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INFINITE LOCALLY FINITE HYPOHAMILTONIAN GRAPHS

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An enumerably infinite graph G is called *hypohamiltonian*, if it has no two-way infinite hamiltonian path, but every vertex deleted subgraph $G-v$ has such a path. Thomassen [4] gives examples of infinite hypohamiltonian graphs. Each of these graphs has a vertex of infinite degree. An infinite graph is called *locally finite* if every vertex has finite degree. Thomassen raised the question if there exist infinite hypohamiltonian graphs that are locally finite. In this paper we show:

THEOREM. *There exists a planar infinite locally finite hypohamiltonian graph.*

The terminology of this paper is that of [4]; additionally, we write $e = (v,w)$ if e is an edge in G with endvertices v and w.

To prove the theorem, we first show that there exists a graph fragment with the following properties:

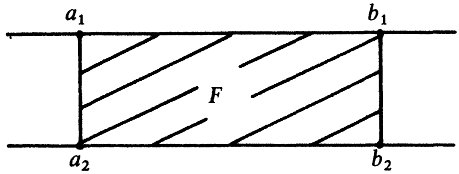

Figure 1.

DEFINITION 1. Let F be a finite graph-fragment with four marked vertices a_1, a_2, b_1, b_2 as shown in Figure 1. F is *suitable*, if the following holds:

1. (i) There exists a hamiltonian path from a_i to b_i, $i = 1,2$, in F.

(ii) There exists a hamiltonian path in F from a_1 to a_2.

(iii) There are two disjoint paths, one from a_1 to b_1 and the other from a_2 to b_2 covering together all vertices of F.

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2. (i) There exists no hamiltonian path from a_1 to b_2 or from a_2 to b_1 in F.

(ii) There exists no hamiltonian path from b_1 to b_2 in F.

(iii) There are no two disjoint paths in F, one from a_1 to a_2 and the other from b_1 to b_2 covering together all the vertices of F.

(iv) There are no two disjoint paths, one from a_1 to b_2 and the other from a_2 to b_1 covering together all the vertices of F.

3. $F - v$ contains at least one hamiltonian path with endvertices either a_1 and b_2 or a_2 and b_1, for each v in $V(F)$.

We say a path ω in F is of type (1,i), if ω is hamiltonian and a_j, b_j, $j \in \{1,2\}$, are endvertices of ω. A path ω in F is of type (1,ii), if ω is hamiltonian and a_1, a_2 are endvertices of ω. Two paths γ_1 with endvertices a_1, b_1 and γ_2 with endvertices a_2, b_2 in F are said to be a *pair of type* (1,iii), if they are disjoint and cover together all vertices of F.

To show that such a fragment exists indeed, we use the graph shown in Figure 2, where H_F^1 and H_F^2, are certain subfragments as defined below.

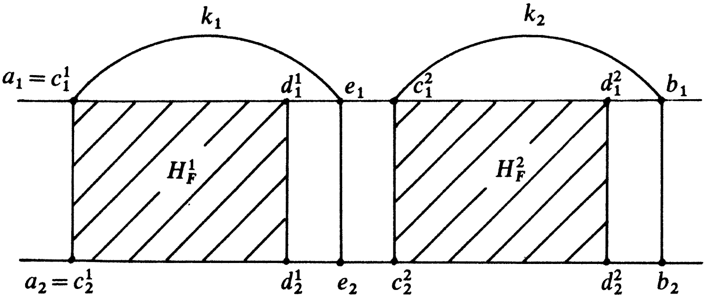

Figure 2.

Definition 2. A finite fragment H_F with marked vertices c_i, d_i, $i = 1,2$, has property E, if the following three conditions hold:

1. (i) There exists a hamiltonian path from c_i to d_i, $i = 1,2$, in H_F.

(ii) There are two disjoint paths, one from c_1 to d_1 and the other from c_2 to d_2 covering together all vertices of H_F.

2. (i) There does not exist a hamiltonian path from c_1 to d_2 or from c_2 to d_1 in H_F.

