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UNIVERSALLY WEAKLY INNER ONE-PARAMETER AUTOMORPHISM GROUPS OF SEPARABLE C*-ALGEBRAS, II

LAWRENCE G. BROWN AND GEORGE A. ELLIOTT

Abstract.
A universally weakly inner one-parameter automorphism group of a quotient of a separable C*-algebra is lifted to an automorphism group which is again of this type. (This answers a question in [2].) The characterization of universally weakly inner one-parameter automorphism groups of a separable C*-algebra given in [2] is shown to hold without the assumption of separability.

1. Introduction.

In [2], a characterization was given of universally weakly inner one-parameter automorphism groups of a separable C*-algebra. This was used to show that a universally weakly inner one-parameter automorphism group of a quotient of a separable C*-algebra can be lifted to a strongly continuous one-parameter automorphism group of the C*-algebra.

The main purpose of the present paper is to show that such a lifting can be chosen to be universally weakly inner. This is done in Section 3.

We also point out, in Section 2, that as far as the characterization of universally weakly inner one-parameter automorphism groups is concerned, the assumption made in [2] that the C*-algebra is separable may be dropped. Note that it is still only in the separable case, however, that we are able to apply this characterization to lift such a group from a quotient.

Finally, we correct some errors in [2].

2. Approximation by inner automorphism groups.

THEOREM 2.1 (cf. 2.1 of [2]). *Let A be a C*-algebra, and let α be a one-parameter automorphism group of A. Then α is universally weakly inner if and only if there exists a net (h_γ) of selfadjoint elements of A with the following properties:*

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(i) $\|\alpha_t(a) - \exp(ith_\gamma)a\exp(-ith_\gamma)\| \to 0$ *for each* $a \in A$, *uniformly for* t *in each compact subset of* $\mathbf{R}$;

(ii) *the net* $((h_\gamma + i)^{-1})$ *is Cauchy in the *-ultrastrong topology on* $A \subset A^{**}$, *and its limit in* A^{**} *has kernel equal to* 0.

PROOF. Modify the proof of Theorem 2.1 of [2] as follows. Let α be universally weakly inner. For each fixed finite subset F of smooth elements of A, construct a net $(h_{n,V}) = (h_{n,V,F})$ as in 2.4 of [2] but with the sequence (a_p) on page 141 of [2] replaced by the finite set F. Order the indices as follows:

$$(n,V,F) \leqq (n',V',F') \Leftrightarrow n \leqq n', \;\; V \supset V', \;\; F \subset F'.$$

The net $(h_\gamma) = (h_{n,V,F})$ verifies (i) and (ii).

3. A lifting theorem.

THEOREM 3.1 (cf. 3.1, 3.2 of [2]). *Let A be a separable* C*-*algebra, let I be a closed two-sided ideal of A, and let α be a universally weakly inner one-parameter automorphism group of the quotient* C*-*algebra A/I. Then there exists a universally weakly inner one-parameter automorphism group of A inducing α in the quotient A/I.*

PROOF. Denote the generator of α by δ. Choose a dense sequence (a_p) in the domain of δ. By 2.5 of [2], as revised in 4.5 below, with A/I in place of A, there exists a sequence (h_n) of selfadjoint elements of A/I such that

(i) $[ih_n, a_p] \to \delta(a_p)$, $\quad p = 1,2,\ldots,$

(ii) $[h_m, h_n] \to 0$ $\quad$ (as $m,n \to \infty$),

(iii) $\alpha_t(a) - \exp(ith_n)a\exp(-ith_n) \to 0$, $\quad t \in \mathbf{R}, a \in A/I$.

Passing to a subsequence we may suppose that

(i)′ $\|[h_{n+1} - h_n, a_p]\| < 2^{-n}$, $\;p = 1,\ldots,n$,

(ii)′ $\|[h_{n+1}, h_n]\| < 2^{-n}$.

Choose an approximate unit (u_i) for I which is a central sequence for A (that is, $[u_i, b] \to 0$ for all $b \in A$), and such that, in addition, each u_i is positive of norm one, and for each $i = 1,2,\ldots,$ $u_{i+1}u_i = u_i$. Such an approximate unit is constructed in [3; page 129, lines 5 to 12].

