MARKOV RANDOM WALKS ON GROUPS

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Abstract.
Let $X$ be a Markov chain on $C$ and $f$ a continuous function from $C$ to a topological group $G$. The process

$$S_0 = e, \quad S_{n+1} = S_n f(X_{n+1}), \quad n = 1, 2, \ldots$$

is called a Markov random walk on $G$. We investigate the set of possible values taken by $S$ and, under some regularity conditions on $X$, we also study the recurrence/transience properties of $S$ by means of an embedded ordinary random walk on $G$. We show, for instance, that if $G$ is a transient group then all Markov random walks on $G$ are transient.

Let $G$ be a multiplicatively written locally compact second countable topological group. The object of this paper is to investigate the sequence of successive products $S_1, S_2, \ldots$ of factors of the form $f(X_1), f(X_2), \ldots$ where the $X_n$'s form a Markov chain on some topological space $C$ and $f$ is a continuous function from $C$ to $G$. In particular, we are interested in the possible values and the recurrence properties of the sequence $(S_n)_{n=0}^\infty$.

Muthsam [8] and Wolff [13] studied the problem for the case $G$ a discrete semigroup and $G$ a compact semigroup, respectively. (In their work $C \subseteq G$ and $S_n$ is simply $X_1 X_2 \ldots X_n$.) We will draw heavily on these two papers.

Niemi and Nummelin [9] studied central limit properties of $(S_n)_{n=0}^\infty$, with $G = \mathbb{R}$ and $f$ an arbitrary measurable function. In our investigation, we will make use of their technique of introducing an artificial recurrent atom for the Markov chain.

1.

Let $C$ be a locally compact second countable Hausdorff space and $(X_n)_{n=0}^\infty$ a Markov chain on $C$ with transition kernel $P$, which we will assume to be Feller, cf. [12, p. 34]. $P_x$ will denote the probability on the
canonical probability space (path space) $C^\infty$ induced by $P$ and the initial condition $X_0 = x$. Let $f$ be a continuous function from $C$ to $G$. The process

$$S_n = \prod_{k=1}^n f(X_k), \quad n = 1, 2, \ldots; \quad S_0 = e$$

will be called a Markov random walk on $G$.

**Example 1.** Let $X$ be a Markov chain on the integers $\mathbb{Z}$, with transition probability matrix $P$. Then

$$S_n = X_1 + X_2 + \ldots + X_n, \quad n = 1, 2, \ldots$$

is a Markov random walk on $\mathbb{Z}$. It reduces to ordinary random walk if the rows of $P$ are all identical (i.e. the $X_i$'s are independent). If $C = \{1, -1\}$ with $P(1,1) = P(-1,1) = \frac{1}{2}$, then $S$ is a symmetric simple random walk.

**Example 2.** Let $C = \{0\} \cup \{n^{-1} | \ 1, 2, \ldots\}$ endowed with the topology of the real line. Let $P$ be defined as follows: $P(0,1) = 1$

$$P(n^{-1}, (n+1)^{-1}) = \alpha_n, \quad P(n^{-1}, 1) = 1 - \alpha_n, \quad n = 1, 2, \ldots$$

where $0 < \alpha_n < 1, \quad n = 1, 2, \ldots$. If $X$ is the random walk on $C$ with the transition probability matrix $P$, then

$$S_n = X_1 + X_2 + \ldots + X_n, \quad n = 1, 2, \ldots$$

is a Markov random walk on the real line $\mathbb{R}$.

