CLASSIFICATION OF ALGEBRAIC SURFACES
WITH SECTIONAL GENUS
LESS THAN OR EQUAL TO SIX.

III: RULED SURFACES WITH $\dim \varphi_{K_X \otimes L}(X) = 2$

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Introduction.

In this paper we have considered the problem of classifying biholomorphically smooth, connected, projective, ruled, non rational surfaces $X$ with smooth hyperplane section $C$ such that the genus $g = g(C)$ is less than or equal to six and $\dim \varphi_L(X) = 2$, where $\varphi_L$ is the map associated to $L = K_X \otimes L$. L. Roth in [12] had given a birational classification of such surfaces.

Let $L = [C]$ for some hyperplane section $C$. From the adjunction formula, see [5], we have that

$$2g - 2 = L \cdot (K_X + L)$$

where by $K_X$ we denote the canonical line bundle on $X$. If $g = 0$ or 1, then $X$ has been classified, see [10]. If $g = 2 \neq h^{1,0}(X)$, by [14, Lemma (2.2.2)] it follows that $X$ is a rational surface. Thus we can assume $g \geq 3$.

Since $X$ is ruled, $h^{2,0}(X) = 0$ and

$$(*) \quad \frac{L \cdot L}{8} + h^{1,0}(X) \leq \frac{g + 1}{2},$$

see [4] and [14, p. 390]. Moreover by the classification of surfaces in $P^2$ and $P^3$, it follows that $h^0(L) \geq 5$. Our classification is essentially based on the adjunction process which has been introduced by the Italian school and which has been particularly studied by A. J. Sommese [14]. Let $\varphi_L = r \circ s$ be the Remmert–Stein factorization of $\varphi_L$. When $\dim \varphi_T(X) = 2$, Sommese, in [14, p. 392], has proved that there exists a pair $(\hat{X}, \hat{L})$ such that:

(a) $X$ is obtained by blowing up a finite set $F$ of points on $\hat{X}$, $\pi: X \to \hat{X}$.
(b) Every smooth hyperplane section $C \in |L|$ is the proper transform of a hyperplane section $\hat{C} \in |\hat{L}|$.

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(c) $\hat{L}$ is ample and spanned off $F$.
(d) $\hat{L}$ is very ample if $H^1(X, L) = 0$.
(e) If $h^{1,0}(X) = 0$, then $\hat{s}$ is an embedding unless there is a smooth hyperelliptic $C \subset |L|$. This can happen only in the cases (2.5.1) and (2.5.2) of [14, p. 394].

Let $L' = K_{\hat{X}} \otimes \hat{L}$ and $\varphi_{L'} = \varphi_{K_{\hat{X}} \otimes L}$. Then $\varphi_{L'} = s$. We call $\hat{X}$ the minimal model of $X$ relative to $L$. It has the property that there is no irreducible curve $\mathcal{P} \subset \hat{X}$ such that $\mathcal{P} \cdot \mathcal{P} = -1$ and $\hat{L} \cdot \mathcal{P} = 1$. We call $(\hat{X}, \hat{L})$ the minimal pair.

Moreover by the construction of $\hat{X}$ in [14] it follows that $\hat{C}$ is smooth. Our main goal is to classify the pairs $(\hat{X}, \hat{L})$.

We shall mention that our classification has a slight overlap with the classification that P. Ionescu [6] has given for projective surfaces of sectional genus less than or equal to four. We have summarized our results in Table 1, where $e$ is, by [5], the invariant which characterizes $\mathcal{P}(E)$. We wish to thank Andrew J. Sommese for suggesting the problem and Alan Howard for helpful discussions about ruled surfaces.

0. Background material.

We have already fixed the meaning of $X, L, C, \hat{X}, \hat{L}, \hat{C}, \hat{L}$ and $L$. We would like to fix now the following notations.

We let $d = L \cdot L, g = g(C) = g(L), d' = \hat{L} \cdot \hat{L}, d' = L \cdot L, g' = g(L), c_1^2 = K_{\hat{X}} \cdot K_{\hat{X}}, c_2^2 = K_{\hat{X}} \cdot K_{\hat{X}}$.

(0.1) Proposition. Let $L$ be a line bundle on a smooth, connected, projective surface $X$. Then:

1. $d' = g' + g - 2$,
2. $dd' \leq 4(g - 1)^2$,
3. $d + d' = c_1^2 + 4(g - 1)$.

The proof follows using the adjunction formula [5, p. 361].

(0.2) Proposition. Let $X$ be a smooth, connected, projective surface embedded by a very ample line bundle $l$ into $\mathbb{P}^4$. Then

$$l \cdot l(l \cdot l - 5) - 10(g(l) - 1) + 12\chi(\mathcal{O}_X) = 2c_1^2.$$  

Proof. See [5, p. 434].
<table>
<thead>
<tr>
<th>$\dim \varphi_L(X)$</th>
<th>$g$</th>
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Table 1.
(0.3) **Proposition.** (Castelnuovo’s inequality [2, p. 234 ff ]; [4].) If $C$ is an irreducible curve embedded in $\mathbb{P}^{l-1}_C$ and $C$ belongs to no linear hyperplane $\mathbb{P}^{l-2}_C$, then with $d$ the degree of $C$ and $g$ the genus:

$$g \leq \left\lfloor \frac{d-2}{l-2} \right\rfloor \left( d - l + 1 - \left\lfloor \frac{d-l}{l-2} \right\rfloor \left( \frac{l-2}{2} \right) \right)$$

where $\left\lfloor \cdot \right\rfloor$ is the least integer function.

(0.4) **Proposition.** Let $X$ be any projective, smooth surface and let

$$0 \to E \to F \to G \to 0$$

be the short exact sequence obtained by tensoring the sequence

$$0 \to [C]^{-1} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

with a line bundle $F$, where $C$ is a curve in $X$. Suppose that:

(a) $G$ is a very ample line bundle on $C$,

(b) $E$ is very ample,

(c) $\ker(H^0(G) \to H^1(E))$ gives an embedding of $C$.

Then $F$ is very ample.

Since the proof is standard we will omit it.

