MATH. SCAND. 126 (2020), 479–492

SYMMETRIC RIEMANN SURFACES WITH NO POINTS FIXED BY ORIENTATION PRESERVING AUTOMORPHISMS

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Abstract

We study the symmetric Riemann surfaces for which the group of orientation preserving automorphisms acts without fixed points. We show that any finite group can give rise to such an action, determine the maximal number of non-conjugate symmetries for such surfaces and find a sharp upper bound on maximal total number of ovals for a set of k symmetries with ovals. We also solve the minimal genus problem for dihedral groups acting on the surfaces described above, for odd genera.

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$. By a *symmetry* of X we understand an antiholomorphic involution σ of X with fixed points. It is well known, by the classical results of Harnack, that the set of points fixed by a symmetry σ consists of $1 \leq k \leq g+1$ disjoint simple closed curves called *ovals*. In addition, there also may be orientation reversing fixed point free involutions in the automorphism group, but in the paper we assume that the symmetries indeed have ovals. We shall focus only on these Riemann surfaces, which admit a symmetry but their conformal automorphisms have no fixed points and the only fixed points lie on the ovals of respective symmetries. By a slight abuse of language, we shall call such actions *fixed point free symmetric* actions or *fpf-symmetric* actions in short. Correspondingly, the surfaces in question shall be called *fixed point free symmetric* or *fpf-symmetric* Riemann surfaces. Note, that we actually consider actions which are in a sense *maximal*, meaning that the corresponding automorphism group is the full automorphism group of the Riemann surface.

The paper consists of three parts. In the first part we prove some general results concerning fpf-symmetric actions, showing that any finite group acts without fixed points as the group of orientation preserving automorphisms on

Supported by Polish National Sciences Center by the grant NCN 2015/17/B/ST1/03235.
Received 3 September 2019. Accepted 17 February 2020.
DOI: https://doi.org/10.7146/math.scand.a-121167

some Riemann surface. We also show that any abstractly oriented finite group yields a fpf-symmetric action, provided that it has an orientation reversing involution. Moreover, for any nontrivial divisor n of $g-1$ we find a fpf-symmetric action of a dihedral group of order $2n$ on a Riemann surface of genus g. We also solve, for fpf-symmetric Riemann surfaces, the minimal genus problem for a dihedral group generated by a pair of non-conjugate symmetries. The problem itself is important, as it allows us to see, by using the Hurwitz-Riemann formula, if it is possible for a surface of given genus g to admit at least a pair of non-conjugate symmetries. Generally speaking, there are obstacles for an action of a group of order n on a Riemann surface of genus g, if g is not large enough and solving the minimal genus problem gives a desired lower bound on g, in terms of n.

The second part of the paper concerns quantitative studies of symmetries. We find the sharp upper bound on the maximal number of non-conjugate symmetries that are compatible with a fpf-symmetric action on a Riemann surface of genus g. This result follows the spirit of reference [5], where such a bound is given in general case. Not surprisingly, also in our case the result depends only on the 2-adic structure of g, but the maximal number of symmetries is strictly smaller than the number obtained in [5]. Moreover, we show that the bound is only attained when the symmetries commute.

In the third part of the paper we study topological properties of the symmetries themselves. The sharp upper bounds on the maximal total number of ovals of k non-conjugate symmetries on a Riemann surface of genus g were found in [18] for $2 \leq k \leq 4$, in [12] for $5 \leq k \leq 8$ and in [10] for $k \geq 9$. Also the group structure in the extremal case was studied, see for example [3] for the case of 3 and 4 symmetries, [4] for 2 symmetries. Let us also mention the papers [2], [13], [15], [14], [11], [16], where the authors also study Riemann surfaces with two non-conjugate symmetries. We follow this thread and give similar results in the case of fpf-symmetric actions together with the group structure in the extremal case.