**Example 3.** Let $\mu$ be a probability distribution on the set of idempotent matrices of the form

$$(1 + \alpha \beta)^{-1} \begin{pmatrix} 1 & \alpha \\ \beta & \alpha \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{R}, \alpha \beta \neq -1$. The behavior of a product

$$X_1 X_2 \ldots X_{n-1} X_n$$

of independent $\mu$-distributed random matrices is conveniently (for most purposes) studied by considering the $(1,1)$ entry of the product:

$$S_n = (1 + \alpha_1 \beta_1)^{-1} (1 + \alpha_1 \beta_2) \ldots (1 + \alpha_{n-1} \beta_n)(1 + \alpha_n \beta_n)^{-1}$$

where the subscript $i$ corresponds to the random matrix $X_i$. As we shall see below $S_n$ is, under mild conditions on the probability $\mu$, a Markov random walk on the multiplicative group $\mathbb{R} \setminus \{0\}$.
In our analysis of the Markov random walks we will make extensive use of the fact that the auxiliary process \( Z_n = (X_n, S_n), \quad n = 0, 1, 2, \ldots \), is a Markov chain on \( C \times G \) with transition kernel

\[
U((x, s), A \times B) = \int 1_A(z) 1_B(sf(z)) P(x, dz)
\]

where \( \mathcal{C} \) and \( \mathcal{G} \) are Borel \( \sigma \)-algebras of \( C \) and \( G \), respectively.

Following [13] we call a \((n+1)\)-tuple \((x_0, x_1, \ldots, x_n)\) a chain of length \(n\) from \(x\) to \(y\) if \(x = x_0, y = x_n\), and \(x_{i+1}\) is in the support of the measure \(P(x_i, \cdot)\), \(i = 0, 1, \ldots, n - 1\). Denote the chain by \(k_x\) and define the functions \(p\) and \(q\) by

\[
p(k_x) = f(x_1) f(x_2) \ldots f(x_n) \quad \text{and} \quad q(k_x) = x_n.
\]

Suppose a sequence \( (k^n_x)_{n=1}^\infty \) of chains has the property that \(q(k^n_x)\) converges to \(u\). If \(w\) is a limit point of the sequence \(p(k^n_x)\), it is necessarily of the form \(w = vf(u)\), since \(p(k^n_x) = v_n f(q(k^n_x))\) for some \(v_n \in G\). Note that the assumptions that \(G\) is a group and \(f\) a continuous function are used in this argument.

For \(x, u \in G\) consider the set of all sequences of chains \(k^n_x\) starting from \(x\) such that \(q(k^n_x)\) converges to \(u\) (the sequence may be finite if there is a chain from \(x\) to \(u\)).

**Definition.** \(\Pi(x, u)\) is the set of all limit points of the sequences \(p(k^n_x)\) such that \(q(k^n_x)\) converges to \(u\).

If \(q(k^n_x) \to u\), then we say that \(x\) leads to \(u\), \(x \leadsto u\). If \(x \leadsto u\) implies \(u \leadsto x\), we say that \(x\) is essential. Note that \(x \leadsto u\) is equivalent to

\[
u \in \text{closure} \left( \bigcup_{n=1}^\infty \text{supp}(P^n(x, \cdot)) \right),
\]

where \(P^n\) is the \(n\)-step transition kernel for \(X\).

We will make the following general assumption about the Markov chain \(X\) (cf. [13]),

(A) \(C\) is an irreducible class and all elements of \(C\) are essential.

(A) guarantees that all elements of \(C\) communicate. I.e. for every \(x, u \in G\) there exists a sequence \((k^n_x)\) of chains starting from \(x\) such that \(q(k^n_x)\) converges to \(u\).

In our Example 1 above \(\Pi(0,0)\) is the set of all sums \(x_1 + x_2 + \ldots + x_{n-1} + 0\) such that the transition probabilities \(P(0,x_1), P(x_1,x_2), \ldots\),
$P(x_{n-1}, 0)$ are all positive. If, for example, $P(0, nd), P(nd, 0), P(0, -md), P(-md, 0) > 0$ for some relatively prime positive integers $m, n,$ and $d$, then $\Pi(0, 0) \equiv \Xi dZ$.  