(ii) There does not exist a hamiltonian path from c_1 to c_2 or from d_1 to d_2 in H_F.

(iii) There do not exist two disjoint paths, one from c_1 to c_2 and the other from d_1 to d_2 covering together all the vertices of H_F.

(iv) There do not exist two disjoint paths, one from c_1 to d_2 and the other from c_2 to d_1 covering together all the vertices of H_F.

3. In $H_F - v$, $v \in V(H_F)$ arbitrarily chosen, at least one of the following statements holds:

(i) There is a hamiltonian path from c_1 to d_2 or from c_2 to d_1.

(ii) There are two disjoint paths, one from c_1 to d_2 and the other from c_2 to d_1 covering together all the vertices of $H_F - v$.

(iii) There are two disjoint paths, one from c_1 to c_2 and the other from d_1 to d_2 covering together all the vertices of $H_F - v$.

(iv) There exists a hamiltonian path from c_1 to c_2.

LEMMA 1. *The fragment F shown in* Figure 2 *is suitable.*

We omit the proof, because it is mainly a matter of routine, using the fact that H_F^i, $i = 1,2$, has property E.

To prove that condition (2) of Definition 1 holds, it is convenient to consider $F - \{k_1, k_2\}$ first, and then to show that the addition of k_1, k_2 does not lead to one of the forbidden paths.

Let K be the graph of Figure 3, which is isomorphic to the hypohamiltonian graph K' of Figure 4 found by Hatzel [1]. This is the

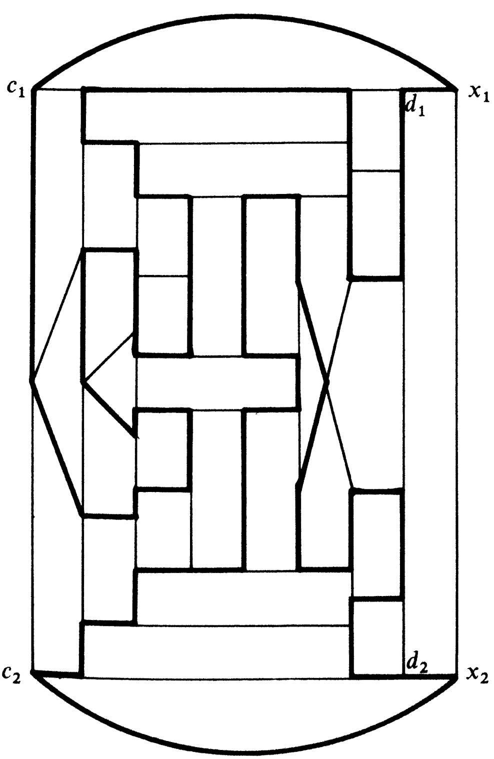

Figure 3.

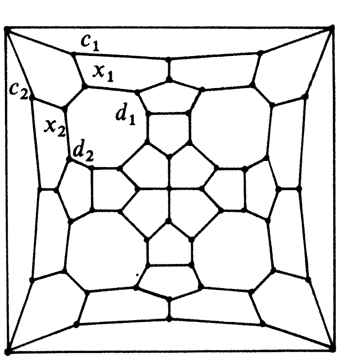

Figure 4.

