

RATIONAL QUARTIC SPECTRAHEDRA

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Abstract

Rational quartic spectrahedra in 3-space are semialgebraic convex subsets in \mathbb{R}^3 of semidefinite, real symmetric (4×4) -matrices, whose boundary admits a rational parameterization. The Zariski closure in $\mathbb{C}\mathbb{P}^3$ of the boundary of a rational spectrahedron is a rational complex symmetroid. We give necessary conditions on the configurations of singularities of the corresponding real symmetroids in $\mathbb{R}\mathbb{P}^3$ of rational quartic spectrahedra. We provide an almost exhaustive list of examples realizing the configurations, and conjecture that the missing example does not occur.

1. Introduction

Spectrahedra are important basic objects in polynomial optimization and in convex algebraic geometry [11]. They are intersections of the cone of positive-semidefinite matrices in the space of real symmetric $(n \times n)$ -matrices by an affine subspace. Quartic spectrahedra are the case of (4×4) -matrices intersected with a 3-dimensional affine space that contains a positive definite matrix. We identify the affine space with \mathbb{R}^3 . The boundary of a quartic spectrahedron has a Zariski closure $V(f_A) \subset \mathbb{R}\mathbb{P}^3$ defined by the determinant $f_A(x) := f_A(x_0, x_1, x_2, x_3)$ of a symmetric matrix $A(x)$, where explicitly

$$A(x) := A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3, \quad (1.1)$$

and each A_i is a real symmetric (4×4) -matrix. Since the matrix $A(x)$ is symmetric, the surface $V(f_A)$ is called a (*real*) *symmetroid*. Similarly, the complex algebraic boundary $V_{\mathbb{C}}(f_A) \subset \mathbb{C}\mathbb{P}^3$ defined by f_A is called a *complex symmetroid* to distinguish it from its real points $V(f_A)$. We say that a real symmetroid $V(f_A)$ is *spectrahedral* if it is the Zariski closure of the boundary of a nonempty spectrahedron, i.e., if $A(x_p)$ is definite for some $x_p \in \mathbb{R}^3$. If the complex algebraic boundary of a quartic spectrahedron is a rational complex symmetroid, we say that the spectrahedron is *rational*. The general quartic symmetroid is not rational.

For a general matrix $A(x)$, the singular points of $V(f_A)$ is a finite set of double points, quadratic singularities called *nodes*. The possible arrangements

of the nodes of general quartic spectrahedral symmetroids were identified by Degtyarev and Itenberg [3] and further investigated by Ottem et al. in [10]. In [10], the authors describe their paper as a “first step towards the classification of all spectrahedra of a given degree and dimension”. A quartic surface with a finite set of nodes, or more generally rational double points, is irrational, in fact birational to a K3-surface. This paper sets out to fill in part of the classification of quartic spectrahedra in 3-space by studying the rational members.

The significant feature of a rational spectrahedron is that its boundary allows a parameterization of rational functions from a subset $X \subset \mathbb{C}\mathbb{P}^2$. This is immediate from the fact that the Zariski closure in $\mathbb{C}\mathbb{P}^3$ of the boundary allows a parameterization of rational functions from $\mathbb{C}\mathbb{P}^2$. We do not provide any explicit boundary parameterizations. *A priori*, X does not have to be a subset of $\mathbb{R}\mathbb{P}^2$ even though the boundary of the spectrahedron is real.

A quartic surface is rational only if it has a triple point, an elliptic double point or is singular along a curve cf. [7], [9]. The first author identified families of rational quartic symmetroids in [5]. We state the results of that paper, after making a note about ranks and quadrics.

At every point $x_p \in V(f_A) \subset \mathbb{C}\mathbb{P}^3$, the matrix $A(x_p)$ has rank at most 3. We say that $x_p \in V(f_A)$ is a *rank- k point*, if $A(x_p)$ has rank k . The symmetroid $V(f_A)$ has a double point at each rank-2 point, and a triple point at each rank-1 point. It may, however, also be singular at rank-3 points. This phenomenon is characterized in Lemma 1.1 by properties of the web of quadrics associated to the symmetroid:

If $y := (y_0, y_1, y_2, y_3)$, then $q_A(x_p) := y \cdot A(x_p) \cdot y^T$ is a quadratic form, and its vanishing $Q_A(x_p) := V(q_A(x_p)) \subset \mathbb{C}\mathbb{P}_y^3$ is a quadric surface. The set $Q_A(x) := \{Q_A(x_p) \mid x_p \in \mathbb{C}\mathbb{P}^3\}$ is called a *web of quadrics*.

LEMMA 1.1 ([6, Lemma 2.13],[13, Lemma 1.1]). *The symmetroid $V_{\mathbb{C}}(f_A)$ has a singularity at a rank-3 point x_p if and only if the web of quadrics $Q_A(x)$ has a basepoint at the singular point of $Q_A(x_p)$.*

The main results of [5] can be summarized as follows:

THEOREM 1.2 ([5]). *The rational complex quartic symmetroids in $\mathbb{C}\mathbb{P}^3$ form irreducible families whose general members are of the following types:*

- A. $V_{\mathbb{C}}(f_A)$ has a triple point and six additional nodes;
- B. $V_{\mathbb{C}}(f_A)$ has a tacnode and six additional nodes, and the web $Q_A(x)$ has two basepoints;
- C. $V_{\mathbb{C}}(f_A)$ is singular along a conic and has four additional nodes, and the web $Q_A(x)$ has four linearly independent basepoints;
- D. $V_{\mathbb{C}}(f_A)$ has rank 2 along a line and has six additional nodes, and the web $Q_A(x)$ has four coplanar basepoints;

E. $V_{\mathbb{C}}(f_A)$ is singular of rank 3 along a line and has four additional nodes, and the web $Q_A(x)$ has one basepoint.

REMARK 1.3. For the general members in the families mentioned in Theorem 1.2, i.e., symmetroids of types A, B, C, D, E, all the additional nodes are rank-2 points and they are isolated. Non-general members can have more nodes that are rank-3 points, or some of the singularities may coincide. In this paper an isolated node will always be a rank-2 point.

REMARK 1.4. No symmetroids of type E of Theorem 1.2 are spectrahedral. A consequence of Lemma 1.1 is that singular rank-3 points appear in complex conjugate pairs on spectrahedral symmetroids. A spectrahedral symmetroid which is singular of rank 3 along a line, is therefore singular along two lines.

PROPOSITION 1.5. *The complex quartic symmetroids in $\mathbb{C}\mathbb{P}^3$ singular of rank 3 along two intersecting lines, L_1, L_2 , form an irreducible family whose general members $V_{\mathbb{C}}(f_A)$ have a rank-2 point at $L_1 \cap L_2$, two isolated nodes and the base locus of the web $Q_A(x)$ is a scheme of length 4 with support in two points.*

PROOF. The statement about the singularities is [5, Proposition 9.3] and the claim about the base locus follows from Lemma 1.1 and [4, Lemma 2.6].

DEFINITION 1.6. A complex quartic symmetroid in $\mathbb{C}\mathbb{P}^3$ singular of rank 3 along two lines intersecting in a rank-2 point, with two additional, isolated nodes is said to be of *type F*.