2. Preliminaries

In the paper we shall use the so called non-euclidean crystallographic groups (*NEC groups* for short). These are just the discrete and cocompact subgroups of the group $\mathscr{G}$ of all isometries of the hyperbolic plane $\mathscr{H}$. The algebraic structure for a NEC group Λ and the geometry of the quotient orbifold $\mathscr{H}/\Lambda$ are encoded in the sequence of symbols called the *signature*:

$$s(\Lambda) = (h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{\ell 1}, \ldots, n_{\ell s_\ell})\}), \quad (2.1)$$

where the integers m_i are called *proper periods*, the brackets $C_i = (n_{i1}, \ldots, n_{is_i})$

are called *the period cycles* and the integers n_{ij} are the *link periods*. The integer h is the *orbit genus* of Λ and the sign corresponds with the orientability of the quotient orbifold of Λ.

A group Λ with signature (2.1) has the presentation with the following generators, usually called the *canonical generators*:

$$x_1, \ldots, x_r, e_i, c_{ij}, \quad 1 \le i \le \ell,\ 0 \le j \le s_i,$$
$$\text{and} \quad a_1, b_1, \ldots, a_h, b_h \text{ if the sign is } +$$
$$\text{or} \quad d_1, \ldots, d_h \text{ otherwise,}$$

and relators:

$$x_i^{m_i}, \quad 1 \le i \le r,$$
$$c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i, \quad 1 \le i \le \ell,\ 1 \le j \le s_i,$$
$$\text{and} \quad x_1 \ldots x_r e_1 \ldots e_\ell a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1}$$
$$\text{or} \quad x_1 \ldots x_r e_1 \ldots e_\ell d_1^2 \ldots d_h^2,$$

according to whether the sign is $+$ or $-$. The last relation is usually called the *long relation*.

The generators x_i are elliptic transformations, c_{ij} are reflections and the only elements of finite order in Λ are conjugates of some powers of x_i or $c_{ij-1}c_{ij}$ while the reflections are conjugates of c_{ij}. The generators e_i are orientation-preserving transformations and are called the *connecting* generators. Finally, a_i, b_i are hyperbolic translations and d_i are glide reflections.

An abstract group with such a presentation can be realized as a NEC group Λ if and only if the value

$$\varepsilon h + \ell - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{\ell}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right),$$

where $\varepsilon = 2$ if the sign is $+$ and $\varepsilon = 1$ otherwise, is positive. This value is equal to the normalized hyperbolic area $\mu(\Lambda)/2\pi$ of an arbitrary fundamental region for such a group and the Hurwitz-Riemann formula

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda),$$

holds true for a subgroup Λ' of finite index in a NEC group Λ.

Now recall that a signature s is called *maximal*, if for every NEC group Λ', containing a NEC group Λ with the signature s and having the same Teichmüller dimension, the equality $\Lambda = \Lambda'$ holds. If the above condition does not hold, then s is a non-maximal signature and the complete lists of these together with their corresponding maximal signatures are given in [1] and [7] (see also [19] and [3]). A NEC group Λ is called *maximal* if there does not exist another NEC group containing it properly. For any maximal signature s, there exists a maximal NEC group with the signature s.

Any compact Riemann surface of genus g can be represented in a form $\mathscr{H}/\Gamma$, where Γ is a torsion free Fuchsian group of genus g. Moreover, in such terms, a finite group G is a group of automorphisms of $X = \mathscr{H}/\Gamma$ if and only if $G = \Lambda/\Gamma$ for some NEC group Λ. If in addition Λ has a maximal signature, then we know that the group G is the full automorphism group for the underlying Riemann surface. We have the so called canonical epimorphism $\theta\colon \Lambda \to G$ and $\Gamma = \ker\theta$.

In this paper we shall assume that the group G^+ of orientation preserving automorphisms acts without fixed points. Recall, that for a NEC group Λ with signature (2.1), the *canonical Fuchsian subgroup* Λ^+ of Λ has signature

$$(\varepsilon h + \ell - 1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{\ell s_\ell})$$

where $\varepsilon = 2$ if the signature of Λ has sign $+$ and $\varepsilon = 1$ otherwise. In our case of fixed point free actions, there are no elliptic generators in Λ^+ and hence the group Λ has signature of the form

$$(h; \pm; [-]; \{(-)^\ell\}). \tag{2.2}$$

For convenience we call such an action to be a fixed point free symmetric one, as there are obviously fixed points – on ovals of the symmetries – but there are no fixed points coming from elliptic elements.