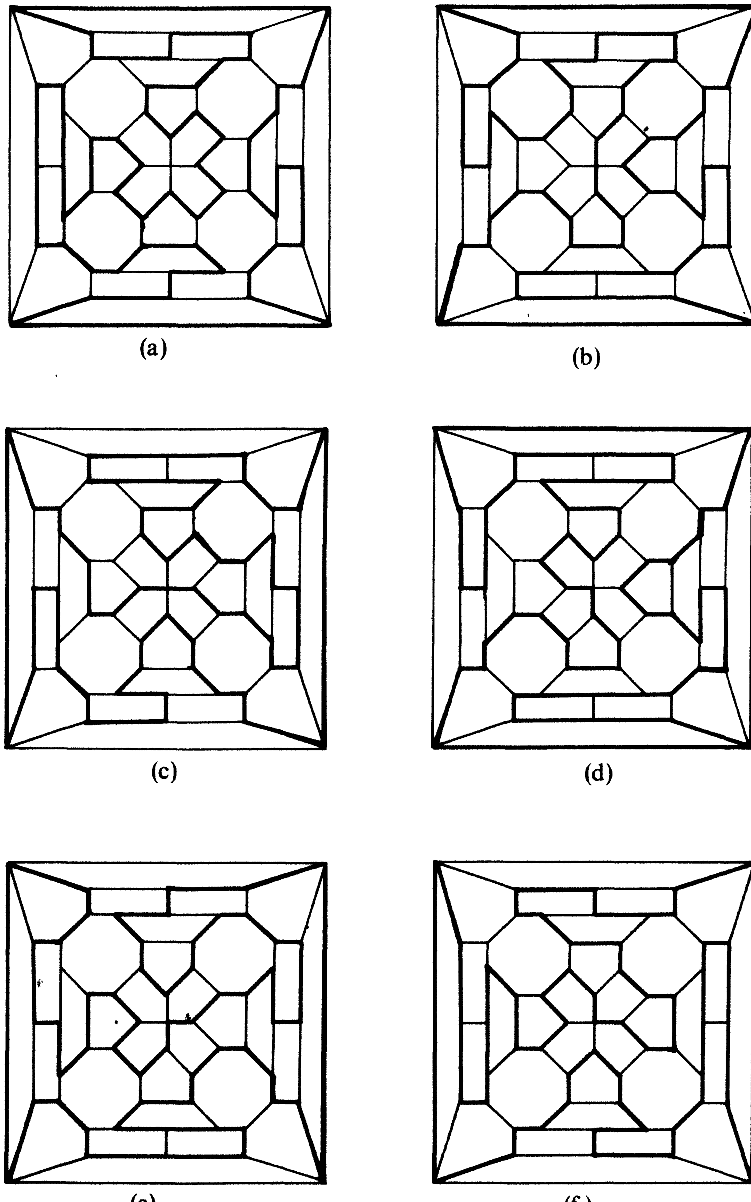

Figure 5, a–f.

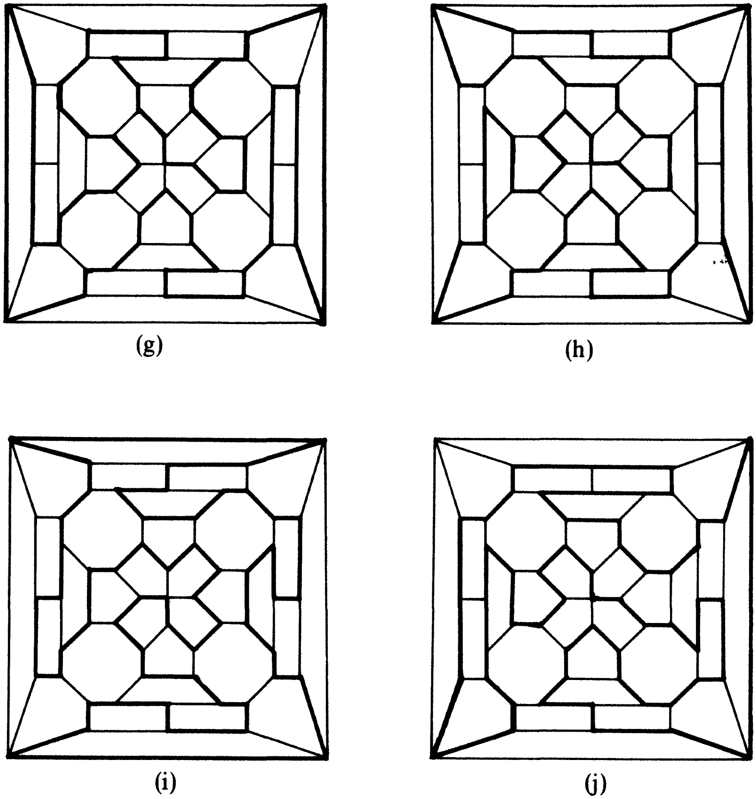

Figure 5, g–j.

