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RELATIVELY 'CLOSABLE' STATES ON C*-ALGEBRAS

NIELS JUUL MUNCH

Abstract.

Let $\mathscr{C}$ be a C*-algebra and φ a state on $\mathscr{C}$. Let $(\mathscr{H}_\varphi, \pi_\varphi, \xi_\varphi)$ be the GNS-representation of $\mathscr{C}$ and denote by S_φ the set of states ψ on $\mathscr{C}$ implemented by vectors in the cone $[\pi_\varphi(\mathscr{C})'_+ \xi_\varphi]$, that is

$$\psi = \omega_\xi \circ \pi_\varphi, \quad \omega_\xi = (\cdot\, \xi, \xi)$$

for some $\xi = \xi_\psi$ in the closure $[\pi_\varphi(\mathscr{C})'_+ \xi_\varphi]$ of $\pi_\varphi(\mathscr{C})'_+ \xi_\varphi$.

We demonstrate that a state ψ lies in S_φ if and only if there is a positive map E of $\mathscr{C}$ into an Abelian von Neumann algebra $\mathscr{A}$ and positive normal functionals (or states) $\bar{\varphi}, \bar{\psi}$ on $\mathscr{A}$, so that $\bar{\varphi}$ is faithful and

$$\varphi = \bar{\varphi} \circ E, \quad \psi = \bar{\psi} \circ E.$$

In particular S_φ is a convex set.

It is further shown that S_φ consists of precisely the states ψ on $\mathscr{C}$ which are 'closable' with respect to φ in the sense that the map

$$\mathscr{H}_\varphi \ni \pi_\varphi(c)\xi_\varphi \to \pi_\psi(c)\xi_\psi \in \mathscr{H}_\psi$$

is well-defined and closable.

1. Introduction and statement of the theorem.

Let φ be a state on a C*-algebra $\mathscr{C}$. The set S_φ of states ψ of the form

$$\psi = \omega_\xi \circ \pi_\varphi, \quad \xi \in [\pi_\varphi(\mathscr{C})'_+ \xi_\varphi]$$

has been considered by Skau in the case where $\mathscr{C}$ is a von Neumann algebra $\mathscr{M}$ and φ is a faithful, normal state on $\mathscr{M}$.

In that case S_φ is surely norm-dense in the set of all normal states $N_{\mathscr{M}}$. Indeed every normal state is implemented by some vector in $\mathscr{H}_\varphi$ (see [1, p. 108]) and each ψ of the form

$$\psi = \omega_{x\xi_\varphi} \circ \pi_\varphi = \omega_{|x|\xi_\varphi} \circ \pi_\varphi, \quad \text{for } x \in \pi_\varphi(\mathscr{M})'$$

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and $|x| = (x^*x)^{1/2}$, belongs to S_φ. Skau proved in [8], however, that if S_φ equals $N_{\mathscr{M}}$ for one normal, faithful φ, then $\mathscr{M}$ is a finite von Neumann algebra and S_φ equals $N_{\mathscr{M}}$ for any such φ. (See also [4]). Theorem 3.1 in [8] also suggests that the relation between a state ψ in S_φ and φ itself may be connected with properties of the densely defined map

$$\pi_\varphi(\xi_\psi, \xi_\varphi): \pi_\varphi(c)\xi_\varphi \to \pi_\varphi(c)\xi_\psi, \quad c \in \mathscr{C}$$

if ξ_ψ is chosen so as to implement ψ.

The following theorem is obtained by considering the positive part in the polar decomposition of $\pi_\varphi(\xi_\psi, \xi_\varphi)$, when this map is closable.

The centralizer of a faithful, normal state φ on $\mathscr{M}$ will be denoted $\mathscr{M}^\varphi$. It can be defined equivalently as the fixed-point algebra of the modular automorphism group σ_t^φ, $t \in \mathrm{R}$, or as the set $\{m_0 \in \mathscr{M} \mid \varphi(mm_0) = \varphi(m_0 m), \forall m \in \mathscr{M}\}$, (see [5, p. 381]). When S is a set in a Hilbert space, $[S]$ will denote the closure of S, as well as the associated orthogonal projection when S is linear.

THEOREM 1.1. *Let φ and ψ be states on a* C*-*algebra $\mathscr{C}$. Then the following are equivalent:*

(a) $\psi \in S_\varphi$.

(b) *There is a positive map E of $\mathscr{C}$ into an Abelian von Neumann algebra $\mathscr{A}$ and positive, normal functionals $\bar\varphi, \bar\psi$ on $\mathscr{A}$ so that $\bar\varphi$ is faithful and*

$$\varphi = \bar\varphi \circ E, \quad \psi = \bar\psi \circ E.$$

These objects can be chosen so that $\|E\| = \|\bar\varphi\| = \|\bar\psi\| = 1$.

If φ is a faithful, normal state on a von Neumann algebra $\mathscr{M}$ and $\psi \in S_\varphi$, then

(c) *there is an Abelian von Neumann subalgebra $\mathscr{A}$ of $\mathscr{M}$, a normal state $\bar\psi$ on $\mathscr{M}$ and a positive, normal map $E: \mathscr{M} \to \mathscr{A}$ so that*

$$\varphi = \varphi \circ E, \quad \psi = \bar\psi \circ E$$

and

$(*)$ $\quad E(m) = m \Leftrightarrow m \in \mathscr{A} \cap \mathscr{M}^\varphi$,

$(**)$ $\quad E(ma) = E(m)a, \quad m \in \mathscr{M}, \quad a \in \mathscr{A} \cap \mathscr{M}^\varphi$,

$$[\pi_\varphi(E(\mathscr{M}))\xi_\varphi] = [\pi_\varphi(\mathscr{A})\xi_\varphi].$$

If $\mathscr{A}_0$ is the von Neumann subalgebra of $\mathscr{A}$ generated by $E(\mathscr{M})$ in (c), then $\mathscr{A}_0 = \mathscr{A}$ precisely when $[\pi_\varphi(\mathscr{A}_0)\xi_\varphi] = [\pi_\varphi(\mathscr{A})\xi_\varphi]$. (See [9] or [1, p. 342]). The last condition therefore implies $\mathscr{A}_0 = \mathscr{A}$.