Finally, a group G is said to be *abstractly oriented* if there is an epimorphism $\varphi\colon G \to Z_2 = \{\pm 1\}$ called an *abstract orientation*. An element g of an abstractly oriented group G with an abstract orientation φ is said to be *orientation preserving* (respectively *orientation reversing*) if $\varphi(g) = +1$ (respectively $\varphi(g) = -1$). Observe that the abstract orientations of G correspond to the subgroups of index 2 in G.

3. Existence of fixed point free symmetric actions

In this part of the paper we determine some conditions on the existence of a fixed point free symmetric action, in terms of the genus of the surface and the order of the group. Before we continue, recall that it was first proved in the work [8], that every finite group can act as a full group of automorphisms of some compact Riemann surface.

PROPOSITION 3.1. *Every finite group H is isomorphic to a group of conformal automorphisms of a fixed point free symmetric Riemann surface.*

PROOF. We shall show that for any finite group H there exists a Riemann surface of some genus g, depending on the order of H, on which H acts without fixed points as the group of orientation preserving automorphisms. Namely, let

$$H = \{h_1, \dots, h_n, e\}$$

be all the elements of a group H, where e is the trivial element. Let $G = H \times Z_2$, where $Z_2 = \langle\sigma\rangle$. Moreover, we define the abstract orientation on G, by letting σ be the orientation reversing generator. Hence σ is a symmetry. Observe that $|G| = 2n + 2$. Now, if $n \geq 3$, let Λ be a NEC group with signature

$$(n; -; [-]; \{(-)\})$$

and we define an epimorphism $\theta\colon \Lambda \to G$ such that $\theta(d_i) = h_i\sigma$ for $i = 1, \dots, n$, $\theta(c_{10}) = \sigma$, $\theta(e_1) = \theta(d_1^2 \dots d_n^2)^{-1}$. By the Hurwitz-Riemann formula

$$\frac{g-1}{n+1} = n - 1$$

and in such a way we get a fixed point free symmetric action on a Riemann surface of genus $g = n^2$, where H is the group of orientation preserving automorphisms. Observe that, as $n \geq 3$, by [7] there are maximal groups Λ, and then the group G is the full automorphism group.

Now for $n = 1$ and $n = 2$ we take Λ to have (maximal) signature $(2; -; [-]; \{(-), (-)\})$. Let first $H = Z_3 = \langle a\rangle$. We take $\theta\colon \Lambda \to Z_3 \times Z_2$ to be defined by mapping the canonical reflections to σ, the connecting generators to 1 and the canonical glide reflections d_1, d_2 to $a\sigma$ and $a^{-1}\sigma$. In such a way we get a fixed point free action on a Riemann surface of genus 7. Similarly, if $H = Z_2$, then taking the same signature Λ and analogous epimorphism, we obtain a fixed point free action on a Riemann surface of genus 5.

Now if $n = 0$, which means that H is trivial, it is enough to take Λ with maximal signature $(0; +; [-]; \{(-)^5\})$ and consider an epimorphism onto Z_2, mapping all the canonical reflections to σ and all the connecting generators to 1. In such a way we obtain a fpf-symmetric action on a Riemann surface of genus 4.

THEOREM 3.2. *Let G be an arbitrary abstractly oriented group having an orientation reversing involution. Then G is isomorphic to the group of all automorphisms of a fixed point free symmetric Riemann surface. The condition that G has an orientation reversing involution is clearly also necessary.*