In this paper, we show with reference to Theorem 1.2 and Definition 1.6:

THEOREM 1.7. *Let $S := V(f_A) \subset \mathbb{R}\mathbb{P}^3$ be a spectrahedral symmetroid, and suppose that the complex symmetroid $V_{\mathbb{C}}(f_A) \subset \mathbb{C}\mathbb{P}^3$ is rational. Let $a \geq 0$ denote the number of real, isolated nodes on S and let $0 \leq b \leq a$ denote the number of nodes on the boundary of the spectrahedron.*

A. *If $V_{\mathbb{C}}(f_A)$ is of type A, then S has a triple point on the boundary of the spectrahedron and $0 \leq b \leq a \leq 6$, with a even.*

B. *If $V_{\mathbb{C}}(f_A)$ is of type B, then either*

- (1) *S has a tacnode on the boundary of the spectrahedron;*
- (2) *S has a tacnode disjoint from the spectrahedron.*

In either case, $0 \leq b \leq a \leq b + 2 \leq 6$, both even.

C. *If $V_{\mathbb{C}}(f_A)$ is of type C, then either*

- (1) *S is singular along a smooth conic section with a real point that lies on the boundary of the spectrahedron and $0 \leq b \leq a \leq b + 2 \leq 4$, both even and $a \geq 2$;*

- (2) S is singular along a smooth conic section with a real point that is disjoint from the spectrahedron and $a = b = 2$ or $a = b = 4$;
- (3) S is singular along a smooth conic section with no real points and $a = b = 2$.

D. If $V_{\mathbb{C}}(f_A)$ is of type D, then S has rank 2 along a line disjoint from the boundary of the spectrahedron and $0 \leq b \leq a \leq b + 2 \leq 6$, both even.

F. If $V_{\mathbb{C}}(f_A)$ is of type F, then either

- (1) S is singular of rank 3 along two conjugated intersecting lines whose intersection point lies on the boundary of the spectrahedron;
- (2) S is singular of rank 3 along two conjugated intersecting lines whose intersection point is disjoint from the spectrahedron.

In either case, $0 \leq a = b \leq 2$ even.

Theorem 1.7 provides necessary conditions for the pair (a, b) to be realized by real, rational spectrahedral symmetroids. This is proven case-by-case in §2 and §2.1. After that, in §2.2, we discuss deformation relations between the symmetroids appearing in Theorem 1.7. We show that symmetroids of type B degenerate into a symmetroid of type F, but that no other types degenerate into each other.

Sufficient conditions for the pair (a, b) are given by explicit examples of rational spectrahedral symmetroids. Section 3 contains examples realizing all solutions of the bounds given by Theorem 1.7, except spectrahedral symmetroids of type D with $(a, b) = (0, 0)$. We conjecture that they do not exist:

CONJECTURE 1.8. *A quartic symmetroid in $\mathbb{C}\mathbb{P}^3$ with a nonempty spectrahedron and a line of rank-2 points has – or is a degeneration of one with – at least two real, isolated nodes.*

ACKNOWLEDGEMENTS. We would like to thank Jan Stevens for the proof of Proposition 2.8, and the anonymous referee for helpful comments that improved the exposition of the paper.

2. Real singularities of rational spectrahedral symmetroids

Before we can prove Theorem 1.7, we need some preliminary results. We restrict the attention to real rational quartic symmetroids $V(f_A)$ with a nonempty spectrahedron, i.e., with $A(x_p)$ definite for some $x_p \in \mathbb{R}^3$. First note that if $A(x_p)$ is definite, then $Q_A(x_p)$ has no real points. So if the quartic spectrahedron of $A(x)$ is nonempty, then the web of quadratic surfaces $Q_A(x)$ has no common real points, i.e., no real basepoints, so they have an even number of complex conjugate basepoints. Therefore, when we consider real singularities of the symmetroid $V(f_A)$, they represent real rank-2 quadrics in a web of quadrics

$Q_A(x)$ with complex basepoints. We say that a real quadric is *semidefinite* resp. *indefinite*, when the associated symmetric matrix is. We begin by specializing the base loci mentioned in Theorem 1.2 to pairs of complex conjugated points.

LEMMA 2.1. *Let $p_1, \bar{p}_1, p_2, \bar{p}_2$, be two pairs of complex conjugate points in $\mathbb{C}\mathbb{P}^3$ that do not all lie in a line. Then a real rank-2 quadric that contains both pairs of points is indefinite if and only if it contains the real lines $\langle p_1, \bar{p}_1 \rangle$ and $\langle p_2, \bar{p}_2 \rangle$.*

PROOF. Assume that $Q := M \cup N$ is a real rank-2 quadric, the union of two planes M and N . If M and N are both real, then Q is indefinite, while if M and N are complex conjugates, then Q is semidefinite.

If M and N each contains only two of the four points $p_1, \bar{p}_1, p_2, \bar{p}_2$, the lemma follows. If M contains exactly three of the points, say p_1, \bar{p}_1, p_2 and is not real, then N must contain $p_1, \bar{p}_1, \bar{p}_2$, so Q is semidefinite. If M contains all four basepoints, M is real, so Q is indefinite.

LEMMA 2.2. *Let $p_1, \bar{p}_1, p_2, \bar{p}_2$, be two pairs of complex conjugate points in $\mathbb{C}\mathbb{P}^3$ that do not all lie in a line, and let $Q(x)$ be the 5-dimensional linear system of all quadratic surfaces with basepoints at these four points.*

If the basepoints are not coplanar, then the rank-2 quadrics in $Q(x)$ form three quadratic surfaces, Q_i, Q_{s1}, Q_{s2} , and four planes, $H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$, where the real quadrics in Q_i are indefinite and the real quadrics in $Q_{s1}, Q_{s2}, H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$ are semidefinite.

If the basepoints are coplanar, then the rank-2 quadrics in $Q(x)$ form three quadratic surfaces, as in the nonplanar case, and in addition a web W , whose real quadrics are indefinite. In this case, the double plane containing the basepoints is a rank-1 quadric that lies in the closure of each component of rank-2 quadrics in $Q(x)$.

PROOF. First, note that the lines $\langle p_1, \bar{p}_1 \rangle$ and $\langle p_2, \bar{p}_2 \rangle$ are real and distinct, so if they intersect, they do so in a real point.

The quadrics in Q_i contain the two lines $\langle p_1, \bar{p}_1 \rangle$ and $\langle p_2, \bar{p}_2 \rangle$. Likewise, the quadrics in Q_{s1} contain the lines $\langle p_1, p_2 \rangle, \langle \bar{p}_1, \bar{p}_2 \rangle$, and the quadrics in Q_{s2} contain the lines $\langle p_1, \bar{p}_2 \rangle, \langle \bar{p}_1, p_2 \rangle$. The quadrics in $H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$ contain the plane $\langle \bar{p}_1, p_2, \bar{p}_2 \rangle, \langle p_1, p_2, \bar{p}_2 \rangle, \langle p_1, \bar{p}_1, \bar{p}_2 \rangle, \langle p_1, \bar{p}_1, p_2 \rangle$, respectively. The first part of the claim follows from Lemma 2.1.

Assume now that the basepoints span a plane M , which is real. Then W consists of all quadrics $Q = M \cup N$, where N is any plane. If Q is real, then N is also real, so Q is indefinite when N is distinct from M . On the other hand, the semidefinite double plane $2M$ is contained in W, Q_i, Q_{s1} and Q_{s2} .