Since the quotient map $A \to A/I$ is open there is a dense sequence (b_p) in A such that

$$a_p = b_p + I, \;\; p = 1,2,\ldots.$$

Choose a sequence (g_n) of selfadjoint elements of A such that

$$g_n + I = h_n, \ n = 1,2,\ldots.$$

Set $g_1 = k_1$. In view of (i)′ and (ii)′ for $n = 1$, choose i_1 large enough that

$$2\|[1-u_{i_1},b]\| \, \|g_2-g_1\| + \|(1-u_{i_1})[g_2-g_1,b](1-u_{i_1})\| < 2^{-1}$$

with $b = b_1$ and $b = k_1$, and set

$$k_1 + (1-u_{i_1})(g_2-g_1)(1-u_{i_1}) = k_2.$$

Then $\|[k_2-k_1,b]\| < 2^{-1}$ with $b = b_1$ and $b = k_1$, that is, (i)′ and (ii)′ hold for $n = 1$ with b_1 in place of a_1, and k_1 in place of h_1. Continuing in this way, we obtain a sequence (k_n) of selfadjoint elements of A such that (i)′ and (ii)′ hold with b_p in place of a_p and k_n in place of h_n.

Hence by [4, Lemma 5], applied to k_n and k_{n+1}, the sequence

$$(\exp(itk_n)b\exp(-itk_n))$$

is Cauchy in norm, first for $b = b_p, p = 1,2,\ldots$, and therefore for any $b \in A$, uniformly for t in any compact subset of R.

The limit defines a strongly continuous one-parameter automorphism group β of A which leaves any closed two-sided ideal of A invariant, and by (iii) above induces α in the quotient A/I (recall that $k_n + I = h_n$).

We shall now show that β is universally weakly inner. Note that the construction above differs from that of [2] only in the more specific choice of the approximate unit (u_i). Actually, to show that β is universally weakly inner, we shall have to reinforce this specific choice of (u_i) by passing to a subsequence.

A universal representation of A is of the form $\pi \oplus \pi_1$ on a Hilbert space $H \oplus H_1$, where $\pi|I$ is a universal representation of I and π_1 is a universal representation of A/I (pulled back to A). Since β induces α on A/I and α is given to be universally weakly inner, it is enough to show that β is weakly inner relative to π. We shall write $a\xi$ instead of $\pi(a)\xi$ for $a \in A$, $\xi \in H$. It is sufficient to show that the sequence $(k_n\xi)$ is convergent for ξ in a dense subset of H, and the bounded sequences $((k_n \pm i)^{-1})$ converge strongly. For then if $v \in I^{**}$ denotes the *-strong limit of $(k_n + i)^{-1}$, both v and v^* have dense image. (To see that the image of v, for example, is dense, note that if $(k_n + i)\xi \to \eta$, then $\xi = (k_n + i)^{-1}(k_n + i)\xi \to v\eta$, so $\xi = v\eta$.) Hence by the Trotter-Kato theorem ([1; Theorem 3.1.26]), there exists a selfadjoint operator k affiliated with I^{**} such that k_n converges to k in the strong resolvent sense, and therefore such that $\exp(itk_n)$ converges strongly to $\exp(itk)$, $t \in \mathrm{R}$.

The sequence $(k_n\xi)$ is eventually constant for each vector ξ in the range of u_j. (In fact it is constant at least from the point $j+1$ on.) It follows on the one hand that the sequence $(k_n\xi)$ converges for each ξ in a dense subset of H, namely, the union of the images of the operators u_j. It follows on the other hand, as we shall show, that $((k_n \pm i)^{-1})$, constructed with respect to a suitable subsequence of (u_j), converges strongly on this Hilbert space. On each vector in the image of $(k_{j+1} \pm i)u_j$, the sequence $((k_n \pm i)^{-1})$ is constant from the point $j+1$ on (as $(k_{j+1} \pm i)u_j = (k_{j+1+p} \pm i)u_j$, $p = 1,2,\ldots$), so to show that the bounded sequence $((k_n \pm i)^{-1})$ converges strongly it is enough to show that the union of the images of $(k_{j+1} \pm i)u_j$, $j = 1,2,\ldots$, is dense.

In other words, it is enough to show that if k denotes the limit of the sequence (k_n) defined on the domain D consisting of the images of the u_j, then $(k \pm i)D$, the image of $k \pm i$ on this domain, is dense. This in turn is equivalent to the essential selfadjointness of the symmetric operator k on the domain D. By Nelson's theorem (see [1; 3.1.21]), this would hold if D were invariant under k and contained a dense set of analytic vectors for k. While this stronger property need not hold for k itself, we shall show that provided that the construction of the sequence (k_n) is based on a suitable subsequence of (u_j), this stronger property holds for a bounded perturbation of k, i.e. for some symmetric operator k' on D such that $k' - k$ is bounded. Then by Nelson's theorem k' is essentially selfadjoint on D; hence also k is essentially selfadjoint on D as desired.