PROOF. In this result we shall prove that there exists a Riemann surface of genus g – depending on the order of G – such that G^+ acts without fixed points on this Riemann surface and G is the full automorphism group. Now let $n \geq 1$ and

$$G = \{g_1, \ldots, g_{2n}, \sigma, e\}$$

be all the elements of G, where σ is an orientation reversing involution, elements $g_1, \ldots, g_n$ reverse orientation and elements $g_{n+1}, \ldots, g_{2n}$ preserve orientation. Let Λ be a NEC group with maximal signature

$$(4n; -; [-]; \{(-)\}).$$

We define an epimorphism $\theta \colon \Lambda \to G$ by taking $\theta(d_{2k-1}) = g_k, \theta(d_{2k}) = g_k^{-1}$ for $k = 1, \ldots, n$ and $\theta(d_{2k-1}) = g_k\sigma, \theta(d_{2k}) = (g_k\sigma)^{-1}$ for $k = n + 1, \ldots, 2n$. Now the connecting generator e_1 is mapped to 1 and the canonical reflection to σ. By the Hurwitz-Riemann formula

$$\frac{g-1}{n+1} = 4n - 1$$

and we obtain a fixed point free symmetric action on a Riemann surface of genus $g = 4n^2 + 3n$. We proved the theorem if G has at least 4 elements.

Now let $G = \mathrm{Z}_2 = \langle\sigma\rangle$. Take a NEC group with maximal signature

$$(3; -; [-]; \{(-)\}$$

and map all the canonical orientation reversing generators to σ. By the Hurwitz-Riemann formula

$$\frac{g-1}{1} = 2$$

and we obtain a fixed point free symmetric action on a Riemann surface of genus 3.

THEOREM 3.3. *Let $g \geq 2$ be an integer and let $n \neq g - 1$ be an arbitrary divisor of $g - 1$. Then there exists a fixed point free symmetric action of a group of order $2n$ on a Riemann surface of genus $g \geq 2$.*

PROOF. Observe first, that if there exists a fixed point free symmetric action of a group of order $2n$ on a Riemann surface of genus g, then by the Hurwitz-Riemann formula n must divide $g - 1$. Our aim is to show that this condition is not only necessary, but also sufficient for the existence of such an action. We shall consider the case $n = g - 1$ in the next theorem, as the maximality of the NEC group is involved. Let $G = \mathrm{D}_n = \langle\sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^n\rangle$ and let Λ be a NEC group with signature

$$(2; -; [-]; \{(-), \overset{(g-1)/n}{\ldots\ldots}, (-)\}).$$

By [7] this signature is maximal. We take an epimorphism $\theta\colon \Lambda \to G$ such that $\theta(d_1) = \tau$ and all the remaining reflections and glide reflection are mapped to σ. All the connecting generators e_i are mapped to 1. In such a way we obtain a fixed point free symmetric action of a dihedral group of order $2n$ on a Riemann surface of genus g.

Let us now introduce a special and important type of a fpf-symmetric action, namely a *maximal fpf-symmetric action*. We shall call a fpf-action maximal, if either the signature of a NEC group Λ in $G = \Lambda/\Gamma$ is maximal, or it is not maximal, but the canonical epimorphism $\theta\colon \Lambda \to G$ cannot be extended to $\theta'\colon \Lambda' \to G'$, where Λ' is a NEC group with maximal signature, containing Λ as a proper subgroup and $G' = G \rtimes \mathbb{Z}_2$. In other words, we demand G to be the full automorphism group for the underlying fpf-symmetric Riemann surface.

THEOREM 3.4. *There is no maximal fpf-symmetric action of a dihedral group of order* $2(g-1)$ *on a Riemann surface of odd genus* $g \geq 2$.

PROOF. Let us assume to a contrary that such an action exists and let $G = D_{g-1} = \langle \sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^{g-1}\rangle$. Now $G = \Lambda/\Gamma$ for some NEC group Λ and a Fuchsian surface group Γ of genus g. By the Hurwitz-Riemann formula, the normalized hyperbolic area of Λ equals 1 and so Λ has one of the following signatures:

$$s_1 = (0; +; [-]; \{(-)^3\}) \quad \text{or} \quad s_2 = (1; -; [-]; \{(-)^2\}).$$

