RANDOM POLYTOPES
AND THE VOLUME-PRODUCT OF
SYMmetric CONVex BODIES

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1. Introduction.
A well known problem in the theory of convex sets is to find a lower bound for the product of volumes $P(K) = V(K)V(K^*)$, where $K$ is an $n$-dimensional symmetric convex body (i.e., bounded, symmetric, convex set in the $n$-dimensional Euclidean space $\mathbb{R}^n$, having non-empty interior) and $K^*$ is the dual body of $K$ with respect to its center of symmetry. An upper bound was obtained by Santalo [10]

$$P(K) \leq \pi_n^2$$

($\pi_n$ denotes the volume of the $n$-dimensional Euclidean unit ball). For an elementary proof of (1) and related results cf. [9].

A widespread conjecture is that

$$P(K) \geq 4^n/n!$$

Equality in (2) is known to hold for

$$K = B_1 = B(l_1^n) = \left\{ x = (x_1, \ldots, x_n); \sum_{i=1}^n |x_i| \leq 1 \right\}$$

and for its dual

$$B_\infty = B(l_\infty^n) = \left\{ x = (x_1, \ldots, x_n); \max_{1 \leq i \leq n} |x_i| \leq 1 \right\}.$$ 

A wider family of convex bodies for which equality holds in (2) was discussed in [9] where (2) was also proved for a special class of convex bodies.

A somewhat weaker problem is whether there is a universal constant $c > 0$ such that $[P(K)]^{1/n} \geq c/n$ for all $n$-dimensional $K$ ((1) implies that $[P(K)]^{1/n} \leq 2\pi e/n$).

In [4] it was shown that

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\[(3) \quad [P(K)]^{1/n} \geq \frac{c}{n \log n}\]

for all \(K\).

In this paper we show how a result of R. Schneider on the limiting expectation of the number of vertices of a random polytope, can be adapted and used to prove (2) for a class of convex bodies – the class of zonoids and for bodies constructed using them. We remark that Mahler [8] proved (2) for \(n = 2\), this becomes a special case of Theorem 2 here since all 2-dimensional symmetric convex bodies are zonoids.

As for the general problem of confirming (2) – as far as we know it is still open. We remark that a proof of (2) was given in [5] but there seems to be some errors in the proof, in particular, the uniqueness which is claimed there is known to be false (cf. [9]). We use here freely, terminology and notations from the theories of convex bodies (cf. [3] and [6]) and Banach spaces (cf. [7]).

2.

In this section we adapt the situation of [11] to our needs.

In \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) (\(n \geq 2\)) let, for \(N \geq n\)

\[G_1 = (H_1, -H_1), \quad G_2 = (H_2, -H_2), \ldots, G_N = (H_N, -H_N)\]

be \(N\) random pairs of hyperplanes, where the hyperplanes \(H_i, -H_i\) in each pair are symmetric with respect to the origin and meet the Euclidean unit ball \(B^n\) whose center is the origin. We denote by \(\mathcal{H}_i\) the “strip”

\[\mathcal{H}_i = \{x \in \mathbb{R}^n; -1 \leq f_i(x) \leq 1\}\]

where \(f_i \in \mathbb{R}^n\) is such that \(\langle f_i, x \rangle = 1\) for all \(x \in H_i\) \(\langle \cdot, \cdot \rangle\) denotes the scalar product of \(\mathbb{R}^n\).

Let \(Y_N\) be the number of vertices of the symmetric (with respect to the origin) polyhedral set

\[\mathcal{H} = \bigcap_{i=1}^{N} \mathcal{H}_i.\]

\(Y_N\) is a real random variable, let \(E(Y_N)\) be its expectation. Assume now that the pairs \(G_i\) are independent, identically distributed and that their distribution is given in the following way: we are given an even probability measure \(\mu\) on \(S^{n-1}\) and define a measure \(\nu\) on the space

\[G^n = \{G = (H, -H); \ H \text{ a hyperplane in } \mathbb{R}^n \text{ which intersects } B^n\}\]

by identifying \(G^n\) with \(S^{n-1} \times [0,1]\) via \(G \leftrightarrow (u, \tau) \in S^{n-1} \times [0,1]\) where \(G = (H_{u,\tau}, -H_{u,\tau})\).
$H_{u,\tau} = \{ x \in \mathbb{R}^n ; \langle x,u \rangle = \tau \}$

and on $S^{n-1} \times [0,1]$, $\nu = (\mu \times \lambda)$ ($\lambda$ - Lebesgue measure on $[0,1]$).