(0.5) **Ruled Surfaces.** Let $X$ be a smooth, connected, projective, geometrically ruled surface, i.e. a fibration $\pi : X \to \bar{C}$, over a curve $\bar{C}$ whose fibres are $\mathbb{P}^1$. Then there exists a rank two vector bundle $E$ (not unique) over $\bar{C}$ and an isomorphism $X = P(E)$, where $P(E)$ denotes the associated projective bundle of $E$. Let $\bar{g}$ be the genus of $\bar{C}$. Let $\sigma$ be a minimal section of $\pi$, there is a line bundle $\mathcal{O}$ on $\bar{C}$ and an extension $E$ of $\mathcal{O}$ by $\mathcal{O}_C$

$$0 \to \mathcal{O}_C \to E \to \mathcal{O} \to 0$$

such that $X = P(E)$ and

$$\mathcal{O} = \sigma^* \mathcal{O}_{P(E)}(1) = \mathcal{O}_{\sigma(C)}(\zeta_E),$$

where $\zeta_E$ is the tautological line bundle.

$$e = - \zeta_E \cdot \zeta_E = - \deg \mathcal{O}$$

is an invariant of the surface $X$. If $E$ is decomposable, then $e \geq 0$ and all the values of $e$ are possible. If $E$ is indecomposable, then

$$-\bar{g} \leq e \leq 2\bar{g} - 2.$$

See [5, p. 376 and 384] and [11, p. 191].
Let \( f \) be a fiber of \( \pi : X \to \overline{C} \). Then every line bundle \( L \) on \( X \) is numerically equivalent to \( \zeta_E^a \otimes \mathcal{L}^b \), that is \( L \equiv \zeta_E^a \otimes \mathcal{L}^b \) for some integers \( a, b \) and \( \mathcal{L} = \mathcal{O}_X(f) \), so

\[
\text{deg } \mathcal{L}|_{\sigma(C)} = 1,
\]

\[
L \cdot L = -a^2 e + 2ab \quad \text{and} \quad 2g(L) - 2 = -a^2 e + ae + 2ab - 2b - 2a + 2a\bar{g}.
\] (0.5.2)

The canonical line bundle \( K_X \) of \( X \) is \( K_X \equiv \zeta_E^{-2} \otimes \mathcal{L}^{(2\bar{g} - 2 - e)} \). Given a line bundle \( \mathcal{U} \) on \( \overline{C} \) we will denote its lift \( \pi^* \mathcal{U} \) on \( X \) again by \( \mathcal{U} \). We have the following propositions:

(0.5.4) **Proposition.** Let \( X \) be a geometrically ruled surface over a curve \( \overline{C} \), with invariant \( e \geq 0 \).

(i) If \( Y \equiv a\zeta_E + b\mathcal{L} \) is an irreducible curve, \( Y \not\equiv \zeta_E, \mathcal{L} \), then \( a > 0 \), \( b \geq a \cdot e \).

(ii) A divisor \( D \equiv a\zeta_E + b\mathcal{L} \) is ample if and only if \( a > 0 \), \( b > a \cdot e \).

**Proof.** See [5, p. 382].

(0.5.5) **Proposition.** Let \( X \) be a geometrically ruled surface over a curve \( \overline{C} \), of genus \( \bar{g} \) and invariant \( e < 0 \).

(i) If \( Y \equiv a\zeta_E + b\mathcal{L} \) is an irreducible curve, \( Y \not\equiv \zeta_E, \mathcal{L} \), then either \( a = 1 \), \( b \geq 0 \) or \( a \geq 2 \), \( b \geq \frac{1}{2}ae \).

(ii) A divisor \( D \equiv a\zeta_E + b\mathcal{L} \) is ample if and only if \( a > 0 \), \( b > \frac{1}{2}ae \).

**Proof.** See [5, p. 382].

The determination of the very ample divisors on a ruled surface with \( \bar{g} \geq 1 \), is more difficult than in the case of a rational ruled surface, i.e. a Hirzebruch surface. There is moreover the following result which is stated as an exercise in [5, p. 385] and it is not too difficult to prove.

(0.5.6) **Proposition.** Let \( X \) be a geometrically ruled surface with invariant \( e \) over an elliptic curve \( e \). Let \( L \equiv \zeta_E \otimes \mathcal{L}^b \). Then

(i) \( L \) is spanned if and only if \( b \geq e + 2 \).

(ii) \( L \) is very ample if and only if \( b \geq e + 3 \).

(0.5.7) **Theorem.** Let \( X = \mathbb{P}(E) \) be a geometrically ruled surface over an elliptic curve \( e \). Then \( L \equiv \zeta_E^a \otimes \mathcal{L}^b \) is very ample if \( a \geq 1 \) and \( b \geq \max_{1 \leq k \leq a} \{3 + ke\} \).

**Proof.** See [8, Theorem (1.6)].
(0.5.8) **Proposition.** Let \( X \) be a geometrically ruled surface over a curve \( \overline{C} \) with \( \overline{g} = g(\overline{C}) \) and invariant \( e \). Let \( L \equiv \xi_E \otimes \mathcal{L}^b \) be a line bundle on \( X \) with \( a > -2 \). Then:

(i) \( h^1(L) = 0 \) for \( b > \begin{cases} ae + 2\overline{g} - 2 + e & \text{if } e \geq 0 \\ \frac{1}{2}ae + 2\overline{g} - 2 & \text{if } e < 0 \end{cases} \)

(ii) \( h^0(L) - h^1(L) = (a + 1)(b + 1 - \overline{g} - ae/2) \).

The proof is a direct application of the Kodaira Vanishing Theorem and the Riemann–Roch Theorem.

By a ruled surface we mean a surface birational to a geometrically ruled surface.

(0.6) **Proposition.** Let \( X \) be a smooth, connected surface and \( L \) an ample line bundle on it. Suppose that \( h^{2,0}(X) = 0 \) and \( L \cdot L = 2g - 2 \). Then \( K_X \) is trivial.

**Proof.** Use [14, p. 382].

(0.7) **Proposition.** Let \( X \) and \( L \) be as above. Suppose that \( L \cdot L = 2g - 2 \) and \( h^0(L|_C) = g \), where \( C \in |L| \).

Then \( K_X \) is trivial.

**Proof.** Use [14, p. 382].

(0.8) **Proposition.** Let \( L \) be an ample and spanned line bundle on a smooth, connected, projective surface \( X \). Assume \( h^0(L) \geq 4, \ L \cdot L \geq 5 \). Then \( K_X \otimes L \) is spanned.

**Proof.** See [15, Theorem (0.8)].

(0.9) **Theorem.** Let \( X \) be a smooth, connected, ruled surface and \( L \) be an ample and spanned line bundle on it. Let \( C \in |L| \), \( g = g(C) = g(L) = 2 \). Suppose that \( h^{1,0}(X) \neq 2 \) and that \( K_X \otimes L \) is spanned. Then \( X \) is rational.