Note that we require at least two period cycles as both of the symmetries σ, τ have ovals. Indeed, as we consider non-conjugate symmetries with fixed points, each of them must be the image of a canonical reflection. By [7] both of these signatures are non-maximal and their maximal pairs are respectively

$$s_1' = (0; +; [-]; \{(2, 2, 2, 2, 2, 2)\}) \quad \text{or} \quad s_2' = (0; +; [2]; \{(2, 2, 2, 2)\})$$

with the index being 2 in both cases. Now we shall show that any epimorphism $\theta\colon \Lambda \to G$ extends to the epimorphism $\theta\colon \Lambda' \to G'$, where Λ' contains Λ and has one of the maximal signatures above and $G' = G \times Z_2 = G \times \langle x\rangle$.

Let first Λ have signature s_1 and generators $c_1, c_2, c_3, e_1, e_2, e_3$. Let also Λ' have signature s_1' and generators $c_0', c_1', c_2', c_3', c_4', c_5'$. It is not hard to see that, without loss of generality, we may assume that $c_1 = c_0'$, $c_2 = c_2'$, $c_3 = c_4'$ and $e_1 = c_5'c_1'$, $e_2 = c_1'c_3'$, $e_3 = c_3'c_5'$. We take $\theta\colon \Lambda \to G$ to be arbitrary epimorphism, without loss of generality we may assume that $\theta(c_1) = \sigma$, $\theta(c_2) = \tau$ and $\theta(c_3) = \rho$, where ρ is a symmetry in G. We define $\theta'\colon \Lambda' \to G'$ as $\theta'(c_0') = \sigma$, $\theta'(c_1') = x$, $\theta'(c_2') = \tau$, $\theta'(c_3') = x\theta(e_2)$, $\theta'(c_4') = \rho$, $\theta'(c_5') = \theta(e_2)^{-1}x\theta(e_3)$. Note that these are all involutions, as the images of the connecting generators are either trivial or a central involution in G, for they must centralize the corresponding canonical reflections of the empty period cycles. Obviously $\theta'|_\Lambda = \theta$ and the action of G is not maximal fpf-symmetric, as it always extends to the action of G', which has fixed points coming from the elliptic elements corresponding to the link periods of Λ'.

Let now Λ have signature s_2 and generators d_1, c_1, c_2, e_1, e_2. Let also Λ' contain Λ and have signature s_2' and generators $x_1', c_0', c_1', c_2', c_3'$. Again we easily see that without loss of generality $c_1 = c_0'$, $c_2 = c_2'$ and $e_1 = x_1'c_3'x_1'^{-1}c_1'$, $e_2 = c_1'c_3'$, $d_1 = c_3'x_1'^{-1}$. We take $\theta\colon \Lambda \to G$ to be arbitrary epimorphism, without loss of generality we may assume that $\theta(c_1) = \sigma, \theta(c_2) = \tau$ and $\theta(d_1) = \rho$ for some arbitrary symmetry ρ in G. We define $\theta'\colon \Lambda' \to G'$ as $\theta'(c_0') = \sigma$, $\theta'(c_1') = x$, $\theta'(c_2') = \tau$, $\theta'(c_3') = x\theta(e_2)$, $\theta'(x_1') = \theta(e_2)^{-1}x\rho$. Again we see that $\theta'|_\Lambda = \theta$ and the action of G is not maximal fpf-symmetric, as it extends to the action of G', which has elliptic fixed points corresponding to the proper and link periods of Λ'.

From the two theorems above we easily derive the following corollary concerning the minimal genus problem for dihedral groups.

Corollary 3.5. *The minimal genus for the fpf-symmetric action of a dihedral group of order* $2n$ *on a Riemann surface of odd genus* g *is* $g = 2n + 1$.

Corollary 3.6. *There is no Riemann surface of even genus admitting at least two non-conjugate symmetries with a maximal fpf-action. Therefore a maximal fixed point free action on a Riemann surface of even genus is compatible with at most one conjugacy class of symmetry.*

We finish this part of the paper with an observation concerning groups generated by at least 4 symmetries.