We assume also that the support of $\mu$ is not contained in any $(n-1)$-dimensional subspace of $\mathbb{R}^n$. $G_i$ are distributed by the probability $\nu$.

The proof of the following theorem is an easy modification of the proof of [11] and we omit it.

**Theorem 1.**

$$\lim_{N \to \infty} E(Y_N) = 2^{-n}n! \ V(Z_\mu) V(Z_\mu^*),$$

where $Z_\mu$ is the convex body whose support function $h(x)$ is given by

$$h(x) = \frac{1}{2} \int_{S^{n-1}} |\langle u,x \rangle| \ d\mu(u).$$

**Definition.** Once the existence of the limit is proved we define

$$E_\mu = \lim_{N \to \infty} E(Y_N).$$

**Remark.** Theorem 1 and [11] show that also $E_\mu = \lim_{N \to \infty} E(X_N)$, where $X_N$ is the number of vertices of the random polyhedral set generated by $N$ random single hyperplanes, rather than pairs of hyperplanes, with the same distribution.

3.

The convex bodies $K = K_\nu$ whose support functions $h_K$ are given by

$$h_K(x) = \int_{S^{n-1}} |\langle u,x \rangle| \ d\nu(u)$$

with a given even nonnegative Borel measure $\nu$ on $S^{n-1}$, are called zonoids. Considered as unit balls of $n$-dimensional Banach spaces, these are exactly the $n$-dimensional Banach spaces whose duals can be embedded isometrically in $L_1$-spaces (cf. [2]).

Let $K = K_\nu$ be a zonoid, in order to estimate $P(K) = V(K) V(K^*)$ we may assume that $\|\nu\| = 1/2$, i.e., that $K = Z_\mu$ ($\mu = 2\nu$) in the terminology of Theorem 1. We get:

**Theorem 2.** a) Let $\mu$ be as in Theorem 1, then $E_\mu \geq 2^n$;

b) Let $K$ be a zonoid of dimension $n$, then $V(K) V(K^*) \geq 4^n/n!$. 

Proof. With the preceding remark it is clear that b) follows from Theorem 1 and a). In order to prove a) it is sufficient to assume that \( \mu \) is atomic with finitely many atoms, i.e.

\[
\mu = \sum_{j=1}^{m} \lambda_j \delta_{u_j} + \sum_{j=1}^{m} \lambda_j \delta_{-u_j}
\]

where \( m \geq n \), \( \lambda_j > 0 \) and \( u_j \in S^{n-1} \) for all \( j \), \( \sum_{j=1}^{m} \lambda_j = 1/2 \) and \( \{u_j\}_{j=1}^{m} \) spans \( \mathbb{R}^n \).

The proof of Theorem 2 is based on three lemmas, the first lemma shows that the polyhedron \( \mathcal{H} \) is a polytope with probability which approaches 1. Throughout this proof the letter \( P \) denotes probability.

Lemma 1.

\[
P(\exists 1 \leq j \leq m \text{ such that } \forall 1 \leq i \leq N, \ H_i \nparallel u_j) \xrightarrow{N \to \infty} 0.
\]

Proof. The set whose probability is to be computed is

\[
A_N = \bigcup_{j=1}^{m} \bigcap_{i=1}^{N} (H_i \nparallel u_j)
\]

hence

\[
P(A_N) \leq \sum_{j=1}^{m} P\left( \bigcap_{i=1}^{N} (H_i \nparallel u_j) \right) = \sum_{j=1}^{m} \left[ P(H_1 \nparallel u_j) \right]^N.
\]

Now

\[
P(H_1 \nparallel u_j) = 1 - P(H_1 \perp u_j)
\]
\[
P(H_1 \perp u_j) = P(\exists -1 \leq \tau \leq 1 \text{ such that } H_1 = H_{u_j,\tau}) = 2\lambda_j
\]

hence

\[
P(A_N) \leq \sum_{j=1}^{m} (1 - 2\lambda_j)^N \xrightarrow{N \to \infty} 0.
\]

An \( n \)-dimensional polyhedral set is said to be simple if all its vertices are intersection points of exactly \( n \) \( (n-1) \)-dimensional faces.