REMARK 2.3. If $V_{\mathbb{C}}(f_A)$ is a symmetroid of type C, then Theorem 1.2 implies that $Q_A(x) \subset Q(x)$. A generic 3-space in $Q(x)$ intersects $H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$

in a point each outside of Q_i , Q_{s1} and Q_{s2} . Since $Q_A(x)$ is a special 3-space that intersects one of the quadratic surfaces Q_i , Q_{s1} or Q_{s2} in a conic section C , it meets $H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$ in points on C [5, Proof of Proposition 4.5]. Hence $H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$ are not relevant in the analysis of the isolated nodes of $V_{\mathbb{C}}(f_A)$.

We now give a preliminary analysis of real singularities for spectrahedral symmetroids with nonisolated singularities.

LEMMA 2.4. *Let $S = V(f_A)$ be a rational quartic spectrahedral symmetroid with nonisolated singularities. Then $V_{\mathbb{C}}(f_A)$ has rank 2 along a real line or a real conic, or it is singular and has rank 3 along two intersecting complex conjugate lines. Furthermore:*

- A. *A line of rank-2 points on S is disjoint from the spectrahedron.*
- B. *A real conic of rank-2 points on S may have no real points, or have a real point and be disjoint from the spectrahedron, or lie on the boundary of the spectrahedron.*

PROOF. If the complex symmetroid $V_{\mathbb{C}}(f_A)$ is singular along a curve, then, by Theorem 1.2, this curve contains a line or a smooth conic section. Furthermore, when $V_{\mathbb{C}}(f_A)$ is singular along a line, the matrix $A(x)$ may have rank 2 or 3 along the line.

In the first case, when $A(x)$ has rank 2 along the line, the quadrics $Q_A(x)$ have four coplanar basepoints and the line is real. By Lemma 2.2, the matrix $A(x)$ is indefinite along the line, so on the real spectrahedral symmetroid $V(f_A)$, the singular line must be disjoint from the spectrahedron.

In the second case, when $A(x)$ has rank 3 along the singular line, the web of quadrics $Q_A(x)$ contains a pencil $L \subset Q_A(x)$ of quadrics that are all singular at a basepoint cf. [5, Proof of Proposition 3.5]. Since this basepoint cannot be real, the complex conjugate is also a basepoint. But then, the complex conjugate pencil $\bar{L} \subset Q_A(x)$ must be distinct from L , and $V_{\mathbb{C}}(f_A)$ must be singular of rank 3 along two complex conjugate lines. If these lines do not intersect, the symmetroid $V_{\mathbb{C}}(f_A)$ is a scroll of lines. The lines of this scroll form a curve of bidegree $(2, 2)$ on a quadratic surface in the Grassmannian of lines in $\mathbb{C}\mathbb{P}_y^3$, so the scroll is birational to an elliptic scroll, i.e., irrational. Therefore, the symmetroid $V_{\mathbb{C}}(f_A)$ that is singular, but of rank 3 along two lines, is rational only if the two lines intersect. When the lines are complex conjugates, they of course intersect in a real point. Thus the real symmetroid $V(f_A)$ is singular at this point.

If $V(f_A)$ is a rational spectrahedral symmetroid singular along a smooth conic section, then, by Theorem 1.2, $A(x)$ must have rank 2 along this curve and the web of quadrics $Q_A(x)$ has two pairs of complex conjugate basepoints

that are linearly independent. Clearly the conic section is real and the *a priori* listed possibilities follow.

We separate the proof of nonexistence of some cases from the proof of Theorem 1.7. First we note that symmetroids of type C.3 belong to a well-known class of real surfaces. A real quartic surface singular along a conic section with no real points in the plane at infinity, is known as a *cyclide* [7, Chapter V].

Not all cyclides are symmetroids. By Theorem 1.2.C, a symmetroid singular along a conic has four additional nodes. The cyclide $V((x_0^2 + x_1^2 + x_2^2)^2 - x_3^4)$ has no singularities outside of the conic $V(x_3, x_0^2 + x_1^2 + x_2^2)$, hence it is not a symmetroid. For cyclides with four additional nodes, we have the following result:

PROPOSITION 2.5 ([7, Article 68]). *If a cyclide has four additional nodes, then at most two of the isolated nodes are real.*

COROLLARY 2.6. *A general, real, quartic symmetroid singular along a smooth conic section with no real points has either two or no real nodes.*

In a paper by Chandru, Dutta and Hoffmann, the authors summarize classical works by Cayley [1] and Maxwell [8]. This is used to produce a classification of the various forms of the cyclides [2, §6]. Apart from the degenerated cases of a cone or a cylinder, the cyclides are divided into three forms, *horned cyclides*, *ring cyclides* and *spindle cyclides*. Of these, only the ring cyclides have no real nodes. The ring cyclides resemble squashed tori and do not bound a convex region. Hence they do not occur as spectrahedral symmetroids. For horned cyclides and spindle cyclides with precisely two real nodes, the nodes connect two components of the real cyclide. If these occur as spectrahedral symmetroids, one of the components is the boundary of the spectrahedron. We conclude:

PROPOSITION 2.7. *Let $S := V(f_A)$ be a real quartic symmetroid with a nonempty spectrahedron that is singular along a real conic section with no real points. Then S has two real nodes, both on the boundary of the spectrahedron.*

We now disprove the existence of all spectrahedral symmetroids of type C with $(a, b) = (0, 0)$.

PROPOSITION 2.8 ([12]). *Let $S := V(f_A) \subset \mathbb{R}\mathbb{P}^3$ be a spectrahedral symmetroid, and suppose that the complex symmetroid $V_{\mathbb{C}}(f_A) \subset \mathbb{C}\mathbb{P}^3$ is of type C. Then S has a real, isolated node.*

PROOF. The symmetroids with a given pair (a, b) form a full-dimensional, Zariski open set in the families of symmetroids of types C.1, C.2 or C.3. Hence

it suffices to show the nonexistence of symmetroids with $(a, b) = (0, 0)$ on a Zariski open set in each of the families.

By Theorem 1.2.C, $Q_A(x)$ has four linearly independent basepoints. Since S is spectrahedral, the basepoints appear in complex conjugate pairs, p_1, \bar{p}_1 and p_2, \bar{p}_2 . After a change of coordinates, we may assume that $p_1 := [1 : i : 0 : 0]$ and $p_2 := [0 : 0 : 1 : i]$. Consider the space $Q(x)$ of all quadrics with $p_1, \bar{p}_1, p_2, \bar{p}_2$ as basepoints. The quadrics in $Q(x)$ have matrices on the form

$$M(x) := \begin{bmatrix} x_{00} & 0 & x_{02} & x_{03} \\ 0 & x_{00} & x_{12} & x_{13} \\ x_{02} & x_{12} & x_{22} & 0 \\ x_{03} & x_{13} & 0 & x_{22} \end{bmatrix}.$$

The quadratic surfaces described in Lemma 2.2 are

$$Q_i = V(x_{00}, x_{22}, x_{02}x_{13} - x_{03}x_{12}),$$

$$Q_{s1} = V(x_{02} - x_{13}, x_{03} + x_{12}, x_{00}x_{22} - x_{12}^2 - x_{13}^2),$$

$$Q_{s2} = V(x_{02} + x_{13}, x_{03} - x_{12}, x_{00}x_{22} - x_{12}^2 - x_{13}^2).$$

By Remark 2.3, we do not have to consider $H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$.