We obtain such a perturbation k' of k as follows. We first change g_1 to a selfadjoint element x_1 of A, within distance 2^{-1} of g_1, such that when (u_j) is replaced by a suitable subsequence,

$$(\mathrm{P}_1) \qquad (1 - u_{j+1})x_1 u_j = 0, \ j \geqq 1.$$

To do this, note that for each j the element

$$g_{1,j} = g_1 - (1 - u_{j+1})g_1 u_j - u_j g_1 (1 - u_{j+1})$$

is selfadjoint, is close to g_1 if j is large, and satisfies

$$(1 - u_{j+2})g_{1,j}u_{j-1} = 0,$$

and hence also

$$(1 - u_{j+2+p})g_{1,j}u_{j-1-q} = 0, \ p,q = 0;1,2,\ldots.$$

Here we use that (u_j) is a central sequence of selfadjoint elements of A such that for all j, $u_{j+1}u_j = u_j$. Now, with $j_1 < j_2 < \ldots$, repeat this construction successively, first with g_{1,j_1} in place of g_1 and with $j = j_2$ to get g_{1,j_1,j_2}, then with g_{1,j_1,j_2} in place of g_1 and with $j = j_3$ to get g_{1,j_1,j_2,j_3}, and so on. If (j_n)

increases sufficiently rapidly, then the sequence $(g_{1,j_1,\ldots,j_r})_{r=1,2,\ldots}$ converges to an element x_1 of A, within distance 2^{-1} of g_1. The preceding equality persists after these substitutions, and in the limit yields

$$(1-u_{j_r+2+p})x_1u_{j_r-1-q}=0,\ p,q=0,1,2,\ldots,r=1,2,\ldots.$$

In particular, passing to a subsequence of (j_n), we have

$$(1-u_{j_{r+1}})x_1u_{j_r}=0,\ r=1,2,\ldots,$$

which, after a change of notation, is just (P_1). We note that the property (P_1), because of the condition $u_{j+1}u_j=u_j$, still holds after passage to a subsequence.

Continuing with the construction of k', we pass again to a subsequence of (u_j), but without changing u_1, and, as above for g_1, we change g_2-g_1 to an element x_2 of A, within distance 2^{-2} of g_2-g_1, so that

$$(\mathrm{P}_2)\qquad (1-u_{j+1})x_2u_j=0,\ j\geqq 2.$$

We then pass again to a subsequence of (u_j), with now both u_1 and u_2 unchanged, and change g_3-g_2 to an element x_3 of A, within distance 2^{-3} of g_3-g_2, so that

$$(\mathrm{P}_3)\qquad (1-u_{j+1})x_3u_j=0,\ j\geqq 3.$$

Continuing in this way, we pass to a single subsequence of (u_j) and obtain a sequence (x_n) of selfadjoint elements of A such that, for all n, $\|(g_n-g_{n-1})-x_n\|\leqq 2^{-n}$, where $g_0=0$, and

$$(\mathrm{P}_n)\qquad (1-u_{j+1})x_nu_j=0,\ j\geqq n.$$

Changing each g_n-g_{n-1} by an element of norm at most 2^{-n} in the construction of k (which is defined as the sum $\Sigma_0^\infty(1-u_n)(g_{n+1}-g_n)(1-u_n)$ on the domain D, where $u_0=g_0=0$) results in changing k by an element $k'-k$ of A of norm at most $\Sigma_0^\infty 2^{-(n+1)}<\infty$. Of course, the k that is being changed is constructed with respect to a subsequence of the original approximate unit (u_j). Note that the properties (P_n) imply that the operator

$$k'=\Sigma_0^\infty(1-u_{i_n})x_{n+1}(1-u_{i_n})$$

defined on the domain D, where we set $u_{i_0}=0$, leaves D (the union of the images of the u_j) invariant.

To ensure that D consists of analytic vectors for k', we must again pass to a subsequence of (u_j) before the construction of k. We shall pass to a subsequence (u_{j_n}) of (u_j) such that with

$$k'=\Sigma_0^\infty(1-u_{i_n})x_{n+1}(1-u_{i_n})\text{ on }D,$$

with (u_{i_n}) any subsequence of (u_{j_n}), constructed as above, and where (x_n) is viewed as a given sequence of selfadjoint elements of A with the properties (P_n), we have for each $\xi \in D$,

$$\|(k')^n \xi\| = O(n!).$$

Then each $\xi \in D$ is analytic for k' (see [1; Definition 3.1.17]; we have $\Sigma \|(k')^n \xi\| \, |z|^n/n! < \infty$ whenever $|z| < 1$).