**Proof.** By the first Lefschetz Theorem, see [1] or [3], \( h^{1,0}(X) \leq 2 \). Thus \( h^{1,0}(X) = 0 \) or 1. Consider the long cohomology sequence associated to the short exact sequence

\[
0 \to K_X \to K_X \otimes L \to K_C \to 0.
\]

The Kodaira Vanishing Theorem, [5], implies \( h^1(K_X \otimes L) = 0 \). By
definition \( h^0(K_C) = g = 2 \). Since \( K_X \otimes L \) is spanned, by restriction, \( K_C \) is also spanned. Therefore

\[
H^0(K_X \otimes L) \to H^0(K_C) \to 0
\]

is exact; otherwise the image of \( \alpha \) would have only one section and this would contradict the fact that \( K_C \) is spanned. Then by (1.1.1) it follows that \( h^{1,0}(X) = 0 \) and hence \( X \) is rational.

1. **The case of dim \( \varphi_L(X) = 2 \) and \( h^{1,0}(X) = 2 \).**

Since \( h^{1,0}(X) = 2 \), \( X \) is a ruled surface over a curve of genus two. By Theorem (0.9), \( g \geq 3 \). Let \( g = 3 \) and consider the long cohomology sequence of

\[
0 \to K_X \to K_X \otimes L \to K_C \to 0.
\]

By the facts that

(a) \( h^{2,0}(X) = 0 \), since \( X \) is a ruled surface,
(b) \( h^0(K_C) = g = 3 \),
(c) \( h^1(K_X \otimes L) = 0 \) by Kodaira Vanishing Theorem,
(d) \( h^{1,0}(X) = 2 \) by hypothesis,

it follows that \( h^0(K_X \otimes L) = 1 \) which contradicts the fact that \( K_X \otimes L \) is spanned by [14, p. 387]. Therefore \( g \geq 4 \).

Now consider \((\hat{X}, \hat{L})\). If \( g = 4 \), by (*) it follows that \( d \leq 4 \) which contradicts \( h^0(L) \geq 5 \) and Castelnuovo’s inequality. Therefore \( g = 5, 6 \).

Again by (*) if \( g = 5 \) and \( \hat{d} \geq 2g - 1 \), then \( h^{1,0}(X) \leq 1 \). Thus if \( g = 5 \), \( \hat{d} = 7 \) or \( \hat{d} = 8 \) and \( h^0(\hat{L}) = 5 \). If \( \hat{d} = 7 \) then, by degree consideration \( X = \hat{X}, L = \hat{L}, \hat{d} = 7 \), and \( h^0(\hat{L}) = h^0(\hat{L}) = 5 \). Therefore by Proposition

(0.2) we have \( c_1^2 = -16 \). Now applying Proposition (0.1) it follows that \( \hat{d}' = -7 \) which gives a contradiction. Now suppose that \( \hat{d} = 8 \). If \( X = \hat{X} \), then \( h^0(L) = h^0(\hat{L}) = 5 \) and by Proposition (0.2), \( c_1^2 = -14 \) which contradicts Proposition (0.1). If \( X \) is made by blowing up one point we get again a contradiction in the same way. Hence \( g = 6 \). Using the fact that \( \hat{d}' + g - h^{1,0}(X) - 2 \) and \( c_1^2 \leq -8 \) we obtain that \( \hat{d} \leq 10 \). So \( 7 \leq \hat{d} \leq 10 \). By Castelnuovo’s inequality if \( \hat{d} = 7, 8 \), then \( h^0(\hat{L}) = 5 \).

Let \( \hat{d} = 7 \). Then \( X = \hat{X}, L = \hat{L}, h^0(L) = 5, \hat{d} = 7 \). By Proposition (0.2), \( c_1^2 = -24 \) which contradicts Proposition (0.1). If \( \hat{d} = 8 \), we get contradictions in the same way in both the cases in which \( X = \hat{X} \) and \( X \) is made by blowing up one point. Therefore

\[
\hat{d} = 9, 10, h^0(\hat{L}) \geq 5.
\]
Using the fact that \( c_1^2 \leq -8 \) and Proposition (0.1), we get contradictions. Thus we can state the following theorem:

(1.1) **Theorem.** There is no smooth, connected, projective, ruled surface such that \( h^{1,0}(X) = 2 \), \( \dim \varphi_L(X) = 2 \) and \( g \leq 6 \).

2. **The case of** \( \dim \varphi_L(X) = 2 \) **and** \( h^{1,0}(X) = 1 \).

We would like to remind that \( h^0(\hat{L}) \geq h^0(L) \geq 5 \) and \( g = 3, \ldots, 6 \). By the long cohomology sequence of

\[
0 \to K_X \to K_X \otimes L \to K_C \to 0,
\]

it follows that \( g = 4, 5, 6 \) and by Castelnuovo’s inequality:

\[
g = 4 \Rightarrow d \geq 6, \\
g = 5, 6 \Rightarrow d \geq 7.
\]

(2.1) **Lemma.** Let \( X \) be a smooth, connected, projective surface such that \( h^{1,0}(X) = 1 \), \( h^{2,0}(X) = 0 \). Let \( L \) be an ample line bundle on it. Suppose that \( K_X \otimes L \) is ample, spanned and \( g' = g(K_X \otimes L) = 1 \). Then \( c_1^2 = 0 \).