For symmetroids of type C.1, $Q_A(x)$ intersects either Q_{s1} or Q_{s2} in a conic section; say Q_{s1} . Then the hyperplane spanned by Q_{s1} and $Q_A(x)$ equals $V(\lambda(x_{02} - x_{13}) + \mu(x_{03} + x_{12}))$ for some constants λ, μ . After conjugating $M(x)$ with the matrix

$$\begin{bmatrix} \lambda & -\mu & 0 & 0 \\ \mu & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we may assume that $\mu = 0$. Thus $x_{02} = x_{13}$ in the hyperplane $\langle Q_{s1}, Q_A(x) \rangle$. If $Q_A(x)$ is sufficiently general, we may after a projective linear transformation assume that $A(x)$ is on the form

$$\begin{bmatrix} x_0 & 0 & x_1 & a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 \\ 0 & x_0 & x_2 & x_1 \\ x_1 & x_2 & x_3 & 0 \\ a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 & x_1 & 0 & x_3 \end{bmatrix}, \quad (2.1)$$

for $a_0, a_1, a_2, a_3 \in \mathbb{R}$. The rank-2 locus of $V_{\mathbb{C}}(f_A)$ is then

$$Q_A(x) \cap Q_{s1} = V(a_0x_0 + a_1x_1 + (a_2 + 1)x_2 + a_3x_3, x_1^2 + x_2^2 - x_0x_3),$$

$$Q_A(x) \cap Q_{s2} = V(x_1, a_0x_0 + (a_2 - 1)x_2 + a_3x_3, x_2^2 - x_0x_3),$$

$$Q_A(x) \cap Q_i = V(x_0, x_3, x_1^2 - a_1x_1x_2 - a_2x_2^2).$$

Assume for contradiction that S has no real, isolated nodes. Then $Q_A(x) \cap Q_{s2}$ and $Q_A(x) \cap Q_i$ are not real. For $a_0 = 1$, this means that the discriminants

$$D_{s2} := (a_2 - 1)^2 - 4a_3 \quad (2.2)$$

$$D_i := a_1^2 + 4a_2 \quad (2.3)$$

are negative. Moreover, for $a_0 = 1$, Sylvester's criterion implies that the conic section $Q_A(x) \cap Q_{s1}$ is positive definite if and only if $D_{s2} + D_i < 0$. In other words, if S has no real, isolated nodes, then $Q_A(x) \cap Q_{s1}$ has no real points, so S is not of type C.1.

For symmetroids of type C.2, $Q_A(x)$ intersects Q_i in a conic section. The hyperplane spanned by Q_i and $Q_A(x)$ equals $V(\lambda x_{00} + \mu x_{22})$ for some constants λ, μ . After rescaling x_{22} , we may assume that $x_{22} = \pm x_{00}$ in this hyperplane. If $x_{22} = -x_{00}$, then Sylvester's criterion implies that the hyperplane contains no definite matrix. Assume therefore that $x_{22} = x_{00}$. If $Q_A(x)$ is sufficiently general, we may after a projective linear transformation assume that $A(x)$ is on the form

$$\begin{bmatrix} x_0 & 0 & x_1 & x_2 \\ 0 & x_0 & x_3 & a_0x_0 + a_1x_2 + a_2x_2 + a_3x_3 \\ x_1 & x_3 & x_0 & 0 \\ x_2 & a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 & 0 & x_0 \end{bmatrix} \quad (2.4)$$

for $a_0, a_1, a_2, a_3 \in \mathbb{R}$. The isolated nodes of $V_{\mathbb{C}}(f_A)$ are then

$$Q_A(x) \cap Q_{s1} = V(x_2 + x_3, a_0x_0 + (a_1 - 1)x_1 - (a_2 - a_3)x_3, x_0^2 - x_1^2 - x_3^2),$$

$$Q_A(x) \cap Q_{s2} = V(x_2 - x_3, a_0x_0 + (a_1 + 1)x_1 + (a_2 + a_3)x_3, x_0^2 - x_1^2 - x_3^2).$$

For $a_0 = 1$, these are not real if the discriminants

$$D_{s1} := a_1^2 + a_2^2 + a_3^2 - 2a_1 - 2a_2a_3 \quad (2.5)$$

$$D_{s2} := a_1^2 + a_2^2 + a_3^2 + 2a_1 + 2a_2a_3 \quad (2.6)$$

are negative. But if $D_{s1} < 0$ and $D_{s2} < 0$, then $D_{s1} + D_{s2} = 2(a_1^2 + a_2^2 + a_3^2) < 0$, which is impossible. Hence at least one of $Q_A(x) \cap Q_{s1}$ and $Q_A(x) \cap Q_{s2}$ consists of real points.

The case for symmetroids of type C.3 is covered by Proposition 2.7.

2.1. Proof of Theorem 1.7

2.1.1. *Type A.* Triple points are rank-1 points, hence the real ones are semi-definite and therefore on the boundary of the spectrahedron.

By Theorem 1.2.A, a general complex symmetroid with a triple point has six nodes. Since a spectrahedral symmetroid is a real surface, the number a of real nodes is even. There are no further restraints on b , the number of real semidefinite nodes, as the examples in Table 1 show.

2.1.2. *Type B.* By Theorem 1.2.B, the web $Q_A(x)$ has two basepoints. Since S is spectrahedral, the basepoints are complex conjugates, p and \bar{p} . Let $Q(x)$ be the 7-dimensional linear system of all quadratic surfaces with p and \bar{p} as basepoints. The rank-2 locus of $Q(x)$ consists of two fourfolds, X_i and X_s . The quadrics in X_i are pairs of planes, where one of the planes contains the line $\langle p, \bar{p} \rangle$. In X_s , the quadrics consist of two planes, where the planes contain one basepoint each. The set $\text{Sing}(X_i) = \text{Sing}(X_s)$ consists of pairs of planes that both contain $\langle p, \bar{p} \rangle$. The real quadrics in $X_i \setminus \text{Sing}(X_i)$ are indefinite and the real quadrics in $X_s \setminus \text{Sing}(X_s)$ are semidefinite. The real quadrics in $\text{Sing}(X_i) = \text{Sing}(X_s)$ are either semidefinite or indefinite.

In the proof of [5, Proposition 7.4], it is shown that the tacnode corresponds to a point in $\text{Sing}(X_i) = \text{Sing}(X_s)$, and that $Q_A(x)$ intersects $X_i \setminus \text{Sing}(X_i)$ in two points and $X_s \setminus \text{Sing}(X_s)$ in four points. The claim follows.

2.1.3. *Type C.* Let C denote the conic section in the singular locus of S . By Theorem 1.2.C, S is the discriminant of a web $Q_A(x)$ of quadrics with four linearly independent basepoints. In the notation of Lemma 2.2, $Q_A(x)$ intersects one of the surfaces Q_i , Q_{s1} or Q_{s2} in C and the remaining two surfaces in two points each. By Remark 2.3, S does not have any other rank-2 points. Hence $a \leq 4$.

Assume first that C contains a real point. If C is disjoint from the spectrahedron, then $C \subset Q_i$. The real quadrics in Q_{s1} and Q_{s2} are semidefinite, so $a = b$. If C lies on the boundary of the spectrahedron, then C is contained in either Q_{s1} or Q_{s2} . It follows that $a \leq b + 2$, since the real quadrics in Q_i are indefinite. We get $a \geq 2$ from Proposition 2.8.

The case where C has no real points is dealt with in Proposition 2.7.

2.1.4. *Type D.* Lemma 2.4 states that a line of rank-2 points is disjoint from the spectrahedron.