Proposition 3.7. *Let* G *be a finite and abstractly oriented group of order* $2n$*, generated by* $k \geq 4$ *non-conjugate orientation reversing involutions. Then the minimal genus for the Riemann surface which admits a fpf-symmetric action of a group* G *is* $g = n(k-2) + 1$.

Proof. Let σ_i, $i = 1, \dots, k$ be the symmetries generating G. Again, we require all of them to have ovals so let Λ be a NEC group with signature $(0; +; [-]; \{(-)^k\})$, maximal by [7], and consider an epimorphism which maps the canonical reflections c_i of the i-th empty period cycle to σ_i, $1 \leq i \leq k$, and the connecting generators e_i to 1. As a result, by the Hurwitz-Riemann formula we obtain a Riemann surface of genus $g = n(k-2) + 1$ having a fpf-symmetric action of a group G. Observe that the genus is minimal, as we have the minimal possible number of period cycles required for all the symmetries to have ovals and thus the area of Λ is minimal.

4. Number of non-conjugate symmetries

In this part of the paper we shall determine the maximal number of non-conjugate symmetries that a Riemann surface of genus g may admit if a fixed point free symmetric action is of our interest. Recall that the sharp upper bound on the number of non-conjugate symmetries, with fixed points, that a Riemann surface of genus g may admit, was found in [5], where the authors prove that

THEOREM 4.1. *Let X be a Riemann surface of genus g. Assume that $g = 2^{r-1}u + 1$ for some odd u and suppose that there are k non-conjugate symmetries of X. Then $k \leq 2^{r+1}$ and this bound is attained if and only if $u \geq 2^{r+1} - 3$.*

Similarly, in our case of a fpf-symmetric action, the bound depends on a 2-adic structure of g. Namely

THEOREM 4.2. *Let X be a Riemann surface, with a fpf-symmetric action, of genus $g = 2^{r-1}u + 1 > 3$ for some odd u and suppose that there are k non-conjugate symmetries of X with fixed points. Then $k \leq 2^{r-1}$ and this bound is attained if and only if $u \geq 2^{r-1} - 1$ and the symmetries commute.*

PROOF. Let G be a 2-group generated by the symmetries. As we know, there exists a NEC group Λ with signature (2.2) and an epimorphism $\theta\colon \Lambda \to G$ such that $G = \Lambda/\Gamma$, $X = \mathscr{H}/\Gamma$ where $\Gamma = \ker\theta$. Now by the Hurwitz-Riemann formula

$$\frac{g-1}{|G|/2} = \varepsilon h - 2 + \ell$$

and the right-hand side is an integer. If the number of symmetries were greater than 2^{r-1}, then

$$\frac{|G|}{2} > 2^{r-1},$$

as G is an abstractly oriented group and all the symmetries reverse orientation. But then the left-hand side of the above equality is clearly not an integer, a contradiction. Therefore indeed the number of symmetries is not greater than 2^{r-1}.

Now if there are exactly 2^{r-1} non-conjugate symmetries acting on a Riemann surface X of genus g, then the 2-group G generated by them satisfies $|G|/2 = 2^{r-1}$ and so $G = Z_2^r$, which proves the commutativity of the symmetries. Also period cycles must hold the condition $\ell \geq 2^{r-1}$. By the Hurwitz-Riemann formula

$$u = \frac{g-1}{2^{r-1}} \geq \ell - 2 \geq 2^{r-1} - 2$$

and the required bound on u follows.

For the sufficient condition, let $u \geq 2^{r-1} - 1$ be an odd integer. We shall construct a Riemann surface of genus $g = 2^{r-1}u + 1$ having exactly 2^{r-1} commuting symmetries. For, let Λ be a NEC group with the maximal signature

$$(0; +; [-]; \{(-)^{u+2}\}$$

and consider an epimorphism $\theta\colon \Lambda \to Z_2^r$ which maps the canonical reflections corresponding to the first 2^{r-1} consecutive empty period cycles to $\sigma_1, \ldots, \sigma_{2^{r-1}}$, where σ_i, $i = 1, \ldots, 2^{r-1}$ are all the orientation reversing involutions in Z_2^r. Now, as $u + 2 \geq 2^{r-1} + 1$ we see that this is indeed possible. The remaining canonical reflections can be mapped to arbitrary symmetries, say σ_1. The canonical generators e_i, $i = 1, \ldots, u + 2$ are all mapped to 1. In such a way we constructed a Riemann surface X of genus $g = 2^{r-1}u + 1$ having 2^{r-1} non-conjugate (and commuting) symmetries, where the group generated by the symmetries yields a fixed point free symmetric action.