In Example 2, $\Pi(1, 1)$ is a subset of the real interval $[1, \infty)$. Note that $\Pi(1, 1)$ is necessarily a semigroup and that it is asymptotically dense. It is also easily seen that $\Pi(0, 1) = \Pi(1, 1)$ but $\Pi(0, 0)$ is empty because all chains with $q(k_0^n)$ close to 0 have 

$$p(k_0^n) \geq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m_n}$$

for some large positive integer $m_n$.

$\Pi(x, x)$ is either empty or a closed subsemigroup of $G$, the invariance semigroup of the element $x \in C$. To see this, notice that $(x, g)$ can be reached by the $Z$-chain from $(x, e)$, if and only if $g \in \Pi(x, x)$, cf. [13]. So if $(x, e) \sim (x, h)$ ($(x, e)$ leads to $(x, h)$) as well we get $(x, e) \sim (x, g)$ and $(x, g) \sim (x, gh)$ whence, by the transitivity of the relation $\sim$, we obtain the desired result $(x, e) \sim (x, gh)$.

**Remark.** The intuitively obvious transitivity of $\sim$ can be seen as follows, cf. [13]: If $g$ is a bounded measurable function on $C \times G$, then we can define

$$Ug(x, s) = \int g(z, sf(z))P(x, dz) \quad (x, s) \in C \times G$$

$(U((x, s), N)$ is to be interpreted as $U_{1,N}(x, s)$ for a set $N \in \mathcal{C} \times \mathcal{G})$.

Using the assumption that $P$ is Feller, the local compactness of $C$ and the continuity of the multiplication in $G$ we can show that $U$ and all its iterates $U^n$ are Feller. Note that the functions $U^n(\cdot, N)$ are then lower semicontinuous for open sets $N$. Suppose $(x, s) \sim (x', s')$ and $(x', s') \sim (x'', s'')$. This means, by definition, that for any neighborhoods $N'$ and $N''$ of $(x', s')$ and $(x'', s'')$, respectively, there are $n'$ and $n''$ with $U^{n'}((x, s), N'), U^{n''}((x', s'), N'') > 0$. Choose $N'$ to be such that $U^{n''}(\cdot, N'') > 0$ on $N'$ to obtain $U^{n'+n''}((x, s), N'') > 0$.

Suppose that there is an essential element $(c, g) \in C \times G$ for the $Z$-chain. Clearly, $(c, g) \sim (c, g)$ so $e \in \Pi(c, c)$. If $h \in \Pi(c, c)$, then $(c, g) \sim (c, gh)$ and, by essentiality, $(c, gh) \sim (c, g)$, that is $h^{-1} \in \Pi(c, c)$. All elements communicating with $(c, g)$ are essential (as well as all $(c, h), h \in G$) so $\Pi(u, u)$ is also a group for all those $u \in C$ for which $\Pi(c, u) \neq \emptyset$. This is, for instance, the case for all

$$u \in \bigcup_{n \geq 1} \text{ support } (P^n(c, \cdot)),$$
a set which is dense in \( C \) by assumption (A). The set of essential elements thus contains the set

\[
\bigcup_{\Pi(c,u) \neq \emptyset} \{u\} \times G,
\]

cf. \[8\], \[13\].

The groups \( \Pi(c,c) \) and \( \Pi(u,u) \) are, in fact, conjugate. To see this, consider the transitions \( (u,e) \simeq (c,l) \sim (u,e) \) or, equivalently, \( (u,e) \sim (c,l) \) and \( (c,e) \sim (u,l^{-1}) \). If \( g \in \Pi(c,c) \), we get \( lgl^{-1} \in \Pi(u,u) \) and, conversely, every element of \( \Pi(u,u) \) can be represented in this way. Hence \( l\Pi(c,c)l^{-1} = \Pi(u,u) \).

**Remark.** A slight modification of Example 2 will make \( \Pi(1,1) \) a group (= R): Introduce a negative element \(-2\), say, and a transition probability \( P(1,-2) > 0 \) and let \( P(-2,1) = 1 \). Then \( \Pi(x,y) = R \) for all \( x,y \in C \). Note that the assumption (B) below is not satisfied by the Markov chain in Example 2 even in its modified form.