By Theorem 1.2.D, S is the discriminant of a web $Q_A(x)$ of quadrics with four coplanar basepoints. Since S has a nonempty spectrahedron, the basepoints occur in two complex conjugate pairs. In the notation of Lemma 2.2,

$Q_A(x)$ intersects W in a line and the surfaces Q_i, Q_{s1}, Q_{s2} in two points each. These are the only singularities on S , so we get that

$$0 \leq b \leq a \leq b + 2 \leq 6,$$

and a, b are even from Lemma 2.2.

2.1.5. *Type F.* It follows from [4, Remark 4.5] that S is the intersection of a 3-space with $S' \subset \mathbb{C}\mathbb{P}^5$, where S' is a quartic spectrahedral symmetroid that is singular of rank 3 along two complex conjugate 3-spaces intersecting in a plane. The claim is immediate from [4, Proposition 4.6], which says that S' is singular along an additional quadratic surface whose real points lie on the spectrahedron.

2.2. *Deformation relations between rational symmetroids*

It is natural to ask whether the different types of rational symmetroids listed in Theorem 1.7 are deformations of each other. That is, does there exist a flat family of symmetroids where one of the symmetroids is of one type, while all the others are of a different type? The following result addresses this:

PROPOSITION 2.9. *Let $V(f_A) \subset \mathbb{R}\mathbb{P}^3$ be a quartic spectrahedral symmetroid and suppose that the complex symmetroid $V_{\mathbb{C}}(f_A) \subset \mathbb{C}\mathbb{P}^3$ is rational. If $V_{\mathbb{C}}(f_A)$ is of type B, then it can degenerate into a symmetroid of type F. This is the only instance of a symmetroid of type A, B, C, D, or F that degenerates into one of the other types. Moreover, a real symmetroid $V(f_A)$ does not degenerate from type B.1 to B.2; from C.1 to C.2 or C.3; or from F.1 to F.2, or vice versa. In addition, a real symmetroid $V(f_A)$ with a real nodes, b of which lie on the spectrahedron, does not degenerate into a symmetroid $V(f_{A'})$ with a' real nodes, b' of which lie on the spectrahedron, if $(a, b) \neq (a', b')$.*

PROOF. We start with the complex symmetroids. In [5, Remark 9.2], it is argued that if S is a quartic surface singular along two intersecting lines, L_1 and L_2 , with the implicit assumption that $L_1 \cap L_2$ is not a triple point, then S satisfies the equation of a tacnodal surface. Hence a symmetroid of type F can arise as a degeneration of surfaces of type B of Theorem 1.2. Consider the explicit deformation

$$M(t) := \begin{bmatrix} \ell_{00} & 0 & \ell_{02} & \ell_{03} \\ 0 & \ell_{00} & -\ell_{03} + t\ell_{12} & \ell_{02} + t\ell_{13} \\ \ell_{02} & -\ell_{03} + t\ell_{12} & a_{22}x_0 + \ell_{22} & a_{23}x_0 + \ell_{23} \\ \ell_{03} & \ell_{02} + t\ell_{13} & a_{23}x_0 + \ell_{23} & a_{33}x_0 + \ell_{33} \end{bmatrix},$$

where ℓ_{ij} are linear forms in x_1, x_2, x_3 , and $a_{22}, a_{23}, a_{33}, t$ are scalars. For all t , the matrix $M(t)$ is on the form (3.2) of a tacnodal symmetroid. For $t = 0$, the matrix $M(0)$ takes the form (3.4) of a symmetroid singular along two lines.

No other type of complex symmetroids is a degeneration of one of the other types. Most of the possible degenerations can be excluded by simple reasons, for instance a surface does not degenerate into one whose singular locus has lower degree or dimension, or into one with only singularities of lower multiplicity. We check the remaining cases:

- A symmetroid of type A is not a degeneration of symmetroids of type B. This is because a web of quadrics $Q_A(x)$ with basepoints does not degenerate into a web with no basepoints.
- Symmetroids of types C or D are not degenerations of symmetroids of type B. Indeed, as noted in the proof of Theorem 1.7, the associated quadric at a tacnode is singular along the line spanned by two basepoints of $Q_A(x)$. In a degeneration of tacnodal symmetroids, one quadric in $Q_A(x)$ is singular along the line spanned by two basepoints. For symmetroids of types C or D, $Q_A(x)$ has four basepoints. Assume that $Q_A(x)$ has four basepoints and recall the notation from Lemma 2.2. If the basepoints are independent and a quadric is singular along the line spanned by two of them, then it lies in one of the intersections $Q_i \cap Q_{s1}$, $Q_i \cap Q_{s2}$ or $Q_{s1} \cap Q_{s2}$. If the basepoints are coplanar and a quadric is singular along the line spanned by two of them, then it lies in one of the intersections $W \cap Q_i$, $W \cap Q_{s1}$ or $W \cap Q_{s2}$. In either case, this implies that the surface has one fewer isolated rank-2 point, than one of type C or D.
- A symmetroid of type C is not a degeneration of symmetroids of type D. This is because a web of quadrics $Q_A(x)$ with linearly dependent basepoints does not degenerate into one with independent basepoints.
- A symmetroid of type F is not a degeneration of symmetroids of types C or D. This is because a symmetroid with a curve in its rank-2 locus does not degenerate into one without a curve in its rank-2 locus.

For the real symmetroids, we note that a positive semidefinite rank-2 matrix has two positive eigenvalues, while an indefinite rank-2 matrix has eigenvalues with different signs. In a degeneration from one into the other, one of the eigenvalues changes to 0, causing the rank to drop. This implies that a symmetroid with singularities on the spectrahedron does not degenerate into one with singularities outside of the spectrahedron, and vice versa.

Finally, in a degeneration of a surface with two complex conjugate nodes into one with two real nodes, the imaginary part of the coordinates of the nodes changes to 0, while the coordinates have the same real part for both

nodes. Hence the two nodes coincide. Thus a symmetroid with a real nodes does not degenerate into one with $a' \neq a$ real nodes.

3. Examples of existence

This section lists matrices that define spectrahedral symmetroids with the different values of (a, b) described by Theorem 1.7. There is one missing instance, namely D with $(a, b) = (0, 0)$. For brevity in the tables, we write $(a_{00}, a_{01}, a_{02}, a_{03}, a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33})$ for the symmetric matrix

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Tables 1, 2 and 4 provide one such 10-tuple for each of the matrices A_2 and A_3 in (1.1). The matrices A_0 and A_1 are fixed, and therefore not listed.

3.1. Spectrahedral symmetroids with a triple point

Triple points correspond precisely to rank-1 points. Thus (1.1) defines a spectrahedral symmetroid with a triple point if A_0 is a positive definite matrix, A_1 is a rank-1 matrix, and A_2, A_3 are any symmetric matrices. To find examples, we let A_0 be the identity matrix I_4 ,

$$A_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and used a pseudorandom generator to draw symmetric matrices A_2 and A_3 with integer entries between -9 and 9 . Doing this, we found all values of (a, b) specified by Theorem 1.7.A. The results are listed in Table 1, a sample image is given in Figure 1.