**Proof.** [15, Corollary (3.4.2)], [16, Theorem (1.3)] or [7, Corollary (2.4)].

(2.2) **Proposition.** Let \( X \) be a smooth, connected, projective, ruled surface such that \( h^{1,0}(X) = 1 \), \( \dim \varphi_L(X) = 2 \), and \( \hat{d} = 2g - 2 \). Then if \( (\hat{X}, \hat{L}) \) exists it has to satisfy the following invariants,

\[
g = 6, \quad \hat{d} = 10, \quad d' = 9, \quad g' = 5, \quad \hat{c}_1^2 = -1, \quad h^0(\hat{L}) = 6.
\]

**Proof.** Since \( \hat{d} = 2g - 2 \) using Clifford’s Theorem, Riemann–Roch’s Theorem, Proposition (0.7) and the long cohomology sequence of

\[
0 \to \mathcal{O}_X \to \hat{L} \to \hat{L}|_C \to 0,
\]

we have that \( h^0(\hat{L}) \leq g \). Therefore, using the fact that \( h^0(\hat{L}) \geq 5 \) we have that \( g = 5 \) or \( 6 \). Assume that \( g = 5 \). Then \( h^0(\hat{L}) = 5 \). By Propositions (0.1), (2.1) and Theorem (0.9), we obtain the following invariants:

\[
d' = 6, \quad g' = 3, \quad \hat{c}_1^2 = -2, \\
d' = 7, \quad g' = 4, \quad \hat{c}_1^2 = -1, \\
d' = 8, \quad g' = 5, \quad \hat{c}_1^2 = 0.
\]

Since \( h^0(L) \geq 5 \), by Castelnuovo’s inequality \( d \geq 7 \). Suppose that \( X = \hat{X} \), that is \( L = \hat{L} \) and \( d = \hat{d} = 8 \). Then by Proposition (02),
c_1^2 = -8 which gives a contradiction. Now suppose that X is obtained by blowing up one point on \( \hat{X} \). Then \( h^0(L) = 5 \) and \( d = 7 \). By Proposition (0.2) we have that \( c_1^2 = -13 \) which contradicts the values that we have obtained for \( \hat{c}_1^2 \). Therefore \( g \neq 5 \). It remains to examine the case in which \( g = 6 \). As we have seen \( h^0(\hat{L}) = 5 \) or 6. Exactly as in the case \( g = 5 \) we obtain the following set of invariants:

\[
\begin{align*}
d' &= 7, \quad g' = 3, \quad \hat{c}_1^2 = -3, \\
d' &= 8, \quad g' = 4, \quad \hat{c}_1^2 = -2, \\
d' &= 9, \quad g' = 5, \quad \hat{c}_1^2 = -1, \\
d' &= 10, \quad g' = 6, \quad \hat{c}_1^2 = 0.
\end{align*}
\]

As in the case in which \( g = 5 \) we see that \( h^0(\hat{L}) \neq 5 \). Thus \( h^0(\hat{L}) = 6 \). Now consider the first set of invariants.

\[
(K_X + L) \cdot (K_X + L) = -2,
\]

which contradicts the fact that \( K_X \otimes L \) is spanned by Proposition (0.8). Also in the last case we obtain a contradiction using the formula

\[
(2.2.1) \quad t(2h^{1,0}(X) - 2) + \frac{t-1}{t} d = 2g - 2,
\]

which is obtained for ruled surfaces which are minimal models using the adjunction formula and the Hurwitz formula, see [5].

Now consider the second set of invariants. By the long cohomology sequence of

\[
(2.2.2) \quad 0 \to K_X \to K_X \otimes L' \to K_C \to 0,
\]

we have that \( h^0(K_X \otimes L) = 3 \). Moreover

\[
(K_X + L') \cdot (K_X + L') = 2.
\]

Since \( \varphi_{K_X \otimes L} \) cannot be an embedding, it follows that it gives a 2:1 branched cover of \( P^2 \). Thus we have a contradiction since 2:1 branched covers of \( P^2 \) have first Betti numbers zero.

(2.3) **Theorem.** Let \((\hat{X}, \hat{L})\) be a minimal pair of a smooth, connected, projective, ruled surface. Suppose that \((\hat{X}, \hat{L})\) satisfy the invariants:

\[
g = 6, \quad d = 10, \quad d' = 9, \quad g' = 5, \\
\hat{c}_1^2 = -1, \quad h^0(\hat{L}) = 6, \quad h^{1,0}(X) = 1.
\]
Then, if \((\hat{X}, \hat{L})\) exists, it has to be made by blowing up one point on a geometrically ruled surface over an elliptic curve such that either \(e = 0\) and
\[
\hat{L} \equiv \zeta_E^{11} \otimes \mathcal{L}^5 \otimes [\mathcal{P}]^{-10} \quad \text{or} \quad e = -1 \quad \text{and} \quad \hat{L} \equiv \zeta_E^7 \otimes \mathcal{L}^{-1} \otimes [\mathcal{P}]^{-5},
\]
where \(\mathcal{P}\) is the irreducible line on \(\hat{X}\) that we obtain, when we blow up a point on a minimal model.

**Proof.** Since \(\zeta_1^2 = -1\), the surface has to be made by blowing up one point over a minimal model. Hence
\[
\hat{L} \equiv \zeta_E^a \otimes \mathcal{L}^b \otimes [\mathcal{P}]^r.
\]
Since the surface is a minimal model relative to \(L\) and \(\mathcal{P} \cdot \mathcal{P} = -1\) we have that \(\hat{L} \cdot \mathcal{P} \geq 2\), that is
\[
2 \leq (a \zeta_E + b \mathcal{L} + r \mathcal{P}) \cdot \mathcal{P} = -r.
\]
Since \(\dim \varphi_L(X) = 2\) we have that \(\hat{L} \cdot f \geq 3\). Hence
\[
3 \leq (a \zeta_E + b \mathcal{L} + r \mathcal{P}) \cdot f = a.
\]
Since \(\varepsilon\) is an elliptic curve and \(\hat{L}\) is ample
\[
\hat{L} \cdot \zeta_E = (a \zeta_E + b \mathcal{L} + r \mathcal{P}) \cdot \zeta_E = -ae + b \geq 1.
\]
Moreover \(K_{\hat{X}} \equiv \zeta_E^{-2} \otimes \mathcal{L}^{-e} \otimes [\mathcal{P}]\), so
\[
K_{\hat{X}} \otimes \hat{L} \equiv \zeta_E^{a-2} \otimes \mathcal{L}^{b-e} \otimes [\mathcal{P}]^{r+1}.
\]
Therefore we have the following system:
\[
r \leq -2, \quad a \geq 3.
\]

(i) \(ae - b \leq -1\),
(ii) \(d = -a^2 e + 2ab - r^2\),
(iii) \(2g - 2 = d + ae - 2b - r\),
(iv) \(d' = -a^2 e + 2ae + 2ab - 4b - r^2 - 2r - 1\),
(v) \(2g' - 2 = d' + ae - 2b - r - 1\).