For any (Feller) Markov chain, the essential classes are closed. If, for example, the group \( G \) is compact or \( \Pi(c,c) = G \), we can then conclude that all elements of \( C \times G \) are essential. This also holds, of course, for the case when

\[(B) \quad C = \bigcup_{n \geq 1} \text{support} \left( P^n(c, \cdot) \right).\]

Conversely, if (B) is true for all \( c \in C \) we can argue as in \[8\] to prove that \( \Pi(c,c) \) is a group (for one and thus for all \( c \in C \)) if and only if \( (c,g) \) is essential for the \( Z \)-chain (for some \( g \in G \)).

We summarize the preceding discussion in

**Theorem 1.** Under assumption (A), if there is an essential element \( (c,g) \) for the \( Z \)-chain the set of essential elements contains the set

\[
\bigcup_{\Pi(c,u) \neq \emptyset} \{u\} \times G.
\]

The invariance semigroup \( \Pi(c,c) \) is a closed subgroup of \( G \). Furthermore, for \( u \) such that \( \Pi(c,u) \neq \emptyset \), \( \Pi(u,u) \) is a group conjugate to \( \Pi(c,c) \).

**Corollary 1.** If, in addition, condition (B) holds for all \( c \in C \), either all or none of the invariance semigroups \( \Pi(c,c) \) are groups. Then all or none, respectively, of the elements of \( C \times G \) are essential for the \( Z \)-chain.
Corollary 2. If $\Pi(c,c) = G$ or if $G$ is compact, then the same holds for any $u \in C$ and all elements of $C \times G$ are essential.

For compact $C$ and $G$ we refer to the thorough discussion in [13].

Remark. It might be worth while to note that the (relative) compactness of $\Pi(c,c)$ will automatically make it a group, since a compact subgroup of a group is, in fact, a subgroup.

2.

If $X$ is stationary, i.e. $X$ has an invariant probability measure $\pi$ as its initial distribution, then $S$ is a special case of what we could call a generalized random walk on $G$, cf. [3], [7]. For a stationary $X$ on a countable state space to admit a unique invariant probability distribution it is necessary and sufficient that $X$ be ergodic, cf. [13, p. 136]. In the case of a general state space we will use the sufficient condition that $X$ be positive Harris, see [12]. In this section, we will study the recurrence properties of such a walk. For the case $G = R$ the criterion for recurrence is particularly simple [3]:

$S$ is recurrent if and only if $E_\pi f(X_1)(= E_\pi S_1) = 0$

($E_\pi$ is expectation with respect to $P_x$ and $E_\pi(\cdot) = \int E_\pi(\cdot \pi(dx)$.)

If $G$ is the direct product of a compact group $K$ and $R$, we have a similar criterion, namely $E_\pi f_2(X_1) = 0$, where $f_2$ is the second component of $f$: If $f(x) = (k,r) \in K \times R$, then $f_2(x) = r$. This result follows from the preceding, since the walk on the compact factor is automatically recurrent and the walk on the second factor $R$ is itself nothing but a generalized random walk on $R$. For $G = C^*$, the multiplicative group of non-zero complex numbers $\simeq T \times R$, where $T$ is the circle group, we thus have the following recurrence criterion

$E_\pi \log |S_1| = 0.$

Consider a group $G$. If all (ordinary) random walks generated by probability measures whose supports are not contained in any proper closed subgroups of $G$ are transient, the group itself is called a transient group. It is well-known that $R^m$ is transient for $m \geq 3$. Under mild conditions Markov random walks are also transient on $R^m$, $m \geq 3$, see [4]. Our aim is to show that the same holds true for all transient groups provided that some additional assumptions are satisfied.