Lemma 2. For all \( N \geq n \)

\[
P(\mathcal{H} \text{ is not simple}) = 0.
\]

Proof. Let \( B_N \) be the set whose probability is to be computed. We have with the notation of Theorem 1:
\[ B_N \subseteq \bigcup_{k=(\epsilon_1 k_1, \ldots, \epsilon_{n+1} k_{n+1}), \epsilon_j = \pm 1} (\epsilon_1 H_{k_1} \cap \ldots \cap \epsilon_{n+1} H_{k_{n+1}} = \{x\} \text{ for some } x) \]

Therefore
\[ P(B_N) \leq 2^{n+1} \binom{N}{n+1} P(H_1 \cap \ldots \cap H_{n+1} = \{x\} \text{ for some } x) \]
\[ = 2^{n+1} \binom{N}{n+1} \int_{H_1 \cap \ldots \cap H_n = \{x\}} P(x \in H_{n+1}) dv(G_1) \ldots dv(G_n). \]

The nature of \( v \) guarantees that for fixed \( x \), \( P(x \in H_{n+1}) = 0 \), hence \( P(B_N) = 0 \).

The third lemma is a result of I. Barany and L. Lovasz.

**Lemma 3.** [1] Let \( K \) be an \( n \)-dimensional symmetric, simple convex polytope. Let \( v \) be the number of vertices of \( K \), then \( v \geq 2^n \).

We complete the proof of Theorem 2 as follows: for fixed \( N \), let \( A_N \) and \( B_N \) be as in the proofs of Lemmas 1 and 2. If \( (G_1, \ldots, G_N) \) is not in \( A_N \cup B_N \). Then \( \mathcal{K} \) satisfies the assumptions about \( K \) that were made in Lemma 3 and therefore \( Y_N \geq 2^n \). By Lemmas 1 and 2
\[ P(A_N \cup B_N) \xrightarrow{N \to \infty} 0 \]
which completes the proof.

### 4. Two corollaries.

It is clear that \( P(K) = P(K^*) \), hence, if \( K \) is such that \( K^* \) is a zonoid, then \( K \) satisfies (2).

**Corollary 1.** Let \( E \) be a \( k \)-dimensional Banach space having a basis \((e_i)_{i=1}^k\) with an unconditional basic constant \( \chi((e_i)) = 1 \) (i.e., \( \| \sum \pm \alpha_i e_i \|_E = \| \sum \alpha_i e_i \|_E \) for all scalars \( \alpha_i \) and all choices of signs). Let \( E_1, \ldots, E_K \) be finite dimensional Banach spaces, each of which is either a subspace of an \( L_1 \) space or a quotient space of a \( C(K) \) (\( K \)-compact) space. Let \( F = (E_1 \oplus \ldots \oplus E_k)_E \) be the direct sum of \( (E_i) \) in the sense of \( E \) (for \( x = (x_1, \ldots, x_k) \in F, x_i \in E_i, \| x \|_F = \| \sum_{i=1}^k x_i \|_E e_i \|_E \)). If \( \dim F = n \) then \( P(B_F) \geq 4^n/n! \) (\( B_F \) is the unit ball of \( F \)).

**Proof.** Use [9, Theorem 15 and Theorem 21] and Theorem 2.
Corollary 2. Let $X_N$ be the number of vertices of a random polyhedral set generated by $N$ independent random hyperplanes in $\mathbb{R}^n$, distributed according to the same probability measure as in Theorem 1. Then $\lim_{N \to \infty} E(X_N) \geq 2^n$ and also

$$2 \leq \left[ \lim_{N \to \infty} E(X_N) \right]^{1/n} \leq \pi.$$

Proof. The first claim follows from Theorem 2 and the remark following the proof of Theorem 1. The right hand side of the second inequality follows from [11] and Stirling's formula.

Remarks added in proof. a) Recently, we proved a uniqueness in Theorem 2: equality in b) (for a zonoid $K$) holds if and only if $K$ is a parallelotope.

b) Recently, J. Bourgain and V. Milman proved that $[P(K)]^{1/n} \geq c/n$ for all $n$-dimensional $K$.

References