The following is immediate from the Theorem.

COROLLARY 1.2. (1) *If λ_n are positive numbers so that $\sum_{n=1}^{\infty} \lambda_n = 1$, then $\psi_n \in S_\varphi$, $n \in \mathbf{N}$, implies that $\psi = \sum_{n=1}^{\infty} \lambda_n \psi_n$ lies in S_φ.*

(2) *If $E: \mathscr{C} \to \mathscr{C}$ is a positive map, then $\psi \in S_\varphi$ and $\|\psi \circ E\| = \|\varphi \circ E\| = 1$ implies $\psi \circ E \in S_{\varphi \circ E}$.*

2. Constructing E.

For every faithful, normal state φ on a von Neumann algebra $\mathscr{M}$ and every Abelian von Neumann subalgebra $\mathscr{A}$, there is positive, normal map E of $\mathscr{M}$ into $\mathscr{A}$ so that $\varphi \circ E = \varphi$ and the conditions ($*$), ($*$ $*$) hold together with $[\pi_\varphi(E(\mathscr{M}))\xi_\varphi] = [\pi_\varphi(\mathscr{A})\xi_\varphi]$.

This E is treated in an independent section.

For the benefit of the reader and to fix notation we recollect the following, which can be found for instance in ([1, pp. 338]).

Suppose that $\mathscr{C}$ is a C*-algebra with unity acting on $\mathscr{H}$, and that $\xi_\varphi \in \mathscr{H}$ is a cyclic vector for $\mathscr{C}$ implementing the state $\varphi = \omega_{\xi_\varphi}$ on $\mathscr{C}$.

Then for every Abelian von Neumann algebra $\mathscr{B}$ contained in $\mathscr{C}'$, there is a (unique) orthogonal measure $\mu_{\mathscr{B}}$ on the state space $E_{\mathscr{C}}$ of $\mathscr{C}$, so that the barycenter of $\mu_{\mathscr{B}}$ is φ and the map $\varkappa_{\mu_{\mathscr{B}}}: L^\infty(E_{\mathscr{C}}, \mu_{\mathscr{B}}) \to \mathscr{B}$ defined by

$$(\varkappa_{\mu_{\mathscr{B}}}(f)c\,\xi_\varphi, \xi_\varphi) = \mu_{\mathscr{B}}(f\hat{c}), \quad c \in \mathscr{C}$$

is an isomorphism. Here, as usual, $\hat{c}(\omega) = \omega(c)$. Put $P_S = [S\xi_\varphi]$, for any set $S \subseteq B(\mathscr{H})$. The map $\varkappa_{\mu_{\mathscr{B}}}$ is related to ξ_φ and $P_{\mathscr{B}}$ by

$$\varkappa_{\mu_{\mathscr{B}}}(\hat{c}_1)c_2\xi_\varphi = c_2 P_{\mathscr{B}} c_1 \xi_\varphi, \quad c_i \in \mathscr{C}$$

In the remainder of this section $\mathscr{M}$ is assumed to be acting on a Hilbert space $\mathscr{H}$ with a cyclic vector ξ_φ implementing the faithful, normal state φ.

However, as seen from the proof, Lemma 2.1. holds also without the assumption that ξ_φ separates $\mathscr{M}$.

LEMMA 2.1. *Assume that $\mathscr{B} \subseteq \mathscr{M}'$ is an Abelian von Neumann algebra and put*

$$\mathscr{M}_0 = \mathscr{M} \vee \mathscr{B}$$

Then a normal conditional expectation E_0 of $\mathscr{M}_0$ onto $\mathscr{B}$ preserving ω_{ξ_φ} is given by

$$E_0(mb) = \varkappa_{\mu^0_{\mathscr{B}}}(mb)\hat{} = \varkappa_{\mu_{\mathscr{B}}}(\hat{m})b, \quad m \in \mathscr{M},\ b \in \mathscr{B}$$

where $\mu^0_{\mathscr{B}}$ (respectively $\mu_{\mathscr{B}}$) is the orthogonal measure on $E_{\mathscr{M}_0}$ (respectively $E_{\mathscr{M}}$) with barycenter ω_{ξ_φ} (respectively φ) corresponding to $\mathscr{B}$ and ξ_φ.

PROOF. Consider the map $E_0: \mathcal{M}_0 \to \mathcal{B}$ given by

$$E_0(m) = \varkappa_{\mu^0_{\mathcal{B}}}(\hat{m})$$

The identities

$$(\varkappa_{\mu^0_{\mathcal{B}}}(\hat{m}_1)m_2\xi_\varphi, \xi_\varphi) = \mu^0_{\mathcal{B}}(\hat{m}_1\hat{m}_2) = (\varkappa_{\mu^0_{\mathcal{B}}}(\hat{m}_2)m_1\xi_\varphi, \xi_\varphi), \quad m_i \in \mathcal{M}_0,$$

show that E_0 is positive, normal and preserves ω_{ξ_φ}.