Introduce the following condition on the transition kernel $P$ of the Markov chain $X$, cf. [9, (1.7)]

(1) $P(x, A) \geq h(x) \nu(A), \ x \in C, \ A \in \mathcal{C},$
where \( \nu \) is a measure on \((C, \mathcal{C})\) and \( h \geq 0 \) a function on \( C \) with \( \int h(x) \pi(dx) > 0 \).

Using the splitting technique described in [9], cf. also [1], we can decompose the chain \( X = (X_1, X_2, \ldots) \) by (a.s. finite) random times \( \tau(1), \tau(2), \ldots \) in such a way that the blocks \( X_{\tau(i-1)+1}, \ldots, X_{\tau(i)} \) and \( X_{\tau(i)+1}, \ldots, X_{\tau(i+1)} \) are independent and identically distributed (i.i.d.). (For a thorough exposition of the technique the reader should consult the fundamental [10]).

**Remark.** For a Harris recurrent chain, \( P^k \) is minorized as in (1) for some positive integer \( k \), see [9]. The choice of \( k = 1 \) is necessary in order to ensure the decomposition of \( X_1, X_2, \ldots \) into independent blocks.

Suppose first that \( C \) is discrete and that \( X \) is positive recurrent and aperiodic. (1) is automatically satisfied with \( h(x) = 1_{\{c\}}(x) \), \( \nu = P(c, \cdot) \), where \( c \) is an arbitrary fixed element of \( C \). The times \( \tau(1), \tau(2), \ldots \) are the successive returns of \( X \) to \( c \). They are all a.s. finite: \( E_\pi \tau(1) = (\pi(c))^{-1} \).

\[
X_{\tau(1)+1} X_{\tau(1)+2} \ldots X_{\tau(n)} = S_{\tau(1)}^{-1} S_{\tau(n)} \quad n = 1, 2, \ldots
\]

are products of i.i.d. random elements of \( \Pi(c,c) \subseteq G \). In other words, \( (S_{\tau(1)}^{-1} S_{\tau(n)})_{n=1}^\infty \) is a random walk on \( \Pi(c,c) \), generating all of \( \Pi(c,c) \). Hence if \( \Pi(c,c) \) is a transient group the random walk is transient. If \( \Pi(c,c) \) is a subsemigroup of \( G \) which is not a group, then the random walk must be transient, too (because any recurrent random walk automatically generates a group, see [12]). The same holds for the process \( (S_{\tau(n)})_{n=1}^\infty \), which differs from \( (S_{\tau(1)}^{-1} S_{\tau(n)})_{n=1}^\infty \) only by its initial distribution. Now, by (an extension to the group case of ) the Main Lemma of [1], we can conclude that the Markov random walk \( (S_n)_{n=1}^\infty \) itself is transient. Hence the desired result – \( \Pi(c,c) \) transient or no group at all -\( \Rightarrow G \) transient - holds for discrete \( C \). (From the discussion in Section 1 we know that the invariance semigroups \( \Pi(c,c) \) are all isomorphic if they are groups).

In the general case the times \( \tau(i) \) are return times to the set, where the function \( h \) is positive, but, in general, not all of them. The difficulty lies in the problem of characterizing the set (subgroup), where the random walk \( (S_{\tau(1)}^{-1} S_{\tau(n)})_{n=1}^\infty \) “lives”, i.e. the smallest closed subgroup containing all the random variables \( S_{\tau(1)}^{-1} S_{\tau(n)} \), \( n = 1, 2, \ldots \). Under certain conditions on \( f \) and \( \nu \), a characterization of this subgroup can be obtained. Suppose, for example, that the support of \( \nu \) has non-empty interior and that the function \( f \) is an open mapping. Then the smallest closed subgroup that could possibly contain \( f(\text{supp}(\nu)) \) is the identity component of \( G \). If \( G \) is connected and transient it follows that \( S \) is transient.