Using (i) and (iii) it follows that
\[
(2.4.1) \quad b \leq -1 - r.
\]
Again by (i)
\[
e \leq b - 1 \over a.
\]
By (2.4.1)

(2.4.2) \[ e \leq \frac{b - 1}{a} \leq \frac{-2 - r}{a}. \]

Now we write (ii) as

\[ 10 + r^2 = -a^2 e + 2ab = a(-ae + 2b). \]

By (iii) the above equality becomes

\[ -ar = 10 + r^2, \]

which implies

\[ a = \frac{10 + r^2}{-r}. \]

Substituting in (2.4.2) we get

\[ e \leq \frac{-2 - r}{a} = \frac{(-2 - r)(-r)}{10 + r^2}. \]

Since \( r^2 < r^2 + 10 \) and \( r \leq -2 \) we have that

\[ r^2 + 2r < r^2 + 10, \]

which implies that \( e < 1 \), that is \( e = -1 \) or 0. Let \( e = 0 \). By (v)

\[ 2b = -r. \]

Substituting in (iv) we obtain

\[ 10 = -r(a + r). \]

Therefore we have the following cases:

(A) \quad a = 7, \quad r = -2, \quad b = 1,

(B) \quad a = 11, \quad r = -10, \quad b = 5,

that is, either

\[ \hat{L} \equiv \zeta_{E}^{7} \otimes \mathcal{L} \otimes [\mathcal{P}]^{-2} \quad \text{or} \quad \hat{L} \equiv \zeta_{E}^{11} \otimes \mathcal{L}^{5} \otimes [\mathcal{P}]^{-10}. \]

Since by [14, p. 393], \( \hat{L}|_{\zeta_{E}} \) has to be very ample we see that case (A) is not possible. Now let \( e = -1 \). By (iii)

\[ a = -2b - r. \]
Substituting in (iv)

\[ br = 5. \]

Since \( r \leq -2 \), and, by (i), \( b \geq -2 \) we have that

\[ a = 7, \quad b = -1, \quad r = -5 \]

that is \( \hat{L} \equiv \zeta_E^7 \otimes \mathcal{L}^{-1} \otimes \mathcal{P}^{-5} \).

(2.5) **Lemma.** There is no geometrically ruled surface \((\hat{X}, \hat{L})\) over an elliptic curve such that \( \hat{L} \) is ample and \( \hat{d} \leq 2g - 2 \).

**Proof.** Use (0.5.3) and Propositions (0.5.4) and (0.5.5).

Now we assume \( \hat{d} \leq 2g - 3 \). We get the following proposition:

(2.6) **Proposition.** There is no smooth, connected, projective, ruled surface such that \( h^{1,0}(X) = 1 \), \( \dim \varphi_L(X) = 2 \), \( \hat{d} \leq 2g - 3 \), \( g = 4, 5 \). In the case in which \( g = 6 \), \((\hat{X}, \hat{L})\) has to satisfy one of the following sets of invariants:

(1) \[ \hat{d} = 9, \quad h^0(\hat{L}) = 6, \quad d' = 9, \quad g' = 5, \quad \hat{c}_1^2 = -2, \]

(2) \[ \hat{d} = 9, \quad h^0(\hat{L}) = 6, \quad d' = 10, \quad g' = 6, \quad \hat{c}_1^2 = -1. \]

**Proof.** Since \( h^0(\hat{L}) \geq 5 \), by Castelnuovo’s inequality \( g \neq 4 \). If \( g = 5 \), then \( \hat{d} = 7, \quad h^0(\hat{L}) = 5 \). If \( g = 6 \), then \( \hat{d} = 7, 8, \quad h^0(\hat{L}) = 5 \), and \( \hat{d} = 9, \quad h^0(\hat{L}) = 5, 6 \). In the case in which \( g = 5 \), by degree considerations \( X = \hat{X} \). Thus \( \hat{L} \) is very ample and we have a contradiction using Propositions (0.2) and (0.1). Now let \( g = 6 \). In the case in which \( \hat{d} = 7 \) by Castelnuovo’s inequality since \( h^0(L) \geq 5 \) we have \( X = \hat{X} \) and we get a contradiction as before. If \( \hat{d} = 8 \), then we can blow up at most one point. Thus \( \hat{d} = 7, \quad h^0(L) = 5 \) and we get again a contradiction as before. If \( X = \hat{X} \), that is \( \hat{d} = 8 \), then \( \hat{L} = L \) is very ample, \( h^0(L) = 5 \) and we get, in the same way, a contradiction.

Now consider the case in which \( \hat{d} = 9 \) and \( h^0(\hat{L}) = 5 \). Again by Castelnuovo’s inequality, since \( h^0(L) \geq 5 \), we can blow up at most two points. If \( \hat{d} = 7, 8 \) we have contradictions as before. If \( X = \hat{X} \), that is \( \hat{d} = 9 \), then by Propositions (0.2) and (0.1) we have \( c_1^2 = -7, \quad d' = 4 \) and \( g' = 0 \) which implies that \( X \) is rational. It remains to consider the case in which \( \hat{d} = 9 \) and \( h^0(\hat{L}) = 6 \). By Propositions (0.1), (0.7) and Theorem (0.9) we obtain the following sets of invariants:
(A) \quad d' = 7, \quad g' = 3, \quad \hat{c}_1^2 = -4,

(B) \quad d' = 8, \quad g' = 4, \quad \hat{c}_1^2 = -3,

(C) \quad d' = 9, \quad g' = 5, \quad \hat{c}_1^2 = -2,

(D) \quad d' = 10, \quad g' = 6, \quad \hat{c}_1^2 = -1,

(E) \quad d' = 11, \quad g' = 7, \quad \hat{c}_1^2 = 0.

By Lemma (2.5), case (E) does not happen. By the long cohomology sequence of

\[ 0 \to K_X \to K_X \otimes L \to K_X \otimes L|_{C'} \to 0 \]

obtained by tensoring with \( K_X \) the short exact sequence

\[ 0 \to \mathcal{O}_X \to L \to L|_{C'} \to 0, \]

where \( C' \in |L| \), we have that, in case (B),

\[ h^0(K_X \otimes L) = 3. \]

Moreover

\[ (K_X + L) \cdot (K_X + L) = \hat{c}_1^2 + 4g' - 4 - d' = 1, \]

which implies that \( \hat{X} = P^2 \). Therefore case (B) cannot happen either. In case (A) we get a contradiction, since

\[ (K_X + L) \cdot (K_X + L) = -3. \]

Therefore (C) and (D) are the only possible cases.