3.2. Spectrahedral symmetroids with a tacnode

Consider the space $Q(x)$ of all quadrics with basepoints p, \bar{p} . After a change of coordinates, we may assume that $p := [1 : i : 0 : 0]$. Then quadrics in $Q(x)$ have matrices on the form

$$\begin{bmatrix} x_{00} & 0 & x_{02} & x_{03} \\ 0 & x_{00} & x_{12} & x_{13} \\ x_{02} & x_{12} & x_{22} & x_{23} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix}. \tag{3.1}$$

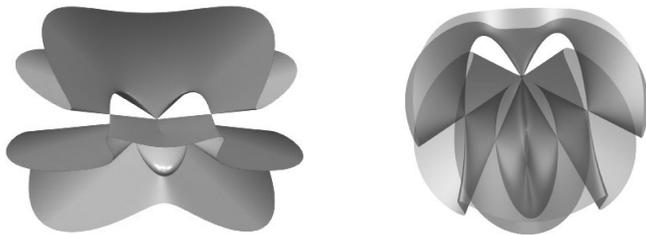
FIGURE 1. A symmetroid of type A with $(a, b) = (6, 4)$.

TABLE 1. Examples of matrices defining spectrahedral symmetroids of type A.

(a, b)	A_2	A_3
(6, 6)	$(-9, -1, 2, -2, -3, 7, 7, 5, 5, -4)$	$(3, -4, -7, 3, -1, -2, -9, 4, 8, 6)$
(6, 5)	$(3, -7, 5, 4, 3, 2, 9, 5, 7, -8)$	$(-3, 5, 5, -5, -3, -6, 3, -2, 0, -7)$
(6, 4)	$(-7, 7, -6, 6, 5, 5, 5, -8, -2, 1)$	$(-3, 8, -2, 1, -4, 1, 5, -8, 8, -7)$
(6, 3)	$(0, 0, 3, -3, 0, -2, 8, 6, 7, 1)$	$(-2, 1, 5, 4, 5, 6, -8, 5, 1, 6)$
(6, 2)	$(9, 7, 2, 3, -5, -9, -2, -2, 3, -5)$	$(-9, 6, -3, 3, -7, 2, -1, 2, -7, 5)$
(6, 1)	$(8, -5, 2, -9, 1, -1, 2, -5, 9, -8)$	$(-9, 6, -3, 3, -7, 2, -1, 2, -7, 5)$
(6, 0)	$(-3, 6, -4, 1, 2, 6, 9, 0, -7, 8)$	$(8, 6, 3, -4, 5, 9, 7, 3, 7, -9)$
(4, 4)	$(1, -6, -6, 4, 6, 2, 5, -5, -1, -8)$	$(-5, 1, -7, 6, 9, 9, 7, -9, -8, -2)$
(4, 3)	$(6, 3, 9, 9, -8, 9, 0, -7, 6, -7)$	$(-1, 3, 3, 3, -9, 5, -6, 5, 4, -9)$
(4, 2)	$(6, 8, -3, 9, 2, -2, -9, 4, 6, 7)$	$(-2, 9, -4, -2, 8, -1, 9, 1, -4, 4)$
(4, 1)	$(2, 9, -1, -8, 1, 0, -1, -8, 6, -2)$	$(2, -6, 0, -6, -5, 2, -1, 6, -1, 9)$
(4, 0)	$(-8, 0, -9, 6, 3, -6, 3, -7, 6, 6)$	$(1, -5, -1, 8, -9, 0, 4, -2, 8, -3)$
(2, 2)	$(4, -4, 1, 6, 3, 2, 0, -5, 1, -3)$	$(4, 3, 9, 5, -6, -3, -5, 2, 0, 4)$
(2, 1)	$(-5, 8, -1, -6, 8, 7, 9, 5, 9, 6)$	$(3, 1, 4, 5, -4, -6, -8, -2, 8, 5)$
(2, 0)	$(-9, -6, 5, -1, 4, 0, 1, 8, 1, 6)$	$(-7, 5, -7, 4, 5, 0, 3, 6, 4, 7)$
(0, 0)	$(-5, 9, 5, 1, 1, -2, 9, -2, 0, 2)$	$(7, -6, 5, 1, -4, 1, 3, -9, 8, -5)$

Using the notation from §2.1.2, a 3-space $Q_A(x) \subset Q(x)$, corresponding to a tacnodal symmetroid, intersects $\text{Sing}(X_i) = V(x_{00}, x_{02}, x_{03}, x_{12}, x_{13})$ in a point. Hence $A(x)$ has the form

$$A(x) := \begin{bmatrix} \ell_{00} & 0 & \ell_{02} & \ell_{03} \\ 0 & \ell_{00} & \ell_{12} & \ell_{13} \\ \ell_{02} & \ell_{12} & a_{22}x_0 + \ell_{22} & a_{23}x_0 + \ell_{23} \\ \ell_{03} & \ell_{13} & a_{23}x_0 + \ell_{23} & a_{33}x_0 + \ell_{33} \end{bmatrix}, \quad (3.2)$$

where each ℓ_{ij} is a linear form in x_1, x_2, x_3 , and $a_{ij} \in \mathbb{R}$. Moreover, if $V(f_A)$ is spectrahedral, we can take $A_1 = A([0 : 1 : 0 : 0])$ to be positive definite.

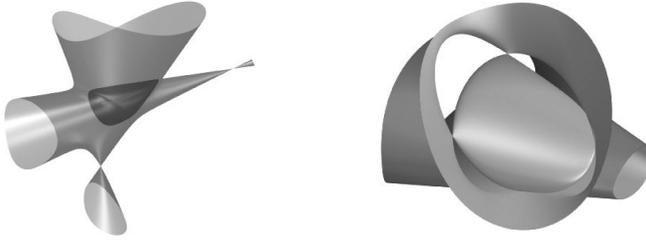


FIGURE 2. A symmetroid of type B.1 with $(a, b) = (6, 4)$.

TABLE 2. Examples of matrices defining spectrahedral symmetroids of types B.1 and B.2.

(a, b)	B.1	B.2
$(6, 4)$	$(6, 0, -3, 6, 6, -6, -4, 0, 3, 6)$ $(3, 0, 5, -8, 3, -3, -4, 2, -5, 8)$	$(8, 0, -6, -8, 8, 7, 1, -7, 4, 7)$ $(6, 0, 2, -2, 6, -7, 6, -1, 9, 7)$
$(4, 4)$	$(0, 0, 4, -4, 0, 3, 3, 1, -8, 4)$ $(5, 0, -2, 4, 5, -5, 2, 1, -5, -8)$	$(4, 0, -8, -2, 4, 5, -3, 5, -8, 9)$ $(3, 0, 2, -4, 3, 0, 4, 0, 2, 6)$
$(4, 2)$	$(8, 0, -5, 8, 8, 8, -3, 5, -3, 2)$ $(-8, 0, 1, 9, -8, -9, 7, -8, 8, 9)$	$(1, 0, 5, -7, 1, 6, 4, -6, -3, 6)$ $(6, 0, 2, -8, 6, 1, -3, 5, 4, -8)$
$(2, 2)$	$(6, 0, -8, -4, 6, 1, 9, -8, 0, 3)$ $(3, 0, -2, 7, 3, 7, 6, -6, 4, 1)$	$(7, 0, 6, 5, 7, 2, -4, 0, -7, 2)$ $(0, 0, -8, 2, 0, 3, 7, 8, -3, 7)$
$(2, 0)$	$(6, 0, 6, -5, 6, 6, -1, 1, -1, 7)$ $(8, 0, 5, -8, 8, -6, 1, 2, -2, -9)$	$(5, 0, 6, -6, 5, -6, 8, -8, 7, 6)$ $(4, 0, 2, -1, 4, 6, -6, -2, 3, 5)$
$(0, 0)$	$(-1, 0, 2, 4, -1, 0, 1, 7, 5, 5)$ $(7, 0, 4, -6, 7, 5, 8, -2, 0, -4)$	$(9, 0, 3, -9, 9, 6, -7, -4, 1, 1)$ $(4, 0, 5, -9, 4, -2, 5, 3, 1, -4)$

To find examples, we let A_1 in (1.1) be the identity matrix I_4 , and

$$A_0 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix}.$$

For symmetroids with a tacnode on the boundary of the spectrahedron, we chose $\alpha = 1$ and $\beta = 2$. Likewise, we chose $\alpha = 1$ and $\beta = -2$ for symmetroids with a tacnode disjoint from the spectrahedron. We used a pseudorandom generator to draw symmetric matrices A_2 and A_3 on the form (3.1) with integer entries between -9 and 9 . Doing this, we found all values of (a, b) specified by Theorem 1.7.B.1 and Theorem 1.7.B.2. The results are listed in Table 2 and a sample image in Figure 2.