smallest known example of a hypohamiltonian graph that is planar. We will demonstrate now that K contains a fragment with property E. Let c_i, d_i, x_i, $i = 1,2$, be the vertices marked in Figure 3, and let K_F be the fragment resulting from K by deletion of the vertices x_1 and x_2 and the corresponding edge. We get:

LEMMA 2. *K_F has property E, where c_i, d_i, $i = 1,2$, are the marked vertices.*

PROOF. We have to show that the conditions (1)–(3) of Definition 2 hold.

(1): (i) holds, because K is hypohamiltonian; the paths demanded in (ii) are contained in the two cycles shown in Figure 3.

(2): The existence of these paths would always yield a hamiltonian cycle in K.

(3): To prove this, we take the presentation K' of Figure 4. Figure 5 (a)–(j) presents a hamiltonian circuit in $K' - v$, for each $v \in V(K')$ (up to symmetries; see [1]). All but the cases (d) and (i) are already presented in [1]. It is left to the reader to check that in each case one of the required paths exists. Considering all possible symmetries, one gets the assumption.

Letting now K_F play the role of H_F and using Lemma 1 we see that the corresponding fragment F is suitable.

After having shown the existence of a suitable fragment, we now are able to prove the Theorem.

Let G be the graph of Figure 6, where each A_i, $i = 1,2,3,\ldots$, is a suitable fragment. We shall show that G is hypohamiltonian. Let a^i_j, b^i_j, $j = 1,2$, $i = 1,2,3,\ldots$, be the marked vertices of the fragment A_i, as indicated in Figure 6. Let δ be a two-way infinite path in G.

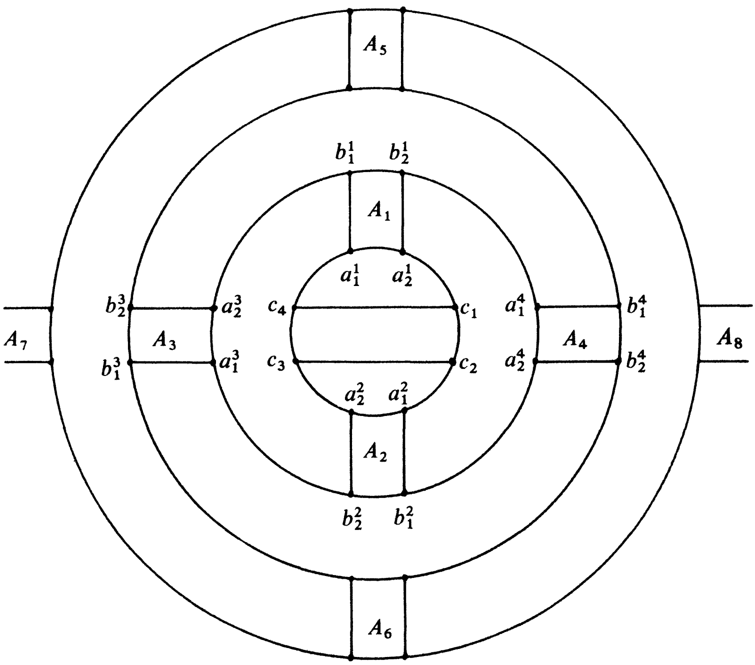

Figure 6.

If $\delta \cap A_i$ consists of all the vertices of A_i, then $\delta \cap A_i$ is a path of type (1,i) or (1,ii), or a pair of paths of type (1,iii). We shall show now that it is not possible to extend these paths to a hamiltonian path in G. There exists an $i \in \{1,2,3,\ldots\}$ such that the intersection $\delta \cap A_i$ is not a pair of type (1, iii), because otherwise $(G - \{c_1, c_2, c_3, c_4\}) \cap \delta$ would consist of four infinite components, which is impossible.

