + A)(-x_n, \gamma) \delta_{x_n + H}.
\]

(This series converges in the total variation norm on \( M(G/H) = l^1(G/H) \) because of (1) and (2).) By (3), we have for every \( \gamma' \in A \)
\[ \hat{\vartheta}(\gamma') = \sum_{n=1}^{\infty} \hat{\lambda}_n(\gamma + A)(-x_n, \gamma)(-x_n + H, \gamma') \]
\[ = \sum_{n=1}^{\infty} \hat{\lambda}_n(\gamma + \gamma')(\gamma + \gamma') \]
\[ = \hat{\mu}(\gamma + \gamma'). \]

Since \( A \) is a compact infinite torsion-free Abelian group, \( P \cap A \) is dense in \( A \) (see [5, Theorem (3.2)]). Thus for \( \gamma' \in A \), there exists a net \( \{ \gamma_x \} \) in \( P \cap A \) such that \( \{ \gamma_x \} \) converges to \( \gamma' \). Since all \( \gamma + \gamma_x \) are in \( P \), we have \( \hat{\vartheta}(\gamma_x) = \hat{\mu}(\gamma + \gamma_x) = 0 \) for all \( x \). Since \( \hat{\vartheta} \) is continuous, we have

\[ \hat{\vartheta}(\gamma') = \lim_{x} \hat{\vartheta}(\gamma_x) = 0. \]

Since this holds for each \( \gamma' \in A \), \( \nu \) must be the zero measure, which is to say that \( \hat{\lambda}_n(\gamma + A) = 0 \) for \( n = 1, 2, \ldots \).

Now let \( \pi \) be the natural homomorphism from \( \hat{G} \) onto \( \hat{G}/A \) and put \( \bar{P} = \pi(P) \). Then \( \bar{P} \) is a subsemigroup of \( \hat{G}/A \) and \( \hat{G}/A = \bar{P} \cup (-\bar{P}) \). We have just shown that \( \hat{\lambda}_n(\gamma + A) = 0 \) for each \( \gamma + A \in \bar{P} \) and \( n = 1, 2, \ldots \). Theorem 3.6 implies that \( (\lambda_n)_n(\gamma + A) = 0 \) for each \( \gamma + A \in \bar{P} \) and \( n = 1, 2, \ldots \). From (4) we conclude that

\[ \hat{\mu}_n(\gamma) = \sum_{n=1}^{\infty} (\lambda_n)_n(\gamma + A)(-x_n, \gamma) \]
\[ = 0 \]

for each \( \gamma \in P \).

**Case II:** \( n > 0 \). We write elements of \( \mathbb{R}^n \oplus X \) as \( (a, \gamma) \) where \( a \in \mathbb{R}^n \) and \( \gamma \in X \). Define

\[ H = (\mathbb{Z}^n \oplus X)^{\perp} (= \mathbb{Z}^n \oplus \{0\}) \]

and put \( P' = P \cap (\mathbb{Z}^n \oplus X) \). Let \( \pi \) be the natural homomorphism from \( G = \mathbb{R}^n \oplus \hat{X} \) onto \( \mathbb{R}^n \oplus \hat{X}/\mathbb{Z}^n \oplus \{0\} \). Let \( \mu \) be a measure in \( M_{P'}(G) \). Fix an element \( (a_0, \gamma_0) \) in \( P \) and define \( \sigma = (-a_0, -\gamma_0)\mu \). Let \( \pi(\sigma) \) denote the image of \( \sigma \) under \( \pi: \pi(\sigma) \) is an element of \( M(\mathbb{R}^n \oplus \hat{X}/\mathbb{Z}^n \oplus \{0\}) \). We have