Now consider the case in which \( \hat{c}_1^2 = -1 \), that is \( \hat{X} \) is made by blowing up one point on a geometrically ruled surface over an elliptic curve. Then the system is:

\[ r \leq -2, \quad a \geq 3 \]

(i) \quad ae - b \leq -1,

(ii) \quad 9 = -a^2 e + 2ab - r^2,

(iii) \quad 10 = 9 + ae - 2b - r,

(iv) \quad 10 = -a^2 e + 2ae + 2ab - 4b - r^2 - 2r - 1,

(v) \quad 10 = 10 + ae - 2b - r - 1.

By (i) and (iii) it follows

\[ (2.6.1) \quad b \leq -r - 2. \]

Again by (i)

\[ e \leq \frac{b - 1}{a}. \]
Thus, using (2.6.1)

\[(2.6.2) \quad e \leq \frac{-r - 3}{a}.\]

By (ii) and (iii) we obtain

\[a = \frac{r^2 + 9}{-r - 1}.\]

Substituting in (2.6.2) we get

\[e \leq \frac{r^2 + 4r + 3}{r^2 + 9}.\]

By \(r^2 < r^2 + 9\) and \(r \leq -2\) it follows that \(e < 1\), that is \(e = -1, 0\).

\[(2.7) \text{ Remark.} \quad \text{If there exist smooth, connected, projective, ruled surfaces} \ (\hat{X}, \hat{L}) \text{ with} \ g = 6 \text{ and} \ h^{1,0}(\hat{X}) = 1 \text{ which satisfy the invariants:} \]

\[d = 9, \quad d' = 10, \quad g' = 6, \quad \hat{c}_1^2 = -1.\]

then \(\hat{X}\) is made by blowing up one point on a geometrically ruled surface over an elliptic curve with invariant \(e = -1, 0\).

Now consider the case in which \(\hat{c}_1^2 = -2\). In this case \(\hat{X}\) is made by blowing up two points on a minimal model. Let \(\mathcal{P}_1\) and \(\mathcal{P}_2\) denote the irreducible lines on \(\hat{X}\) that we obtain, when we blow up two points on a minimal model.

We have that either

\[\mathcal{P}_1 \cdot \mathcal{P}_1 = \mathcal{P}_2 \cdot \mathcal{P}_2 = -1, \quad \mathcal{P}_1 \cdot \mathcal{P}_2 = 0,\]

or

\[\mathcal{P}_1 \cdot \mathcal{P}_1 = -2, \quad \mathcal{P}_2 \cdot \mathcal{P}_2 = -1, \quad \mathcal{P}_1 \cdot \mathcal{P}_2 = +1.\]

In the first case we have:

\[\hat{L} \equiv \zeta_E^2 \otimes \mathcal{L}^b \otimes [\mathcal{P}_1]^{r_1} \otimes [\mathcal{P}_2]^{r_2};\]

\[\hat{K}_\hat{X} \equiv \zeta_E^{-2} \otimes \mathcal{L}^e \otimes [\mathcal{P}_1] \otimes [\mathcal{P}_2]\]

and

\[\hat{K}_\hat{X} \otimes \hat{L} \equiv \zeta_E^{-2} \otimes \mathcal{L}^{b-e} \otimes [\mathcal{P}_1]^{r_1+1} \otimes [\mathcal{P}_2]^{r_2+1}.\]
Thus:

\[ r_1 \leq -2, \quad r_2 \leq -2, \quad a \geq 3, \]

(i) \[ ae - b \leq -1, \]
(ii) \[ 9 = -a^2 e + 2ab - r_1^2 - r_2^2, \]
(iii) \[ 10 = 9 + ae - 2b - r_1 - r_2, \]
(iv) \[ 9 = -a^2 e + 2ae + 2ab - 4b - r_1^2 - r_2^2 - 2r_1 - 2r_2 - 2, \]
(v) \[ 8 = 9 + ae - 2b - r_1 - r_2 - 2. \]

By (i) and (iii) it follows

\[ b \leq -r_1 - r_2 - 2. \]

Again by (i) we have

\[ e \leq \frac{-1 + b}{a}. \]

Using (2.7.1) we get

\[ e \leq \frac{-r_1 - r_2 - 3}{a}. \]

By (ii) and (iii) we obtain

\[ a = \frac{9 + r_1^2 + r_2^2}{-r_1 - r_2 - 1}. \]

Substituting in (2.7.2) we get

\[ e \leq \frac{r_1^2 + r_2^2 + 2r_1 r_2 + 2r_1 + 2r_2 + 7}{r_1^2 + r_2^2 + 9}. \]

Since \( r_1 \leq -2 \) for \( i = 1, 2 \), again by Schwartz's Lemma, it follows that

\[ e \leq 2 \cdot \frac{(r_1^2 + r_2^2 + r_1 + r_2 + \frac{7}{2})}{r_1^2 + r_2^2 + 9}. \]

Hence \( e = -1, 0, 1 \).

Again in the second case we get \( e = -1, 0, 1 \).

We would like to state the following

(2.8) **Remark.** If there exist smooth, connected, ruled surfaces \((\hat{X}, \hat{L})\) with \( g = 6 \), \( \dim \varphi_L(X) = 2 \), and \( h^{1,0}(\hat{X}) = 1 \), which satisfy the invariants:

\[ d = 9, \quad d' = 9, \quad g' = 5, \quad c_1^2 = -2, \]
then $\tilde{X}$ is made by blowing up two points on a geometrically ruled surface over an elliptic curve with invariant $e = -1, 0, 1$.

Finally we can assume $d \geq 2g - 1$. Let $g = 6$. By

$$(K_{\tilde{X}} + \tilde{L}) \cdot (K_{\tilde{X}} + \tilde{L}) \geq g + h^{2,0}(X) - h^{1,0}(X) - 2,$$

it follows that $d \leq 17$. By the long cohomology sequence of

$$0 \to \mathcal{O}_{\tilde{X}} \to \tilde{L} \to \tilde{L}|_{\mathcal{C}} \to 0,$$

and by Riemann–Roch's Theorem, $h^1(\tilde{L}) = 0$ or 1 and $h^0(\tilde{L}) \geq 6$. By the long cohomology sequence of

$$0 \to K_{\tilde{X}} \to K_{\tilde{X}} \otimes \tilde{L} \to K_{\mathcal{C}} \to 0,$$

it follows that $h^0(K_{\tilde{X}} \otimes \tilde{L}) = 5$. In the same way we have that:

- if $g = 5$, $d \leq 14$, $h^1(\tilde{L}) = 0, 1$, $h^0(\tilde{L}) \geq 5$, $h^0(K_{\tilde{X}} \otimes \tilde{L}) = 4$,

- if $g = 4$, $d \leq 11$, $h^1(\tilde{L}) = 0, 1$, $h^0(\tilde{L}) \geq 5$, $h^0(K_{\tilde{X}} \otimes \tilde{L}) = 3$.