Without loss of generality, let $n \equiv 1 \pmod 2$ for Lemma 3 and Lemma 4.

LEMMA 3. (i) *If* $\delta \cap A_n$ *is a path of type* (1, i), *then* $\delta \cap A_{n+1}$ *does not contain a pair of type* (1, iii) *as subgraph.*

(ii) $\delta \cap A_n$ *and* $\delta \cap A_{n+1}$ *are not of type* (1, ii), *both.*

The proof is easy and therefore left to the reader.

If δ contains all vertices of A_n and A_{n+1}, we have to distinguish between three different cases for the intersections $\delta \cap A_n$ and $\delta \cap A_{n+1}$. All other cases are symmetric. We shall show now that in none of these cases δ is a hamiltonian path.

LEMMA 4. (i) *If* $\delta \cap A_n$ *is a path of type* (1, i) *in* A_n *with endvertices* a_2^n *and* b_2^n *and if* $\delta \cap A_{n+1}$ *is a path of type* (1, i) *in* A_{n+1} *with endvertices* a_1^{n+1} *and* b_1^{n+1}, *then* δ *is not a hamiltonian path in* G.

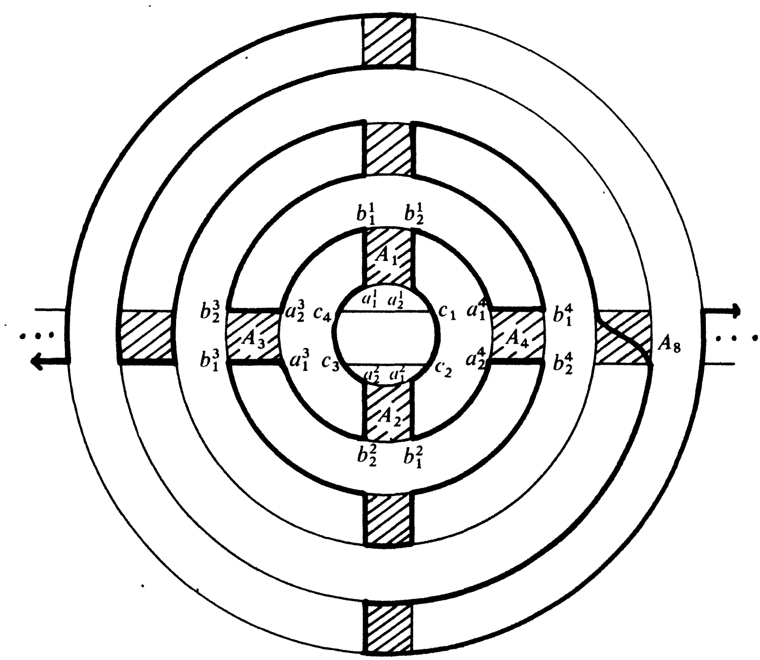

Figure 7.