REMARK 4.3. From [6] and [17] it follows that there is no fixed point free symmetric action of the group Z_2 on a Riemann surface of genus 2 and there is no fpf-symmetric action of the group Z_2^2 on a Riemann surface of genus 3. However, in the proof of Theorem 3.3 we already showed that there is such an action of the group Z_2 on the surface of genus 3. It follows that the number of symmetries in such a case is at most 1.

5. Total number of ovals of symmetries

In this section we shall determine the maximal possible total number of ovals that k non-conjugate symmetries on a Riemann surface of genus g may admit, with the assumption that the action is fixed point free. Recall that, without the assumption of the action being fpf-symmetric, the sharp upper bounds on a maximal total number of ovals of k non-conjugate symmetries on a Riemann surface of genus g and results on the group structure in the extremal case were found for $2 \leq k \leq 4$ in [18] (see also [4], [3]), for $5 \leq k \leq 8$ in [12] and, finally, for $k \geq 9$ in [10].

The next two results are crucial for this part of the paper

THEOREM 5.1 (Gromadzki [9]). *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with the group G of all automorphisms of X, let $G = \Lambda/\Gamma$ for some NEC group Λ and let $\theta\colon \Lambda \to G$ be the canonical epimorphism. Then the number of ovals of a symmetry σ of X equals*

$$\sum [C(G, \theta(c_i)) : \theta(C(\Lambda, c_i))],$$

where the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to σ.

THEOREM 5.2 (Singerman [19]). *Let c_0, e be the canonical generators corresponding to an empty period cycle of a NEC group Λ with signature* (2.2). *Then the centralizer $C(\Lambda, c_i)$ equals $\langle c_0\rangle \times \langle e\rangle = Z_2 \times Z$.*

THEOREM 5.3. *Let X be a Riemann surface of genus g having k non-conjugate symmetries such that the group G generated by the symmetries gives rise to a fixed point free symmetric action on X. Then the total number of ovals of the symmetries is not greater than $g-1+2^k$. Furthermore, the bound is attained if and only if $G = Z_2^k$ or $G = D_n \times Z_2^{k-2}$ for some even integer $n > 2$. In both cases, if the bound is attained, then $g = (\ell-2)\cdot\frac{|G|}{2}+1$, where $\ell \geq \max\{k, 4\}$ is an integer.*

PROOF. First, let us observe that the total number of ovals does not exceed

$$g - 1 + |G|,$$

where $G = \Lambda/\Gamma$ is the group generated by the symmetries. Indeed, let Λ be a NEC group with signature (2.2) and $\theta\colon \Lambda \to G$ be the canonical epimorphism of the action. As $X = \mathscr{H}/\Gamma$ is a Riemann surface of genus g, by the Hurwitz-Riemann formula we have

$$\frac{g-1}{|G|/2} \geq \ell - 2$$

and in turn $\ell \leq \frac{2(g-1)}{|G|} + 2$. Now our k non-conjugate symmetries have fixed points and so $\ell \geq k$. Observe also that, by Theorems 5.1 and 5.2, a reflection corresponding to the empty period cycle contributes with at most $|G|/2$ ovals to the respective symmetry. Now, as there are k symmetries we see that the total number of ovals does not exceed $k|G|/2$ and

$$k|G|/2 \leq \ell|G|/2 \leq g - 1 + |G|.$$

Although it may seem that the structure of the 2-group G and its order play a key role in the formula above, we shall prove that in fact the maximal total number of ovals does not exceed $g-1+2^k$ and the bound is attained if and only if $G = Z_2^k$ or $G = D_n \times Z_2^{k-2}$ for some even integer $n > 2$.