We can formulate this sufficient condition for transience slightly more
generally. The distribution of the random group element $f(Z)$, where $Z$ is distributed according to $v$, is a measure $vf^{-1}$ on the Borel sets of $G$. If the support of $vf^{-1}$ generates all of $G$, then $(S_{\tau(n)}^{-1} S_{\tau(n)})_{n=1}^\infty$ lives on $G$ proper, i.e., there is no closed subgroup a.s. containing all the products. Hence this random walk is transient if $G$ is a transient group. As before, it follows that $(S_n)_{n=1}^\infty$ is transient, too.

We summarize our results in a

**Theorem 2.** Let $X$ be positive Harris satisfying (1). If

(i) $C$ is discrete and the invariance semigroups $\Pi(c,c)$, $c \in C$, are either no groups at all or transient groups or

(ii) the smallest closed subgroup containing the support of the measure $vf^{-1}$ $(vf^{-1}(A) = v(f^{-1}(A)), A \in \mathcal{G})$ is $G$ itself and $G$ is transient,

then $S$ is transient.

We saw above that for $G = \mathbb{R}$ or $K \times \mathbb{R}$ the stationarity of $X$ is enough to furnish us with a recurrence criterion, namely $E_\pi S_1 = 0$ and $E_\pi f_2(X) = 0$, respectively. For other groups, we will again use the decomposition of $X_1, X_2, \ldots$ into independent blocks. If $(S^{-1}_{\tau(1)} S_{\tau(n)})_{n=1}^\infty$ is recurrent, then a fortiori $(S_n)_{n=1}^\infty$ is.

A random walk $S$ on $\mathbb{R}^2$ is recurrent if and only if $ES_1 = 0$ and $ES_1^2 < \infty$, cf. [12, p. 98]. Hence our Markov random walk on $\mathbb{R}^2$ will be recurrent if and only if

$$E_\pi(S_{\tau(2)} - S_{\tau(1)}) = 0 \quad \text{and} \quad E_\pi|S_{\tau(2)} - S_{\tau(1)}|^2 < \infty$$

(assuming, as before, that the process does not live on a smaller subgroup). The first condition reduces to $E_\pi S_1 = 0$, since $E_\pi S_\tau = (E_\pi \tau)(E_\pi S_1)$ (with $\tau = \tau(1)$). The second condition may be written

$$E_\pi \left| \sum_{i=0}^{\tau} f(X_i) \right|^2 < \infty.$$

If $f$ is bounded, $E_\pi \tau^2 < \infty$ will guarantee the recurrence of $S$.

**Theorem 3.** Let $X$ be as in Theorem 2. If

(i) $C$ is discrete, the invariance semigroups $\Pi(c,c) = \mathbb{R}$ or $\mathbb{R}^2$ and $E_\pi S_1 = 0$ and, in addition, in the $\mathbb{R}^2$-case,

$$E_c \left| \sum_{i=1}^{\tau} f(X_i) \right|^2 < \infty \quad \text{for some} \quad c \in C,$$

where $\tau$ is the first return time to $c$, or
(ii) $G = \mathbb{R}$ and $E\pi S_1 = 0$ or

(iii) $G = \mathbb{R}^2$, $E\pi S_1 = 0$ and

$$E|\sum_{i=0}^{\tau(1)} f(X_i)|^2 < \infty,$$

where $\tau(1), \tau(2), \ldots$ are the times discussed above decomposing the chain $X_1, X_2, \ldots$ into i.i.d. blocks,

then the Markov random walk $S$ is recurrent.

The conditions given are also necessary (in (ii) and (iii) provided that the distribution of $S_{\tau(2)} - S_{\tau(1)}$ is not supported by a proper closed subgroup).

**Remark.** If $G$ is compact (or $\Pi(c, c)$ in the discrete case) then $S$ is, of course, always recurrent.

3.

For all Harris chains condition (1) is valid for some $\pi^k$. However, it is crucial for the decomposition of $X$ into i.i.d. blocks that $k = 1$. This limitation is a considerable drawback, when we want to investigate Markov random walks, where the increment $S_{n+1} - S_n$ is a function of two (or more) consecutive values, e.g., $S_{n+1} - S_n = f(X_n, X_{n+1})$.