Let $b \in \mathcal{B}$. As $P_{\mathcal{B}} \in \mathcal{B}'$ and $b \in \mathcal{M}'$ it holds for all $m \in \mathcal{M}$ that

$$\varkappa'_{\mu^0_{\mathcal{B}}}(\hat{b})(m\xi_\varphi) = mP_{\mathcal{B}}b\xi_\varphi = b(m\xi_\varphi).$$

The vector ξ_φ is cyclic for $\mathcal{M}$, consequently $\varkappa'_{\mu^0_{\mathcal{B}}}(\hat{b}) = b$. Tomiyamas theorem ([10, p. 131]) now implies that E_0 is a conditional expectation, i.e.

$$E_0(mb) = E_0(m)b = \varkappa_{\mu^0_{\mathcal{B}}}(\hat{m})b, \quad m \in \mathcal{M}, \ b \in \mathcal{B}.$$

Finally

$$\varkappa_{\mu^0_{\mathcal{B}}}(\hat{m}_1)m_2\xi_\varphi = m_2P_{\mathcal{B}}m_1\xi_\varphi = \varkappa_{\mu_{\mathcal{B}}}(\hat{m}_1)m_2\xi_\varphi, \quad m_i \in \mathcal{M}$$

and it follows that $\varkappa_{\mu^0_{\mathcal{B}}}(\hat{m}) = \varkappa_{\mu_{\mathcal{B}}}(\hat{m}), \ m \in \mathcal{M}$.

Recall that the modular objects J and $\Delta_\varphi^{1/2}$ arise from the polar decomposition of the map $m\xi_\varphi \to m^*\xi_\varphi$. Here Δ_φ is an injective operator implementing the modular group $\sigma_t^\varphi = \Delta_\varphi^{it} \cdot \Delta_\varphi^{-it}$ and J is a conjugation on $\mathcal{H}$ so that $\mathcal{M}' = J\mathcal{M}J$.

The following lemma is undoubtedly well-known, but I know no explicit reference and therefore include a proof.

LEMMA 2.2. *For $m \in \mathcal{M}$ the equality $Jm\xi_\varphi = m^*\xi_\varphi$ holds if and only if $m \in \mathcal{M}^\varphi$.*

PROOF. Due to the separating ability of ξ_φ, the property $m \in \mathcal{M}^\varphi$ is equivalent to

$$\Delta_\varphi^{it}m\xi_\varphi = m\xi_\varphi, \quad t \in \mathrm{R}$$

which in turn, by Stone's Theorem and the functional calculus of self-adjoint operators, amounts to the statement

$$\Delta_\varphi^{1/2}m\xi_\varphi = m\xi_\varphi$$

The Lemma now follows by applying J to both sides of this equation.

PROPOSITION 2.3. *Let $\mathcal{A}$ be an Abelian von Neumann subalgebra of $\mathcal{M}$. Then a positive normal map $E: \mathcal{M} \to \mathcal{A}$ so that $\varphi \circ E = \varphi$ and*

$(*)\quad E(m) = m \Leftrightarrow m \in \mathscr{A} \cap \mathscr{M}^{\varphi}$,

$(**)\quad E(ma) = E(m)a,\ m \in \mathscr{M},\ a \in \mathscr{A} \cap \mathscr{M}^{\varphi}$,

$[E(\mathscr{M})\xi_{\varphi}] = [\mathscr{A}\xi_{\varphi}]$

is given by

$$E(m) = J\varkappa_{\mu_{JAJ}}(\hat{m})^* J,\quad m \in \mathscr{M}$$

where μ_{JAJ} *and* $\varkappa_{\mu_{JAJ}}$ *are defined relative to* ξ_{φ}.

Proof. We use the results and notation of Lemma 2.1. with $B = JAJ$ and

$$\mathscr{M}_0 = \mathscr{M} \vee J\mathscr{A}J.$$

Clearly E is positive, normal and preserves φ.

If $E(m) = m$ then $m = a \in \mathscr{A}$. Now $E(a) = a$ is equivalent to $\varkappa_{\mu_{J\mathscr{A}J}}(a^*)\hat{} = JaJ$ or $\varkappa_{\mu_{J\mathscr{A}J}}(a^*)\hat{}\,\xi_{\varphi} = JaJ\xi_{\varphi}$. But this condition is identical to $P_{J\mathscr{A}J}a^*\xi_{\varphi} = Ja\xi_{\varphi}$ and

$$P_{\mathscr{A}}Ja^*\xi_{\varphi} = a\xi_{\varphi}$$

as $P_{J\mathscr{A}J} = JP_{\mathscr{A}}J$. In the case $a \in \mathscr{M}^{\varphi}$ this is true by Lemma 2.2. If $E(a) = a$ the symmetric condition $P_{\mathscr{A}}Ja\xi_{\varphi} = a^*\xi_{\varphi}$ also holds and

$$\begin{aligned}
&\|Ja\xi_{\varphi} - a^*\xi_{\varphi}\|^2 \\
&= (Ja\xi_{\varphi} - a^*\xi_{\varphi},\ Ja\xi_{\varphi} - a^*\xi_{\varphi}) \\
&= (a\xi_{\varphi},\ a\xi_{\varphi} - Ja^*\xi_{\varphi}) - (Ja\xi_{\varphi} - a^*\xi_{\varphi},\ a^*\xi_{\varphi}) \\
&= 0.
\end{aligned}$$

So $a \in \mathscr{M}^{\varphi}$.