\[ (\pi(\sigma))((m, \gamma)) = \hat{\mu}((a_0, \gamma_0) + (m, \gamma)) \]
\[ = 0 \]
for all \((m, \gamma) \in P'\) because \((a_0, \gamma_0) + (m, \gamma) \in P\). Note that the dual group of \(\mathbb{R}^n \oplus \hat{\mathbb{R}}/\mathbb{Z}^n \oplus \{0\}\) is \((\mathbb{Z}^n \oplus \{0\})^I = \mathbb{Z}^n \oplus X\). Since \(\mathbb{Z}^n \oplus X\) is a group dealt with in Case I, we have

\[
(\pi(s))_a((m, \gamma)) = 0
\]

for all \((m, \gamma) \in P'\). The group \(\mathbb{Z}^n \oplus \{0\}\) is countable and so if \(E\) is a Borel subset of \(G\) with \(m_\mathcal{G}(E) = 0\), we have \(m_\mathcal{G}(E + (\mathbb{Z}^n \oplus \{0\})) = 0\). This implies that \(\pi(s)\) is singular. Since \(\pi(L^1(G)) = L^1(G/H)\) if \(\pi\) is the natural homomorphism of \(G\) onto \(G/H\), it follows that \(\pi(s) = (\pi(s))_a\).

Combine this with (5) to obtain

\[
\hat{\mu}_s((a_0, \gamma_0)) = \hat{\delta}_s((0, 0))
= (\pi(s))_a((0, 0))
= ((\pi(s))_a)^*((0, 0))
= 0.
\]

Since \((a_0, \gamma_0)\) is an arbitrary element of \(P\), we have \(\hat{\mu}_s((a, \gamma)) = 0\) for all \((a, \gamma) \in P\).

**Remark 4.1.** We may use Theorem C and the argument in the proof of Lemma 3.1 to obtain the following special case of Theorem D.

Let \(G\) be a locally compact Abelian group with torsion-free dual group \(\hat{G}\) and let \(P\) be a subsemigroup of \(\hat{G}\) such that \(P \cup (-P) = \hat{G}\). If \(\mu\) is a measure in \(M_{P'}(G)\), then \(\mu_a\) and \(\mu_s\) are also in \(M_{P'}(G)\).

5. **Proof of Theorem D.**

To prove Theorem D, we will make use of two fundamental facts about locally compact Abelian groups.

By [6, Theorem (A.15) and Theorem (25.32)(a)], we can find a divisible locally compact Abelian group \(D\) such that \(D\) contains \(G\) as an open subgroup. We define

\[
\bar{P} = \{\gamma \in \hat{D}; \gamma|G \in P\}.
\]

It is obvious that \(\bar{P}\) is a subsemigroup of \(\hat{D}\) and \(\hat{D} = \bar{P} \cup (-\bar{P})\). Let \(\mu\) be a measure in \(M_{P'}(G)\). It suffices to prove that \(\hat{\mu}_s(\gamma) = 0\) for all \(\gamma \in \bar{P}\). We will regard \(\mu\) as being a measure in \(M(D)\). Since \(G\) is open in \(D\), \(\mu_a\) and \(\mu_s\) are respectively the absolutely continuous and singular parts of \(\mu\) with respect
to $m_D$. Our first aim is to prove that $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \in \bar{P}$ when $\mu$ is regarded as a measure in $M(D)$. If $\gamma \in \bar{P}$, then $\gamma|G$ is in $P$ and therefore

$$\hat{\mu}(\gamma) = \int_D (-x, \gamma) d\mu(x)$$

$$= \int_G (-x, \gamma|G) d\mu(x)$$

$$= 0.$$

Since $\bar{D}$ is torsion-free (see [6, Theorem (24.23)]), Remark 4.1 gives us $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \in \bar{P}$.

Next we take an element $\gamma$ in $P$. There is an element $\gamma_0$ of $\bar{P}$ such that $\gamma_0|G = \gamma$. We find that

$$\hat{\mu}_s(\gamma) = \int_G (-x, \gamma) d\mu_s(x)$$

$$= \int_G (-x, \gamma_0|G) d\mu_s(x)$$

$$= \int_D (-x, \gamma_0) d\mu_s(x) = 0.$$

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