The process $S$ is still a Markov random walk: just consider $\hat{X}_n = (X_n, X_{n+1}) \in C \times C$, cf. [11], but property (1) is lost in general.

$$P(\hat{X}_{n+1} \in A \times B \mid \hat{X}_n = (x_1, x_2)) =$$

$$\delta_{x_2}(A) P(X_{n+2} \in B \mid X_{n+1} = x_2) \geq \delta_{x_2}(A) h(x_2) \nu(B)$$

which cannot be written in the form (1) unless, of course, $\pi\{x_2\} > 0$. The two-step transition is again more regular,

$$P(\hat{X}_{n+2} \in A \times B \mid \hat{X}_n = (x_1, x_2)) \geq h(x_2)(\int_A \nu(dy) h(y)) \nu(B).$$

Random walks on certain groups, notably the group of rigid motions on $\mathbb{R}^d$, have been investigated using the theory of semi-Markov processes, cf. [4]. It can also be regarded as a Markov random walk on $\mathbb{R}^d$. The essential features of random walks on the semigroup of real $n \times n$-matrices of rank $k < n$ may be studied by viewing (an embedded) Markov random walk on the general linear group $GL(k, \mathbb{R})$, see [5] or the survey [6].

Our Example 3 is a simple example of such a walk, the product of
random projections of $\mathbb{R}^2$ on 1-dimensional subspaces. Almost all of these projections can be represented as matrices of the form
\[
\begin{pmatrix}
(1 + \alpha \beta)^{-1} & (1 + \alpha \beta)^{-1} \alpha \\
\beta(1 + \alpha \beta)^{-1} & \beta(1 + \alpha \beta)^{-1} \alpha
\end{pmatrix}
\]
where $\alpha, \beta \in \mathbb{R}, \, \alpha \beta \neq -1$. The projection is orthogonal if its nullspace, spanned by the vector $(-\alpha, 1)$, and its range, spanned by $(1, \beta)$, are orthogonal; thus the criterion for orthogonality is $\alpha = \beta$.

The product of two projections $P_1$ and $P_2$ is not, in general, a projection itself (below, the subscripts $i$ refer to the matrix $P_i, \; i = 1, 2$):
\[
P_1 P_2 = (1 + \alpha_1 \beta_1)^{-1} (1 + \alpha_1 \beta_2) (1 + \alpha_2 \beta_2)^{-1} \begin{pmatrix} 1 & \alpha_2 \\ \beta_1 & \beta_1 \alpha_2 \end{pmatrix}.
\]
Suppose $X = (X_1, X_2, \ldots)$ is a stationary Markov chain on the set of matrices of the same type as $P$ above. The first element of the product matrix $X_1 X_2 \ldots X_n$ is a Markov random walk on the multiplicative semigroup $\mathbb{R}$. Call that element $S_n$, then
\[
S_n = \prod_{k=1}^{n} f(X_{k-1}, X_k), \; S_0 = 1, \; S_1 = (1 + \alpha_1 \beta_1)^{-1},
\]
where
\[
f(X_{k-1}, X_k) = (1 + \alpha_{k-1} \beta_k) (1 + \alpha_k \beta_k)^{-1}, \; k = 2, 3, \ldots.
\]
Assume that $P_\pi(f(X_k, X_{k+1}) = 0) = 0$ for all $k$ (where $\pi$ is the invariant probability measure for the chain $X$). Then $S$ is a Markov random walk on the multiplicative group $\mathbb{R} \setminus \{0\}$. Hence it is recurrent if and only if
\[
E_\pi \log |f(X_0, X_1)| = \int \log |f(x, y)| \pi(dx) P(x, dy) = 0.
\]

Acknowledgement. The generous support of the Academy of Finland (Science Research Council) is gratefully acknowledged. The suggestions of an anonymous referee resulted in numerous improvements and clarifications.

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