Conversely assume again $a \in \mathscr{A} \cap \mathscr{M}^{\varphi}$. Then for all $m \in \mathscr{M}$

$$\begin{aligned}
\varkappa_{\mu_{J\mathscr{A}J}}(ma)\hat{}\,\xi_{\varphi} &= P_{J\mathscr{A}J}ma\xi_{\varphi} \\
&= P_{J\mathscr{A}J}mJa^*J\xi_{\varphi} \\
&= \varkappa_{\mu^0_{J\mathscr{A}J}}(mJa^*J)\hat{}\,\xi_{\varphi}
\end{aligned}$$

and

$$\begin{aligned}
\varkappa_{\mu_{J\mathscr{A}J}}(ma)\hat{} &= \varkappa_{\mu^0_{J\mathscr{A}J}}(mJa^*J)\hat{} \\
&= \varkappa_{\mu_{J\mathscr{A}J}}(\hat{m})Ja^*J
\end{aligned}$$

This implies $E(ma) = E(m)a$, $m \in \mathscr{M}$, $a \in \mathscr{A} \cap \mathscr{M}^{\varphi}$.

Finally

$$E(m)\xi_{\varphi} = P_{\mathscr{A}}Jm^*\xi_{\varphi}$$

and

$$[E(\mathcal{M})\xi_\varphi] = [\mathcal{A}\xi_\varphi].$$

It may be of some interest to note that the constructed $E = E_\varphi$ has a certain continuity property with respect to the faithful state φ when $\mathcal{A}$ is fixed. Namely

$$\|\varphi_n - \varphi\| \xrightarrow[n\to+\infty]{} 0 \text{ implies } E_{\varphi_n}(m) \xrightarrow[n\to+\infty]{} E_\varphi(m), \; m \in \mathcal{M},$$

in the strong topology.

First cyclic vectors ξ_{φ_n} implementing φ_n can be chosen in the natural cone $P = [\Delta_\varphi^{1/4}\mathcal{M}_+ \xi_\varphi]$, (see [1, p. 108]). The inequality $\|\xi_{\phi_n} - \xi_\varphi\|^2 \leqq \|\varphi_n - \varphi\|$ yields $\xi_{\varphi_n} \xrightarrow[n\to+\infty]{} \xi_\varphi$. This in turn implies that

$$P_n = [J\mathcal{A}J\xi_{\varphi_n}] \xrightarrow[n\to+\infty]{} P = [J\mathcal{A}J\xi_\varphi]$$

in the strong topology, (see [9, Lemma 5.1]). As the modular conjugation J_n associated with each ξ_{φ_n} equals J (see [1, p. 106]) one has

$$E_{\varphi_n}(m)(Jm_0\xi_{\varphi_n}) = Jm_0P_n(m^*\xi_{\varphi_n}), \quad m, m_0 \in \mathcal{M}$$

from which the stated continuity follows.

Recall that by the theorem in [11], there is a conditional expectation E of a von Neumann algebra $\mathcal{M}$ onto a von Neumann subalgebra $\mathcal{N}$ preserving a faithful, normal state φ if and only if $\sigma_t^\varphi(\mathcal{N}) = \mathcal{N}$, $t \in \mathrm{R}$. Also note for later use, that any conditional expectation E_0 with domain $\mathcal{M}_0$ and range $\mathcal{N}_0$ preserving the state ω_{ξ_φ}, satisfies $E_0(m_0)\xi_\varphi = P_{\mathcal{N}_0}(m_0\xi_\varphi)$, $m_0 \in \mathcal{M}_0$. In particular the above mentioned E is unique.

The following will be useful in section 3.

Proposition 2.4. *Let $\hat{E}$ be the conditional expectation of $\mathcal{M}$ onto $\mathcal{A} \cap \mathcal{M}^\varphi$ preserving φ and let $E: \mathcal{M} \to \mathcal{A}$ be any normal positive map satisfying the conditions $\varphi \circ E = \varphi$, $(*)$ and $(**)$.*

For $m \in \mathcal{M}$ denote by $K(m)$ the weak convex closure of $\{E^n(m)\}_{n=1}^\infty$. Then

$$\mathcal{A} \cap \mathcal{M}^\varphi \cap K(m) = \{\hat{E}(m)\}.$$

Proof. The set $K(m)$ is compact in the weak topology. Therefore by the Markov–Kakutani theorem there exists a fixedpoint for E in $K(m)$. This fixedpoint lies in $\mathcal{A} \cap \mathcal{M}^\varphi$ by assumption.

However, because of $(**)$, $\varphi(a_0a) = \varphi(ma)$ holds for any $a_0 \in \mathcal{A} \cap \mathcal{M}^\varphi \cap K(m)$, $a \in \mathcal{A} \cap \mathcal{M}^\varphi$. This shows that $\mathcal{A} \cap \mathcal{M}^\varphi \cap K(m) = \{a_0\}$ is a singleton set and that $m \to a_0$ is linear.

Now clearly the map $m \to a_0$ coincides with $\hat{E}$.

It follows from the ergodic theorem in [6] that actually $\hat{E}(m)$ is the "almost uniform", and hence strong, limit of the Cesaro sums

$$s_n(m) = \frac{1}{n} \sum_{k=1}^{n} E^k(m)$$

In closing this section we note, that for a general von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$, the crucial property assuring existence of a positive map E of $\mathcal{M}$ into $\mathcal{N}$ so that $E(m)\xi_\varphi = P_{\mathcal{N}} J m^* \xi_\varphi$, is not Abeliannes of $\mathcal{N}$ but a certain tracial property. The map E_0, which will be used in the sequel, does however require Abeliannes.

PROPOSITION 2.5. *Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then the following are equivalent:*

(1) *There is a positive map E of $\mathcal{M}$ into $\mathcal{N}$ so that*

$$E(m)\xi_\varphi = P_{\mathcal{N}} J m^* \xi_\varphi, \quad m \in \mathcal{M}$$

(2) *The functional $\varphi|_{\mathcal{N}}$ is a trace.*

Also the following conditions are equivalent.