3.3. Spectrahedral symmetroids with a double conic

To find examples of symmetroids $V(f_A)$ of type C.1, we can take $A(x)$ to be on the form (2.1). For $a_0 = 1$, we get $(a, b) = (4, 2)$ if both discriminants (2.2) and (2.3) are positive, $(a, b) = (2, 2)$ if (2.2) is positive and (2.3) is negative, and $(a, b) = (2, 0)$ if (2.2) is negative and (2.3) is positive. It remains to check that $A(x)$ contains a definite matrix to conclude that $V(f_A)$ is in fact spectrahedral. In particular, the following examples are spectrahedral:

- $a_0 := 1, a_1 := 0, a_2 := 1, a_3 := -1$ gives a symmetroid with $(a, b) = (4, 2)$;
- $a_0 := 1, a_1 := 0, a_2 := -2, a_3 := -4$ gives a symmetroid with $(a, b) = (2, 2)$, see Figure 3;
- $a_0 := 1, a_1 := -3, a_2 := 0, a_3 := 1$ gives a symmetroid with $(a, b) = (2, 0)$.

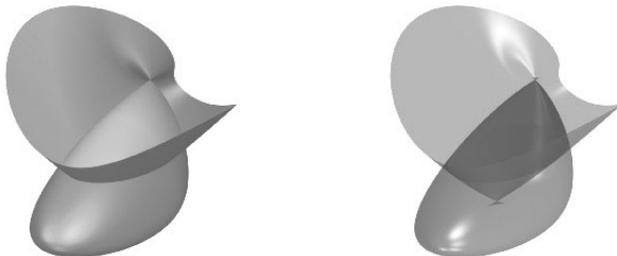


FIGURE 3. A symmetroid of type C.1 with $(a, b) = (2, 2)$.

To find examples of symmetroids $V(f_A)$ of type C.2, we can take $A(x)$ to be on the form (2.4). For $a_0 = 1$, we get $(a, b) = (4, 4)$ if both discriminants (2.5) and (2.6) are positive, and $(a, b) = (2, 2)$ if only one of (2.5) and (2.6) is positive. It remains to check that $A(x)$ contains a definite matrix to conclude that $V(f_A)$ is in fact spectrahedral. In particular, the following examples are spectrahedral:

- $a_0 := 1, a_1 := 3, a_2 := 0, a_3 := 0$ gives a symmetroid with $(a, b) = (4, 4)$, see Figure 4;
- $a_0 := 1, a_1 := 1/2, a_2 := 0, a_3 := 0$ gives a symmetroid with $(a, b) = (2, 2)$.

To find examples of symmetroids $V(f_A)$ of type C.3, we can again take $A(x)$ to be on the form (2.1). For $a_0 = 1$, we get the correct type if the discriminant D_{s^2} from (2.2) is positive, D_i from (2.3) is negative and $D_{s^2} < -D_i$. It remains to check that $A(x)$ contains a definite matrix to conclude that $V(f_A)$ is in fact spectrahedral. In particular, (2.1) gives a spectrahedral symmetroid of type C.3 for $a_0 := 1, a_1 := 2, a_2 := -5$ and $a_3 := 6$. See Figure 5.

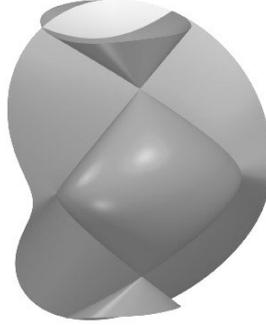


FIGURE 4. A symmetroid of type C.2 with $(a, b) = (4, 4)$.

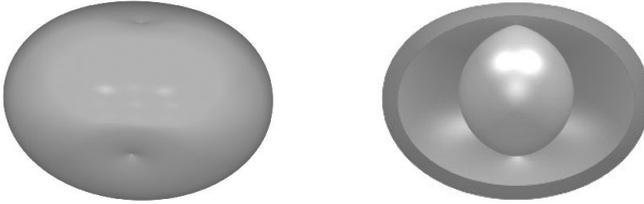


FIGURE 5. A symmetroid of type C.3 with $(a, b) = (2, 2)$. The surface is known as a “spindle cyclide”.

3.4. Spectrahedral symmetroids with rank 2 along a double line

Consider the space $Q(x)$ of quadrics with coplanar basepoints $p_1, \bar{p}_1, p_2, \bar{p}_2$. After a change of coordinates, we may assume that $p_1 := [1 : i : 0 : 0]$ and $p_2 := [1 : 0 : i : 0]$. Then the quadrics in $Q(x)$ have matrices on the form

$$\begin{bmatrix} x_{00} & 0 & 0 & x_{03} \\ 0 & x_{00} & x_{12} & x_{13} \\ 0 & x_{12} & x_{00} & x_{23} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix}.$$

If $Q_A(x) \subset Q(x)$ is a generic 3-space, we may after a projective linear transformation assume that $A(x)$ is on the form

$$\begin{bmatrix} x_0 & 0 & 0 & x_1 \\ 0 & x_0 & x_2 & x_3 \\ 0 & x_2 & x_0 & a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 \\ x_1 & x_3 & a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 & b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 \end{bmatrix}, \quad (3.3)$$

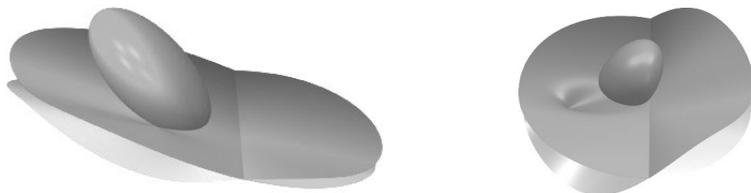


FIGURE 6. Two symmetroids of type D. The surface to the left has $(a, b) = (2, 2)$ and the surface to the right has $(a, b) = (2, 0)$.

TABLE 3. Examples of parameters for (3.3) that define spectrahedral symmetroids of type D.