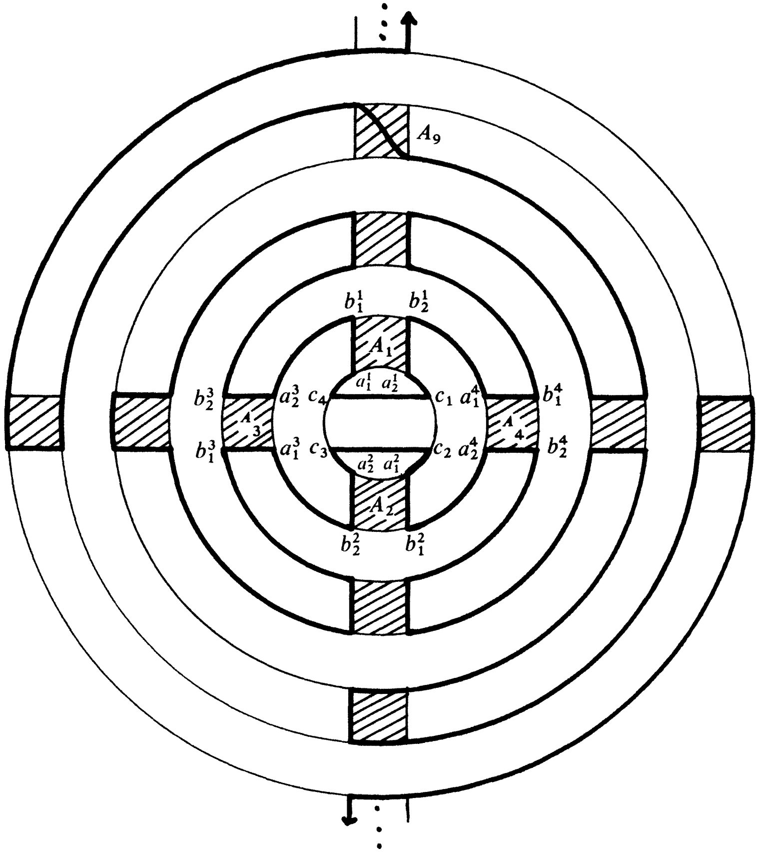

Figure 8.

(ii) *If* $\delta \cap A_n$ *is a path of type* (1, ii) *in* A_n *and* $\delta \cap A_{n+1}$ *is a pair of type* (1, iii) *in* A_{n+1}, *then* δ *is not a hamiltonian path in* G.

(iii) *If* $\delta \cap A_1$, $1 = n, n+1$, *is a hamiltonian path in* A_1 *with endvertices* a_j^1 *and* b_j^1, $j \in \{1,2\}$, *then* δ *is not a hamiltonian path in* G.

For the proof consider the possibilities for $\delta \cap A_{n+2}$ and $\delta \cap A_{n+3}$ in (i). This yields a contradiction to Definition 1. Part (ii) can be reduced to case (i) and (iii) can be proved by induction.

Now, it follows immediately:

LEMMA 5. *The graph* G *is not hamiltonian.*

It remains to show:

LEMMA 6. *The graph* $G - v$ *is hamiltonian for each* $v \in V(G)$.

If $v \in \{c_j | j = 1, \ldots, 4\}$ holds, it is easy to determine a hamiltonian path in $G - v$. If $v \in V(A_i)$ for some $i \in \{1,2,3,\ldots\}$, there exists a hamiltonian path in $A_i - v$ from a_1^i to b_2^i or from a_2^i to b_1^i, by Definition 1. One has to distinguish between two different cases:

(a) $i \equiv 0 \pmod 4$ or $i \equiv 3 \pmod 4$

(b) $i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4$.

Figure 7 (Figure 8) indicates the hamiltonian path in case $v \in V(A_8)$ ($v \in V(A_9)$). The detailed proof of Lemma 6 is straightforward and therefore left to the reader.

Lemma 5 and Lemma 6 yield the assumption of the Theorem. Obviously, G is planar, since the fragment F is planar. There are only vertices of degree 3 or 4 in G.

COROLLARY. *There are infinitely many planar infinite locally finite hypohamiltonian graphs.*

The Corollary is proved by applying Theorem 4.1 of [4] to G and to an arbitrary finite planar hypohamiltonian graph H. Obviously, G and H are satisfying the conditions of Theorem 4.1. of [4], and since there are infinitely many planar hypohamiltonian graphs [4], the Corollary holds.

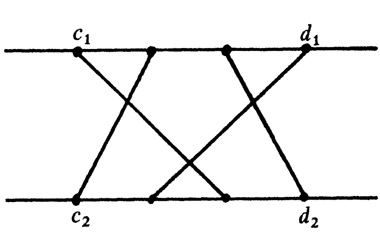

Figure 9.

REMARK. Deleting an edge and its endvertices from the Petersen-graph, as shown in Figure 9, yields a fragment P which has property E (compare [2], [3]), but is not planar. The use of this fragment as H_F yields a nonplanar hypohamiltonian locally finite graph.

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