(3) *There is a conditional expectation E_0 of $\mathcal{M} \vee J\mathcal{N}J$ onto $J\mathcal{N}J$ preserving ω_{ξ_φ}.*

(4) *$\mathcal{N}$ is Abelian.*

PROOF. (1) $\Rightarrow$ (2). First compute for $n \in \mathcal{N}$, $m \in \mathcal{M}$:

$$\begin{aligned}
\varphi(nE(m)) &= (P_{\mathcal{N}} J m^* \xi_\varphi, n^* \xi_\varphi) \\
&= (Jm^* J \xi_\varphi, n^* \xi_\varphi) \\
&= (n\xi_\varphi, JmJ\xi_\varphi) \\
&= \varphi(E(m^*)^* n) \\
&= \varphi(E(m)n).
\end{aligned}$$

So $E(m) \in \mathcal{N}^\varphi$ and $\mathcal{N}_0 \subsetneq \mathcal{N}^\varphi$, where $\mathcal{N}_0$ denotes the von Neumann algebra generated by $E(\mathcal{M})$. In combination with $P_{\mathcal{N}_0} = P_{\mathcal{N}}$, this implies $\mathcal{N}_0 = \mathcal{N}$. Indeed let $\hat{E}$ be the conditional expectation of $\mathcal{N}$ onto $\mathcal{N}_0$ preserving φ. Then

$$\hat{E}(n)\xi_\varphi = P_{\mathcal{N}_0}(n\xi_\varphi) = n\xi_\varphi, \quad n \in \mathcal{N}$$

and $n = \hat{E}(n) \in \mathcal{N}_0$, $n \in \mathcal{N}$. Viz $\mathcal{N}_0 = \mathcal{N}^\varphi = \mathcal{N}$.

(2) $\Rightarrow$ (1). Let E be the composition of maps indicated below.

$$\mathcal{M} \xrightarrow{J(\cdot)^*J} \mathcal{M}' \xrightarrow{P_{\mathcal{N}}(\cdot)P_{\mathcal{N}}} P_{\mathcal{N}}\mathcal{M}'P_{\mathcal{N}} \hookrightarrow P_{\mathcal{N}}\mathcal{N}'P_{\mathcal{N}} = (\mathcal{N}P_{\mathcal{N}})' \xrightarrow{j} \mathcal{N}P_{\mathcal{N}} \xrightarrow{\iota} \mathcal{N}.$$

Here j is the natural anti-*-isomorphism, and τ is the inverse of the reduction map. Clearly E is positive and

$$E(m)\xi_\varphi = j(P_{\mathcal{N}} Jm^* JP_{\mathcal{N}})\xi_\varphi = P_{\mathcal{N}} Jm^* \xi_\varphi, \quad m \in \mathcal{M}$$

(3) ⇒ (4). The composition $E(m) = JE_0(m)^*J$, $m \in \mathcal{M}$, satisfies the condition in (1), hence $\varphi|_{\mathcal{N}}$ and $\omega_{\xi_\varphi}|_{J\mathcal{N}J}$ are traces. Now compute for $m \in \mathcal{M}$ and $n \in J\mathcal{N}J$:

$$\begin{aligned}\omega_{m\xi_\varphi}(n^*n) &= (m^*n^*nm\xi_\varphi, \xi_\varphi)\\ &= (n^*m^*mn\xi_\varphi, \xi_\varphi)\\ &= (n^*E_0(m^*m)n\xi_\varphi, \xi_\varphi)\\ &= (nn^*E_0(m^*m)\xi_\varphi, \xi_\varphi)\\ &= (nn^*m^*m\xi_\varphi, \xi_\varphi)\\ &= \omega_{m\xi_\varphi}(nn^*)\end{aligned}$$

As ξ_φ is cyclic for $\mathcal{M}$, $\mathcal{N}$ is Abelian.

(4) ⇒ (3). Lemma 2.1.

3. The states in S_φ.

For any two states φ and ψ on $\mathscr{C}$ there is a representation π of $\mathscr{C}$ on a Hilbert space $\mathscr{H}$ with vectors ξ_φ and ξ_ψ implementing φ and ψ. For one such π take $\pi = \pi_\varphi \oplus \pi_\psi$.

If $\varphi(c^*c) = 0$ implies $\psi(c^*c) = 0$ let, in any of these situations, $\pi(\xi_\psi, \xi_\varphi)$ denote the operator which is zero on the orthogonal complement $[\pi(\mathscr{C})\xi_\varphi]^\perp$ of $[\pi(\mathscr{C})\xi_\varphi]$ and sends $\pi(c)\xi_\varphi$ to $\pi(c)\xi_\psi$.

LEMMA 3.1. *For a state ψ on $\mathscr{C}$ so that $\varphi(c^*c) = 0$ implies $\psi(c^*c) = 0$, the following are equivalent:*

(1) $\psi \in S_\varphi$.

(2) *For any choice of π, ξ_φ, and ξ_ψ, $\pi(\xi_\psi, \xi_\varphi)$ is closable.*

(3) *For some choice of π, ξ_φ, and ξ_ψ, $\pi(\xi_\psi, \xi_\varphi)$ is closable.*

PROOF. (1) ⇒ (2). The reduction of π to $P = [\pi(\mathscr{C})\xi_\varphi]$ is (unitarily equivalent to) π_φ, and by the assumption (1) there is an implementing vector ξ'_ψ in $[P\pi(\mathscr{C})'_+\xi_\varphi]$. For any vector $\eta = \pi(c)\xi_\varphi + (I-P)\eta$

$$(\pi(\xi'_\psi, \xi_\varphi)\eta, \eta) = (\pi(c^*c)\xi'_\psi, \xi_\varphi) \geqq 0.$$

Hence $\pi(\xi'_\psi, \xi_\varphi)$ is symmetric and closable. The closure of the map

$$\pi(c)\xi_\psi \to \pi(c)\xi'_\psi, \quad [\pi(\mathscr{C})\xi_\psi]^\perp \to 0$$

is a partial isometry u in $\pi(\mathscr{C})'$ so that

$$\pi(\xi'_\psi, \xi_\varphi) = u\pi(\xi_\psi, \xi_\varphi).$$

It follows readily that $\pi(\xi_\psi, \xi_\varphi)$ is closable.