Let us assume that there are at least 4 non-central symmetries among our k symmetries. Now observe that a reflection whose image under θ is non-central, contributes with at most $|G|/4$ ovals to the respective symmetry. Therefore the total number of ovals does not exceed

$$4\cdot\frac{|G|}{4} + (\ell-4)\cdot\frac{|G|}{2} = g - 1$$

and the proof in this case is finished.

Now assume that there are exactly three non-central symmetries among the k symmetries and at least two of them have centralizer in G of order at most $|G|/4$. Then the total number of ovals does not exceed

$$2 \cdot \frac{|G|}{8} + \frac{|G|}{4} + (\ell - 3) \cdot \frac{|G|}{2} = g - 1$$

and the proof is also finished.

If there are exactly three non-central symmetries among the k symmetries and all of them have centralizers in G of order $|G|/2$, then it is easy to see that the subgroup H generated by these three symmetries has order 16, being a semi-direct product of D_4 and Z_2. Therefore the order of G is not larger than the order of the group $H \times Z_2^{k-3}$ being equal to 2^{k+1}. We see that obviously the number of ovals does not exceed

$$3 \cdot \frac{|G|}{4} + g - 1 - \frac{|G|}{2} = g - 1 + \frac{1}{4} \cdot 2^{k+1} = g - 1 + 2^{k-1}.$$

This finishes the proof in this case.

Now if there are exactly two non-central symmetries among our k symmetries, then the order of the group G equals the one of the group $D_n \times Z_2^{k-2}$ being equal to $n2^{k-1}$ and the number of ovals does not exceed

$$2 \cdot \frac{|G|}{n} + g - 1 \leq g - 1 + 2^k.$$

If all the symmetries commute, then $|G| \leq 2^k$ and the theorem follows.

From all the above inequalities we see that actually the bound can be attained only if the group generated by the symmetries is Z_2^k or $D_n \times Z_2^{k-2}$.

For the attainment, assume first that all the symmetries commute and so $G = Z_2^k$ and let our symmetries be $\sigma_1, \ldots, \sigma_k$. Now if $k \geq 4$, then we take a NEC group Λ with maximal signature

$$(0; +; [-]; \{(-)^\ell\}),$$

where $\ell \geq k \geq 4$ is an integer and an epimorphism which maps the first k canonical reflections respectively to $\sigma_1, \ldots, \sigma_k$ and the remaining ones to σ_1. All the connecting generators are mapped to 1. As all the symmetries commute, by the Theorems 5.1 and 5.2, any canonical reflection contributes $|G|/2$ ovals to the respective symmetry. By the Hurwitz-Riemann formula, we have a configuration of k commuting symmetries on a Riemann surface of genus $g = 2^{k-1}(\ell - 2) + 1$. Therefore the total number of ovals is

$$\ell \cdot \frac{|G|}{2} = g - 1 + 2^k.$$

Now if $k = 3$ or $k = 2$, to assure the maximality of the NEC signature, we take ℓ to be at least 4, leading to the desired configuration on a Riemann surface of genus greater than 9 and 5 respectively.

Finally, note that if $G = \mathrm{D}_n \times \mathrm{Z}_2^{k-2}$ with $n > 2$, we obtain the desired configuration by taking Λ and θ as above. By the Hurwitz-Riemann formula

$$\frac{g-1}{n2^{k-2}} = \ell - 2$$

and so $g = 2^{k-2}n(\ell - 2) + 1$; here we remember that σ_1, σ_2, say, are the non-central symmetries and so the first two canonical reflections contribute, by Theorems 5.1 and 5.2, $|G|/n = 2^{k-1}$ ovals to σ_1 and σ_2. All the remaining ones contribute, as before, $n2^{k-2}$ ovals. Also in this case we obtain the configuration of k symmetries with $g - 1 + 2^k$ ovals in total.

ACKNOWLEDGEMENTS. The author thanks the referees for all the valuable remarks and suggestions, which helped to improve the final version of the paper.

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