Set

$$(1 - u_{i_{n-1}})x_n(1 - u_{i_{n-1}}) = y_n.$$

Then with

$$k' = y_1 + y_2 + \dots \text{ on } D,$$

$k' - k$ is bounded, where k is defined as above with respect to the subsequence (u_{i_n}) of (u_{j_n}). Provided $j_n > n$, we have

$$(1 - u_{j+1})y_n u_j = 0 \text{ for all } j;$$

if $j \geqq n$ this holds by (P_n), and if $j \leqq n-1$ then $j < j_{n-1} \leqq i_{n-1}$ and so $(1 - u_{i_{n-1}})u_j = 0$. Thus, if $\xi \in \operatorname{image} u_j$, then, for any n,

$$y_n \xi \in \operatorname{kernel}(1 - u_{j+1}) \subset \operatorname{image} u_{j+1}.$$

Moreover, if $\xi \in \operatorname{image} u_j$, then $y_n \xi = 0$ when $j_n > j$, so

$$k'\xi = (y_1 + \dots + y_q)\xi$$

where q is such that $j_{q+1} > j$. Then also

$$(k')^r \xi = (y_1 + \dots + y_q)^r \xi$$

for $1 \leqq r \leqq j_{q+1} - j$. Similarly,

$$(k')^{(j_{q+1}-j)+r}\xi = (y_1 + \dots + y_{q+1})^r (y_1 + \dots + y_q)^{j_{q+1}-j}\xi$$

for $1 \leqq r \leqq j_{q+2} - j_{q+1}$. Furthermore,

$$\begin{aligned}(k')^{(j_{q+p}-j)+r}\xi = (y_1 + \dots + y_{q+p})^r (y_1 + \dots + y_{q+p-1})^{j_{q+p}-j_{q+p-1}} \dots \\ \dots (y_1 + \dots + y_q)^{j_{q+1}-j}\xi\end{aligned}$$

for $1 \leqq r \leqq j_{q+p+1} - j_{q+p}$. It follows that if the sequence (j_n) is chosen so that $j_{n+1} - j_n$ is sufficiently large for all n, then for any ξ belonging to $\operatorname{image} u_j$ for some j,

$$\|(k')^n \xi\| = O(n!).$$

It is not difficult to see that this holds if, for example,

$$j_{n+1} - j_n \geqq \|x_1\| + \ldots + \|x_{n+1}\|.$$

(This property will persist to subsequences.)

4. Corrigenda.

4.1. Theorem 2.1 of [2] should be corrected as follows: Condition 2.1 (ii) of [2] should be expanded to coincide with Condition 2.1 (ii) above.

4.2. The proof of Lemma 2.2 of [2] should be corrected as follows: in place of the last line of the proof, put "valid for $k = \varepsilon h$."

4.3. The proof of Lemma 2.3 of [2] should read as follows: "We may suppose that $f_1, \ldots, f_p \geqq 0$, and that $\|k\| \leqq 1$. We shall show that in this case the net (k_λ) may be chosen in the unit ball of A. By Dini's theorem, the set of selfadjoint elements of the unit ball of A^{**} verifying this strengthened conclusion in place of k is monotone closed. Hence by work of Kadison and Pedersen (see, for example, [4; page 956, lines 2 to 10]), this is the set of all selfadjoint elements of the unit ball of A^{**}."

4.4. The justification of the step taken in the first paragraph of 2.4 of [2] is the Trotter-Kato theorem, [3, Theorem 3.1.26] (not the result referred to in [2]).

4.5. The conclusion of Theorem 2.5 of [2] should be strengthened as follows (for the application of this theorem to the lifting theorems both of [2] and of the present paper): Add to the properties 2.5 (i) and 2.5 (ii) of [2] the property

(iii) $$\alpha_t(a) - \exp(ith_n)a\exp(-ith_n) \to 0, \quad t \in \mathrm{R}, a \in A.$$

Add to the proof of Theorem 2.5 of [2] the sentence

"As above, 2.5 (iii) follows from (a) and (c)."

4.6. The following corrections should be made to the proof of Theorem 3.1 of [2]. On page 143, line -5 add "and 2.5 (iii)". On page 143, replace line -3 by "(ii)′ $\|[h_{n+1}, h_n]\| < 2^{-n}$". On page 144, line 14, replace a_p by b_p. On page 144, omit lines 13 to 15, omit the last two lines of Section 3.1, insert "by 2.5 (iii)" after "and" in the preceding line, and add a period at the end of the line.

4.7. Condition (i) on the sequence (h_n) in Problem 3.3 of [2] should be replaced by the following condition:

(i) $$[ih_n, a] \to b \Leftrightarrow a \in \text{domain}\,\delta \text{ and } \delta(a) = b.$$

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DEPARTMENT OF MATHEMATICS,
PURDUE UNIVERSITY,
WEST LAFAYETTE, INDIANA 47907
U.S.A.

MATEMATISK INSTITUT
KØBENHAVNS UNIVERSITET
UNIVERSITETSPARKEN 5
2100 KØBENHAVN Ø
DENMARK