(3) $\Rightarrow$ (1). The operator $\pi(\xi_\psi, \xi_\varphi)$ is affiliated with $\pi(\mathscr{C})'$, hence so is the closure. Therefore the partial isometry u in the polar decomposition $\pi(\xi_\psi, \xi_\varphi)^- = uH$ lies in $\pi(\mathscr{C})'$ together with all spectral projection of H. (Cf. [3, p. 336]).

Again put $P = [\pi(\mathscr{C})\xi_\varphi]$ and $\pi_\varphi = \pi|_P$. Clearly $H = HP = PH$. Therefore $\xi'_\psi = H\xi_\varphi$ lies in $[\pi_\varphi(\mathscr{C})'_+ \xi_\varphi] = [P\pi(\mathscr{C})'_+ \xi_\varphi]$ and as $u^*u = [\operatorname{Ran} H]$

$$\begin{aligned} \omega_{\xi'_\psi} \circ \pi_\varphi &= (\pi(\cdot)H\xi_\varphi, H\xi_\varphi) \\ &= (\pi(\cdot)u^*uH\xi_\varphi, H\xi_\varphi) \\ &= \omega_{\xi_\psi} \circ \pi \\ &= \psi. \end{aligned}$$

Proof of Theorem 1.1. First consider the case where φ is a faithful, normal state on a von Neumann algebra $\mathscr{M}$ and $\psi \in S_\varphi$.

Choose ξ_ψ in the natural cone $P^{1/4} = [\Delta_\varphi^{1/4}\pi_\varphi(\mathscr{M})_+ \xi_\varphi]$ (see [1, p. 108]) and let $\pi_\varphi(\xi_\psi, \xi_\varphi)^- = uH$ be the polar decomposition.

In the following we suppress the (normal, faithful) representation π_φ.

Take $\mathscr{B}$ to be the von Neumann algebra in $\mathscr{M}'$ generated by the spectral projections of H and put $\mathscr{A} = J\mathscr{B}J$. Let $E: \mathscr{M} \to \mathscr{A}$ be defined as in Proposition 2.3. by

$$E(m) = J\varkappa_{\mu_{\mathscr{B}}}(\hat{m})^*J, \; m \in \mathscr{M}$$

relative to the vector ξ_φ.

Clearly E preserves φ. Put $\bar{\psi} = \omega_{JH\xi_\varphi}$. The following calculations are formally verified by introducing suitable limits.

$$\begin{aligned} \bar{\psi}(E(m)) &= (\varkappa_{\mu_{\mathscr{B}}}(\hat{m})H\xi_\varphi, H\xi_\varphi) \\ &= (\varkappa_{\mu^0_{\mathscr{B}}}(HmH)^\wedge \xi_\varphi, \xi_\varphi) \\ &= (HmH\xi_\varphi, \xi_\varphi) \\ &= (mu^*uH\xi_\varphi, H\xi_\varphi) \\ &= (m\xi_\psi, \xi_\psi) \\ &= \psi(m). \end{aligned}$$

This proves (c). In the general case choose ξ_ψ in $[\pi_\varphi(\mathscr{C})'_+ \xi_\varphi]$ and put $P = [\pi_\varphi(\mathscr{C})'\xi_\varphi]$. The positive map

$$E_0 : \pi_\varphi(c) \to P\pi_\varphi(c)P$$

mapping $\pi_\varphi(\mathscr{C})$ into the von Neumann algebra $\mathscr{M} = P\pi_\varphi(\mathscr{C})''P$ on $P\mathscr{H}_\varphi$ preserves ω_{ξ_φ} and ω_{ξ_ψ}. The vector ξ_φ is cyclic and separating for $\mathscr{M}$ and ξ_ψ lies in $[\mathscr{M}'_+\xi_\varphi] = [\pi_\varphi(\mathscr{C})'_+\xi_\varphi]$. We therefore obtain the result in (b) by applying (c) to $\mathscr{M}$, ω_{ξ_φ}, ω_{ξ_ψ} and composing the E occurring there with $E_0 \circ \pi_\varphi$.

To prove the implication (b) $\Rightarrow$ (a) it can be assumed that $\bar{\varphi}$ is a state and that $\mathscr{A} \simeq L^\infty(X,\mu)$ is represented on the space $\mathscr{H} = L^2(X,\mu)$, where $\bar{\varphi}(a) = \int a\,d\mu$. Here $\mathscr{A}_* \simeq L^1(X,\mu)$. Let $a_0 \in L^1(X,\mu)_+$ correspond to $\bar{\psi}$, $\bar{\psi}(a) = \int a a_0\,d\mu$, and consider a_0 as a positive self-adjoint operator on $\mathscr{H}$ affiliated with $\mathscr{A}$. (Cf. [7, p. 259]).

As $\mathscr{A}$ is Abelian, E is completely positive and ([10, Theorem 3.6., p. 194]) ensures the existence of a representation π of $\mathscr{C}$ on a space $\mathscr{H}_1$, a bounded operator $V : \mathscr{H} \to \mathscr{H}_1$ so that

$$E = V^*\pi(\cdot)V, \quad \mathscr{H}_1 = [\pi(\mathscr{C})V\mathscr{H}]$$

and a normal representation $\rho : \mathscr{A}' = \mathscr{A} \to \pi(\mathscr{C})'$ satisfying $\rho(a)V = Va$, $a \in \mathscr{A}$.