Let $g = 6$. By Propositions (0.1), (2.1) and Theorem (0.9) we have that:

- $d = 11$, $d' = 7$, $\hat{c}_1^2 = -2$, $g' = 3$,
- $d = 11$, $d' = 8$, $\hat{c}_1^2 = -1$, $g' = 4$,
- $d = 11$, $d' = 9$, $\hat{c}_1^2 = 0$, $g' = 5$,
- $d = 12$, $d' = 7$, $\hat{c}_1^2 = -1$, $g' = 3$,
- $d = 12$, $d' = 8$, $\hat{c}_1^2 = 0$, $g' = 4$,
- $d = 13$, $d' = 7$, $\hat{c}_1^2 = 0$, $g' = 3$.

Let $g = 5$. In the same way we have:

- $d = 9$, $d' = 6$, $\hat{c}_1^2 = -1$, $g' = 3$,
- $d = 9$, $d' = 7$, $\hat{c}_1^2 = 0$, $g' = 4$,
- $d = 10$, $d' = 6$, $\hat{c}_1^2 = 0$, $g' = 3$,
- $d = 12$, $d' = 4$, $\hat{c}_1^2 = 0$, $g' = 1$.

Let $g = 4$. In the same way we have:

- $d = 7$, $d' = 5$, $\hat{c}_1^2 = 0$, $g' = 3$,
- $d = 9$, $d' = 3$, $\hat{c}_1^2 = 0$, $g' = 1$. 

Now consider the cases in which $\hat{X}$ is a minimal model, i.e. $\hat{c}_1^2 = 0$. We have obtained the following cases:

\begin{align*}
g = 6, & \quad d = 11, \quad d' = 9, \quad g' = 5, \\
d = 12, & \quad d' = 8, \quad g' = 4, \\
d = 13, & \quad d' = 7, \quad g' = 3, \\
d = 15, & \quad d' = 5, \quad g' = 1, \\
g = 5, & \quad d = 9, \quad d' = 7, \quad g' = 4, \\
d = 10, & \quad d' = 6, \quad g' = 3, \\
d = 12, & \quad d' = 4, \quad g' = 1, \\
g = 4, & \quad d = 7, \quad d' = 5, \quad g' = 3, \\
d = 9, & \quad d' = 3, \quad g' = 1.
\end{align*}

Let $g = 6, \quad d = 11, \quad d' = 9, \quad g' = 5$. By (0.5.3),

\[ a = 11, \quad b = \frac{11e + 1}{2}. \]

By Propositions (0.5.4) and (0.5.5) we get that $e = -1, \quad a = 11, \quad b = -5$. Let $g = 6, \quad d = 12, \quad d' = 8, \quad g' = 4$. By (0.5.3)

\[ a = 6, \quad b = \frac{6e + 2}{2}. \]

As before we get $e = -1, \quad a = 6, \quad b = -2$. Let $g = 6, \quad d = 13$. As before we get $a = \frac{13}{3}$ which is a contradiction. Let $g = 6, \quad d = 15$. Then:

\begin{align*}
e = 0, & \quad a = 3, \quad b = \frac{5}{2}, \quad \text{contradiction}, \\
e = 1, & \quad a = 3, \quad b = 4, \\
e = 2, & \quad a = 3, \quad b = \frac{11}{2}, \quad \text{contradiction}, \\
e = -1, & \quad a = 3, \quad b = 1.
\end{align*}

Let $g = 5, \quad d = 9$. Then

\[ e = -1, \quad a = 9, \quad b = -4. \]

Let $g = 5, \quad d = 10$. Then

\[ e = -1, \quad a = 5, \quad b = -\frac{3}{2}, \quad \text{contradiction}. \]
Let \( g = 5, \; d = 12 \). Then
\[
\begin{align*}
  e = 0, & \quad a = 3, \quad b = 2, \\
  e = 1, & \quad a = 3, \quad b = \frac{7}{2}, \text{ contradiction}, \\
  e = -1, & \quad a = 3, \quad b = \frac{1}{2}, \text{ contradiction}.
\end{align*}
\]

Let \( g = 4, \; d = 7 \). Then
\[
\begin{align*}
  e = -1, & \quad a = 7, \quad b = -3.
\end{align*}
\]

Let \( g = 4, \; d = 9 \). Then
\[
\begin{align*}
  e = 0, & \quad a = 3, \quad b = \frac{3}{2}, \text{ contradiction}, \\
  e = -1, & \quad a = 3, \quad b = 0.
\end{align*}
\]

Since by [14, p. 392], \( \mathcal{L}_k \) has to be very ample, the cases \( g = 6, \; d = 15, \; \mathcal{L} \equiv \zeta_k^2 \otimes \mathcal{L}^4, \; e = 1 \) and \( g = 5, \; d = 12, \; \mathcal{L} \equiv \zeta_k^2 \otimes \mathcal{L}^2, \; e = 0 \), cannot happen. Since by Proposition (0.5.8) we can compute \( h^0(\mathcal{L}) \), we see that in the case \( g = 4, \; d = 7 \) it follows that \( h^0(\mathcal{L}) = 4 \) which contradicts \( h^0(\mathcal{L}) \geq 5 \). Thus we can state the following proposition:

(2.9) PROPOSITION. Let \( X \) be a smooth, connected, projective, ruled surface and \( L \) a very ample line bundle on it. Suppose that \( \bar{X} \) is a minimal model, \( h^{1,0}(X) = 1, \; \dim \varphi_L(X) = 2 \) and \( \bar{d} \geq 2g - 1 \). Then \((\bar{X}, \mathcal{L})\) has to be one of the following surfaces:

\begin{enumerate}
  \item \( e = -1, \quad \mathcal{L} \equiv \zeta_k^{11} \otimes \mathcal{L}^{-5}, \; g = 6, \; d = 11, \; d' = 9, \; g' = 5, \quad h^0(\zeta_k^{11} \otimes \mathcal{L}^{-5}) = 6, \)
  \item \( e = -1, \quad \mathcal{L} \equiv \zeta_k^2 \otimes \mathcal{L}^{-2}, \; g = 6, \; d = 12, \; d' = 8, \; g' = 4, \quad h^0(\zeta_k^2 \otimes \mathcal{L}^{-2}) = 7, \)
  \item \( e = -1, \quad \mathcal{L} \equiv \zeta_k^3 \otimes \mathcal{L}, \; g = 6, \; d = 15, \; d' = 5, \; g' = 1, \quad h^0(\zeta_k^3 \otimes \mathcal{L}) = 10. \)
  \item \( e = -1, \quad \mathcal{L} \equiv \zeta_k^2 \otimes \mathcal{L}^{-4}, \; g = 5, \; d = 9, \; d' = 7, \; g' = 4, \quad h^0(\zeta_k^2 \otimes \mathcal{L}^{-4}) = 5. \)
  \item \( e = -1, \quad \mathcal{L} \equiv \zeta_k^3 \otimes \mathcal{L}, \; g = 4, \; d = 9, \; d' = 3, \; g' = 1, \quad h^0(\zeta_k^3 \otimes \mathcal{L}) = 6. \)
\end{enumerate}

REMARK. We do not know if those \( \mathcal{L} \) are very ample.

Now consider the case when \( \bar{X} \) is made by blowing up one point over a geometrically ruled surface over an elliptic curve, i.e. when \( \bar{c}_1^2 = -1 \). We have to examine the following cases:
(A) \( g = 6, \ d = 11, \ d' = 8, \ g' = 4, \)
(B) \( g = 6, \ d = 12, \ d' = 7, \ g' = 3, \)
(C) \( g = 5, \ d = 9, \ d' = 6, \ g' = 3. \)

In case (C) as usual we compute that \( h^0(K_X \otimes L) = 2 \) and 
\[
(K_X + L) \cdot (K_X + L) = 1,
\]
which gives a contradiction, since if \( \dim \varphi_{K_X \otimes L}(X) = 1 \), then
\[
(K_X + L) \cdot (K_X + L) = 0.
\]

Thus we have to examine the usual systems in cases (A) and (B).

\[
r \leq -2, \ a \geq 3,
\]

(i) \( ae - b \leq -1, \)
(ii) \( d = -a^2 e + 2ab - r^2, \)
(iii) \( 2g - 2 = d + ae - 2b - r, \)
(iv) \( d' = -a^2 e + 2ae + 2ab - 4b - r^2 - 2r - 1, \)
(v) \( 2g' - 2 = d' + ae - 2b - r - 1. \)

As usual \( e = -1, 0, 1. \) Consider case (A). By (v) we get

\[
(2.9.1) \quad \text{if } e = 0, \quad 2b = -r + 1, \\
(2.9.2) \quad \text{if } e = -1, \quad 2b = -a - r + 1, \\
(2.9.3) \quad \text{if } e = 1, \quad 2b = a - r + 1.
\]

Substituting (2.9.1), (2.9.2), and (2.9.3) in (ii) or (iv) we get:

\[
11 = -ae + a - e^2.
\]

Consider case (B). By (v) we get

\[
(2.9.4) \quad \text{if } e = 0, \quad 2b = -r + 2, \\
(2.9.5) \quad \text{if } e = -1, \quad 2b = -a - r + 2, \\
(2.9.6) \quad \text{if } e = 1, \quad 2b = a - r + 2.
\]

Substituting (2.9.4), (2.9.5), and (2.9.6) in (ii) we get:

\[
12 = -ar + 2a - r^2.
\]

We can state the following lemma.

\[2.10 \text{ LEMMA. Let } (\hat{X}, \hat{L}) \text{ be a minimal pair of a smooth, connected, projective, ruled surface such that } \hat{X} \text{ is made by blowing up one point over a geometrically ruled surface over an elliptic curve with invariant } e. \text{ Suppose that } d \geq 2g - 1. \text{ Then } g = 6 \text{ and } (\hat{X}, \hat{L}) \text{ has to be one of the following:} \]
(1) \( e = -1, \ 0, 1, \ d = 11, \ d' = 8, \ g' = 4, \)

(2) \( e = -1, \ 0, 1, \ d = 12, \ d' = 7, \ g' = 3. \)

Now consider the case in which \( \hat{X} \) is made by blowing up two points over a geometrically ruled surface over an elliptic curve, that is \( \hat{c}_1^2 = -2. \) We have that

\[ g = 6, \ d = 11, \ d' = 7, \ g' = 3. \]

As in the previous case we have that

\[ h^0(K_{\hat{X}} \otimes L) = 2 \text{ and } (K_{\hat{X}} + L) \cdot (K_{\hat{X}} + L) = -1 \]

which gives a contradiction, since \( K_{\hat{X}} \otimes L \) is spanned by Proposition (0.8). We can finally state the following theorem.

(2.11) **Theorem.** Let \((\hat{X}, \hat{L})\) be a minimal pair of a smooth, connected, projective, ruled surface such that \( \hat{d} \geq 2g - 1. \) Then the pair \((\hat{X}, \hat{L})\), if it exists, has to satisfy one of the following sets of invariants:

(1) \( g = 6, \ d = 11, \ d' = 8, \ g' = 4, \ \hat{c}_1^2 = -1, \ e = -1, 0, 1, \)

(2) \( g = 6, \ d = 12, \ d' = 7, \ g' = 3, \ \hat{c}_1^2 = -1, \ e = -1, 0, 1, \)

where \( e \) is the invariant of the minimal model.

Moreover if \( \hat{X} \) is a minimal model then it has to be one of the following:

(3) \( g = 6, \ e = -1, \ \hat{L} \equiv \zeta_E^{11} \otimes \mathcal{L}^{-5}, \ d = 11, \ d' = 9, \ g' = 5, \)

(4) \( g = 6, \ e = -1, \ \hat{L} \equiv \zeta_E^{6} \otimes \mathcal{L}^{-2}, \ d = 12, \ d' = 8, \ g' = 4, \)

(5) \( g = 6, \ e = -1, \ \hat{L} \subset \zeta_E^{3} \otimes \mathcal{L}, \ d = 15, \ d' = 5, \ g' = 1, \)

(6) \( g = 5, \ e = -1, \ \hat{L} \equiv \zeta_E^{0} \otimes \mathcal{L}^{-4}, \ d = 9, \ d' = 7, \ g' = 4, \)

(7) \( g = 4, \ e = -1, \ \hat{L} \equiv \zeta_E^{3}, \ d = 9, \ d' = 3, \ g' = 1. \)

**REFERENCES**

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