(a, b)	a_0	a_1	a_2	a_3	b_0	b_1	b_2	b_3
(6, 4)	0	1	0	1	0	0	0	1
(4, 4)	0	1	0	-1	1	0	1	1
(4, 2)	0	1	-1	2	0	0	0	1
(2, 2)	0	1	0	1	0	0	-1	1
(2, 0)	0	1	-1	1	0	0	0	1

for $a_i, b_i \in \mathbb{R}$. Using the notation from Lemma 2.2, the rank-2 locus of $Q_A(x)$ is

$$Q_A(x) \cap W = V(x_0, x_2),$$

$$Q_A(x) \cap Q_i = V(x_0, x_1, b_2x_2^2 + (-2a_2 + b_3)x_2x_3 - 2a_3x_3^2),$$

$$Q_A(x) \cap Q_{s1} = V(x_0 - x_2, a_1x_1 + (a_0 + a_2)x_2 + (a_3 - 1)x_3, \\ x_1^2 - b_1x_1x_2 - (b_0 + b_2)x_2^2 - b_3x_2x_3 + x_3^2),$$

$$Q_A(x) \cap Q_{s2} = V(x_0 + x_2, a_1x_1 - (a_0 - a_2)x_2 + (a_3 + 1)x_3, \\ x_1^2 + b_1x_1x_2 - (b_0 - b_2)x_2^2 + b_3x_2x_3 + x_3^2).$$

Hence the matrix $A(x)$ defines a symmetroid $V(f_A)$ of type D.

For $a_1 = 1$, the reality of the isolated nodes is determined by the discriminants

$$D_i := (2a_2 + b_3)^2 + 8b_2a_3,$$

$$D_{s1} := (a_3(2a_0 + 2a_2 + b_1) - 2a_0 - 2a_2 - b_1 - b_3)^2 \\ - 4(a_3^2 - 2a_3 + 2)((a_0 + a_2)(a_0 + a_2 + b_1) - b_0 - b_2),$$

$$D_{s2} := (a_3(-2a_0 + 2a_2 - b_1) - 2a_0 + 2a_2 - b_1 + b_3)^2 \\ - 4(a_3^2 + 2a_3 + 2)((a_0 - a_2)(a_0 - a_2 + b_1) - b_0 + b_2).$$

More precisely,

- $(a, b) = (6, 4)$ if D_i, D_{s1} and D_{s2} are positive;
- $(a, b) = (4, 4)$ if D_{s1} and D_{s2} are positive and D_i is negative;
- $(a, b) = (4, 2)$ if D_i is positive and either D_{s1} or D_{s2} is positive;
- $(a, b) = (2, 2)$ if either D_{s1} or D_{s2} is positive and D_i is negative,
- $(a, b) = (2, 0)$ if D_i is positive and D_{s1} and D_{s2} are negative;
- $(a, b) = (0, 0)$ if D_i, D_{s1}, D_{s2} are negative.

It remains to check that $A(x)$ contains a definite matrix to conclude that $V(f_A)$ is in fact spectrahedral. Spectrahedral examples are given in Table 3, except $(a, b) = (0, 0)$. Some images are given in Figure 6.

REMARK 3.1. If we let $a_0 := 3, a_1 := 1, a_2 := 0, a_3 := 1, b_0 := 0, b_1 := 0, b_2 := -1, b_3 := 0$ then (3.3) defines a symmetroid with rank 2 along a line and no real, isolated nodes. By Sylvester’s criterion, it is not spectrahedral. We have not been able to find a spectrahedral symmetroid of type D with $(a, b) = (0, 0)$, nor prove its nonexistence.

3.5. Spectrahedral symmetroids with rank 3 along two double lines

After a change of coordinates, Proposition 1.5 implies that if $V_C(f_A)$ is of type F, then $Q_A(x)$ is contained in the 7-space $Q(x)$ defined by (3.1). The discriminant of $Q(x)$ is singular of rank 3 along the complex conjugate 4-spaces $H_4 := V(x_{00}, x_{02} - ix_{12}, x_{03} - ix_{13})$ and $\overline{H}_4 := V(x_{00}, x_{02} + ix_{12}, x_{03} + ix_{13})$. In order to find examples of 3-spaces $Q_A(x)$ corresponding to symmetroids of type F, we consider the 5-space $H := V(x_{02} - x_{03}, x_{03} + x_{13})$. Because H intersects H_4 and \overline{H}_4 in a 3-space each, a generic 3-space $Q_A(x) \subset H$ corresponds to a symmetroid of type F. We therefore consider matrices on the form

$$A(x) := \begin{bmatrix} \ell_{00} & 0 & \ell_{02} & \ell_{03} \\ 0 & \ell_{00} & -\ell_{03} & \ell_{02} \\ \ell_{02} & -\ell_{03} & \ell_{22} & \ell_{23} \\ \ell_{03} & \ell_{02} & \ell_{23} & \ell_{33} \end{bmatrix},$$

where ℓ_{ij} are linear forms in x_0, x_1, x_2, x_3 . In order to make $A(x)$ resemble (3.2), we let $A_0 = A([1 : 0 : 0 : 0])$ correspond to the intersection between the lines in the singular locus of $V(f_A)$. Hence we reduce to

$$A(x) := \begin{bmatrix} \ell_{00} & 0 & \ell_{02} & \ell_{03} \\ 0 & \ell_{00} & -\ell_{03} & \ell_{02} \\ \ell_{02} & -\ell_{03} & a_{22}x_0 + \ell_{22} & a_{23}x_0 + \ell_{23} \\ \ell_{03} & \ell_{02} & a_{23}x_0 + \ell_{23} & a_{33}x_0 + \ell_{33} \end{bmatrix}, \quad (3.4)$$

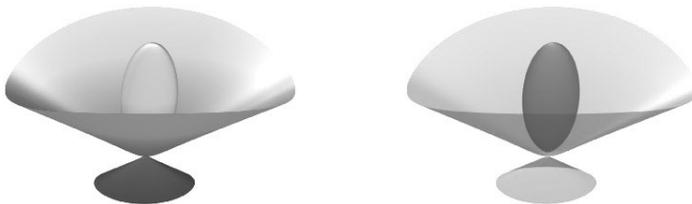
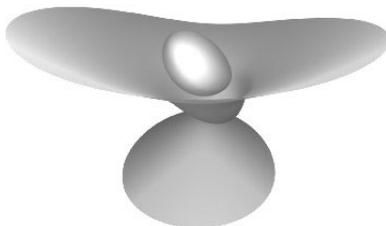
FIGURE 7. A symmetroid of type F.1 with $(a, b) = (0, 0)$.FIGURE 8. A symmetroid of type F.2 with $(a, b) = (0, 0)$.

TABLE 4. Examples of matrices defining spectrahedral symmetroids of types F.1 and F.2.

(a, b)	F.1	F.2
$(2, 2)$	$(0, 0, 1, 0, 0, 0, 1, 0, 1, 0)$ $(0, 0, 0, 1, 0, -1, 0, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0, 1, 0, 1, 0)$ $(0, 0, 0, 1, 0, -1, 0, 0, 0, 3)$
$(0, 0)$	$(0, 0, 1, 0, 0, 0, 1, 1, 1, 0)$ $(0, 0, 0, 1, 0, -1, 0, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0, 1, 0, 1, 0)$ $(0, 0, 0, 1, 0, -1, 0, 0, 0, 0)$

where ℓ_{ij} are linear forms in x_1, x_2, x_3 . Furthermore, if $V(f_A)$ is spectrahedral, we can take $A_1 = A([0 : 1 : 0 : 0])$ to be a definite matrix.

We chose A_1 to be the positive definite matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

For type F.1, we let $a_{22} := 1$, $a_{23} := 1$, $a_{33} := 3$ in A_0 . For type F.2, we let $a_{22} := 1$, $a_{23} := 1$ and $a_{33} := 0$. Table 4 shows A_2 and A_3 realizing all values of (a, b) specified by Theorem 1.7.F.1 and Theorem 1.7.F.2. Some corresponding images are given in Figures 7 and 8.

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