Let $\eta_{\bar{\varphi}}$ be constant 1. The assumptions imply

$$\omega_{V\eta_{\bar{\varphi}}} \circ \pi = \bar{\varphi} \circ E = \varphi \quad \text{and} \quad \omega_{Va_0^{1/2}\eta_{\bar{\varphi}}} \circ \pi = \psi.$$

Take $\xi_\varphi = V\eta_{\bar{\varphi}}$ and

$$\xi_\psi = Va_0^{1/2}\eta_{\bar{\varphi}} = \rho(a_0^{1/2})V\eta_{\bar{\varphi}} = \rho(a_0^{1/2})\xi_\varphi.$$

The operator $\pi(\xi_\psi, \xi_\varphi)$ acts on $\pi(\mathscr{C})\xi_\varphi$ as does the closed (self-adjoint) $\rho(a_0^{1/2})$, consequently $\pi(\xi_\psi, \xi_\varphi)$ is closable.

Retaining the notation of the above proof, we make a few comments regarding $\mathscr{A}$.

For a norm-dense set of states on $\mathscr{C}$, the "standard" states of [2], the projection P is equal to I. For such a φ and any ψ in S_φ therefore the $\mathscr{A}$ occurring in Theorem 1.1 can be assumed to lie in $\pi_\varphi(\mathscr{C})''$. Generally the $\mathscr{A}$ constructed above embeds into $\pi_\varphi(\mathscr{C})'$ via the isomorphism $\pi_\varphi(\mathscr{C})'P \to \pi_\varphi(\mathscr{C})'$. In the case, where $\mathscr{C} = \mathscr{M}$ is a von Neumann algebra and φ is normal and faithful, any $\mathscr{A}$, which in addition to the conditions in Theorem 1.1 (b) satisfies

$$[E(\mathscr{M})] = L^2(X,\mu), \quad E(I) = 1$$

can be embedded into $\mathscr{M}$ via a positive, injective map λ so that $\varphi \circ \lambda \circ E = \varphi$. It suffices to find a similar map λ_0 into $\mathscr{M}'$. Consider the bounded operator $u: m\xi_\varphi \to E(m) \in L^2(X,\mu)$. When a is a positive function in $L^\infty(X,\mu)$

$$(u^* a, m\xi_\varphi) = \bar{\varphi}(E(m)a) \geqq 0$$

for all positive m in $\mathscr{M}$. Hence $u^* a \in [\mathscr{M}'_+ \xi_\varphi]$. As $u\xi_\varphi = 1$ and $u^* 1 = \xi_\varphi$, the reasoning in ([1, p. 226]) demonstrates that actually $u^* a \in \mathscr{M}'_+ \xi_\varphi$. Consequently there is a positive map λ_0 of $\mathscr{A}$ into $\mathscr{M}'$ given by $u^* a = \lambda_0(a)\xi_\varphi$. Now injectivity of λ_0 is derived from injectivity of u^*, and $\omega_{\xi_\varphi} \circ \lambda_0 = \bar{\varphi}$ is obvious. Hence $\varphi \circ J\lambda_0(\cdot)^* J \circ E = \varphi$. If it is further assumed that $a_0 \in \operatorname{Ran} u$ there is in addition to this a functional χ on $\mathscr{M}'$ so that $\chi \circ \lambda \circ E = \psi$. Simply choose $\eta \in \mathscr{H}$ satisfying $u\eta = a_0$ and put $\chi = (\cdot\, \xi_\varphi, \eta)$.

As to the relation between φ and ψ, a more concrete representation can be given on the positive real axis. Let γ be the measure $\mu \circ a_0^{-1}$ on $[0,\infty[$. Then for any given a in $L^\infty(X,\mu)$ obviously

$$|\mu((f \circ a_0)a)| \leqq \|a\|_\infty \|f\|_1, \quad f \in L^1(\gamma).$$

There is therefore a function $E_1(a)$ in $L^\infty(\gamma)$, so that

$$\int_X (f \circ a_0) a\, d\mu = \int_{[0,\infty[} f E_1(a)\, d\gamma, \quad f \in L^1(\gamma)$$

and composing E with E_1, one obtains a new positive map E of $\mathscr{C}$ into $L^\infty(\gamma)$ so that

$$\varphi(c) = \int_{[0,\infty[} E(c)(x)\, d\gamma(x), \quad \psi(c) = \int_{[0,\infty[} x E(c)(x)\, d\gamma(x), \quad c \in \mathscr{C}$$

and so that $E(\mathscr{C})$ is dense in $L^2(\gamma)$. Regrettably these conditions do not guarantee uniqueness of the measure γ, nor of the algebra $\mathscr{A} = L^\infty(\gamma)$.

Two normal, faithful states φ and ψ on a von Neumann algebra $\mathscr{M}$ are said to "commute" if their modular automorphism groups commute $\sigma_t^\varphi \circ \sigma_s^\psi = \sigma_s^\psi \circ \sigma_t^\varphi$, $\forall s,t \in \mathrm{R}$. This is (cf. [5, p. 383]) equivalent to the existence of a positive, self-adjoint, injective operator h_ψ, affiliated with $\mathscr{M}^\varphi$, so that

$$\psi(m) = \varphi(h_\psi m h_\psi), \quad m \in \mathscr{M}$$

(or symmetrically $\varphi = \psi(h_\varphi \cdot h_\varphi)$).

The latter condition is used for a general (non-faithful) ψ in the next proposition.

Recall that if ξ is a cyclic and separating vector for $\mathscr{M}$, then $P_\xi^{1/4} = [\Delta_\xi^{1/4} \mathscr{M}_+ \xi]$ and $P_\xi^{1/2} = [\mathscr{M}'_+ \xi]$.

Proposition 3.2. *Assume that φ is a faithful, normal state for a von*

Neumann algebra $\mathcal{M}$ Then the following are equivalent for a normal state ψ:

(1) *There is a positive, self-adjoint operator h_ψ affiliated with $\mathcal{M}^\varphi$ so that*

$$\psi(m) = \varphi(h_\psi m h_\psi), \quad m \in \mathcal{M}$$

(2) *The state ψ has an implementing vector*

$$\xi_\psi \in P^{1/4}_{\xi_\varphi} \cap P^{1/2}_{\xi_\varphi}$$

with respect to π_φ.

(3) *Condition* (c) *of Theorem* 1.1. *holds with $\bar{\psi} = \psi$.*

PROOF. (1) $\Rightarrow$ (2). The vector $\xi_\psi = \pi_\varphi(h_\psi)\xi_\varphi$ implements ψ. We are through after verifying the implication

$$h \in (\mathcal{M}^\varphi)_+ \Rightarrow \pi_\varphi(h)\xi_\varphi \in P^{1/4}_{\xi_\varphi} \cap P^{1/2}_{\xi_\varphi}.$$

First

$$J\pi_\varphi(h)J\xi_\varphi = \pi_\varphi(h)^*\xi_\varphi = \pi_\varphi(h)\xi_\varphi$$

and $\pi_\varphi(h)\xi_\varphi \in P^{1/2}_{\xi_\varphi}$. Next $\Delta^{1/2}_\varphi(\pi_\varphi(h)\xi_\varphi) = \pi_\varphi(h)\xi_\varphi$ implies that

$$\pi_\varphi(h)\xi_\varphi = \Delta^{1/4}_\varphi \pi_\varphi(h)\xi_\varphi.$$

Hence $\pi_\varphi(h)\xi_\varphi \in P^{1/4}_{\xi_\varphi}$.

(2) $\Rightarrow$ (3). If $\pi_\varphi(\xi_\psi, \xi_\varphi)$ is essentially self-adjoint this follows from the proof of Theorem 1.1.

The vectors $\xi_\psi \pm i\xi_\varphi$ are both separating for $\pi_\varphi(\mathcal{M})$ in that

$$\begin{aligned} &\omega_{\xi_\varphi \pm i\xi_\varphi}(\pi_\varphi(m)) \\ &\quad = (\psi + \varphi)(m) \pm i((\pi_\varphi(m)\xi_\varphi, \xi_\psi) - (\pi_\varphi(m)\xi_\psi, \xi_\varphi)) \\ &\quad = (\psi + \varphi)(m) \end{aligned}$$

Consequently $J(\xi_\psi \pm i\xi_\varphi) = \xi_\psi \mp i\xi_\varphi$ are both separating for $\pi_\varphi(\mathcal{M})'$ and cyclic for $\pi_\varphi(\mathcal{M})$. This implies that

$$\operatorname{Ran}(\pi_\varphi(\xi_\psi, \xi_\varphi) \pm iI) = \pi_\varphi(\mathcal{M})(\xi_\psi \pm i\xi_\varphi)$$

are dense in $\mathcal{H}$. Essential self-adjointness follows (cf. [7, p. 257]).

(3) $\Rightarrow$ (1). This is seen from Proposition 2.4. When $\bar{\psi} = \psi$, $\psi \circ \hat{E} = \psi$ holds. If a_0 is the Radon–Nikodym derivative of ψ with respect to φ on $\mathcal{A} \cap \mathcal{M}^\varphi$. (Cf. the proof of Theorem 1.1.) one finds

$$\begin{aligned} \varphi(a_0^{1/2} m a_0^{1/2}) &= \varphi(a_0^{1/2} \hat{E}(m) a_0^{1/2}) \\ &= \psi(\hat{E}(m)) \\ &= \psi(m). \end{aligned}$$

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REFERENCES

1. O. Bratteli and D. Robinson, *Operator algebras and quantum statistical mechanics*, I: *C*- and W *-algebras, symmetry groups, decomposition of states*, Springer-Verlag, Berlin - Heidelberg - New York, 1979.
2. C. C. Chu, *On standard elements and tensor products of compact convex sets*, J. London Math. Soc. (2) 14 (1976), 71–78.
3. T. Kato, *Perturbation theory for linear operators* (Grundlehren Math. Wiss. 132), Springer-Verlag, Berlin - Heidelberg - New York, 1966.
4. H. Kosaki, *Positive cones associated with a von Neumann algebra*, Math. Scand. 47 (1980), 295–307.
5. G. K. Pedersen, *C*-algebras and their automorphism groups* (London Math. Soc. Monographs 14), Academic Press, London - New York, 1979.
6. D. Petz, *Ergodic theorems in von Neumann algebras*, Preprint, Hungarian Academy of Science, no. 55, Budapest, 1981.
7. M. Reed and B. Simon, *Methods of modern mathematical physics*, I: *Functional analysis*, Academic Press, New York - San Francisco - London, 1972.
8. C. F. Skau, *Positive self-adjoint extensions of operators affiliated with a von Neumann algebra*, Math. Scand. 44 (1979), 171–195.
9. C. F. Skau, *Finite subalgebras of a von Neumann algebra*, J. Funct. Anal. 25 (1977), 211–235.
10. M. Takesaki, *Theory of operator algebras*, I, Springer-Verlag, Berlin - Heidelberg - New York, 1979.
11. M. Takesaki, *Conditional expectations in von Neumann algebras*, J. Funct. Anal. 9 (1972), 306–321.

INSTITUTT FOR MATEMATIKK
UNIVERSITETET I TRONDHEIM, NTH
7034 TRONDHEIM
NORWAY

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