INTEGRABLE-ERGODIC C*-DYNAMICAL SYSTEMS
ON ABELIAN GROUPS

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Abstract.
In this paper we introduce a notion of combined integrability and ergodicity for actions of locally compact, abelian groups on C*-algebras. We prove that for a fixed, second countable group $G$, the set $[G]$ of all covariantly non-isomorphic, integrable-ergodic and faithful C*-dynamical systems $(\mathcal{A}, G, \beta)$, can be classified by means of $H^2_b(\hat{G}, T)$, the second Borel-cohomology group over $\hat{G}$. This is a direct generalization of a result of D. Olesen, G. K. Pedersen, and M. Takesaki on ergodic systems over compact, abelian groups.

Introduction.
In [10], D. Olesen, G. K. Pedersen, and M. Takesaki classify all ergodic and faithful W*-dynamical systems $(\mathcal{M}, G, \alpha)$ on a fixed, compact, abelian group $G$.

They show that the set $[G]$ of covariantly non-equivalent, ergodic and faithful W*-dynamical systems over $G$ admits a multiplication, so that $[G]$, $\times$ is isomorphic to $\chi^2(\hat{G}, T)$, the group of anti-symmetric bicharacters of $\hat{G}$. The classification in the C*-case then became trivial, since they could prove that under the conditions of ergodicity and faithfulness, we have a one-to-one correspondence between W*- and C*- systems.

These results, as far as the W*-case is concerned, have been generalized in a number of different ways. In [13], A. Wassermann was successful in giving the classification for systems on non-abelian groups. H. H. Zettl, in [15], shows how the ergodicity of the action can be weakened down to the condition that the fixed-point algebra $\mathcal{M}^*$ is contained in the centre $Z(\mathcal{M})$ of $\mathcal{M}$. Finally, turning the attention towards locally compact, abelian groups it is proved in [4] that here the proper setting for the W*-classification theorem is that of integrable, ergodic and faithful systems.

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In this paper, we concentrate on the C*-case. We introduce a notion of integrable-ergodic C*-dynamical systems, and – by showing that to each such system corresponds a unique system of [4] – we classify them by means of $H^2_b(\hat{G}, T)$, the second Borel-cohomology group of $\hat{G}$. The conclusion will be that every integrable-ergodic (in short I-E) and faithful C*-dynamical system $(\mathcal{A}, G, \beta)$ on a second countable, locally compact, abelian group $G$, is of the form $(\mathcal{C}^{\text{r,co}}(\hat{G}), \text{ad} v)$, where $\mathcal{C}^{\text{r,co}}(\hat{G})$ is the twisted, reduced group C*-algebra of $\hat{G}$, and $(v_s f)(p) = \langle s, p \rangle f(p), \ f \in L^2(\hat{G}), s \in G, \ p \in \hat{G}$.

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1. Definition and basic facts.

Throughout these notes $G$ will denote a second countable, abelian, locally compact Hausdorff group with Haarmeasure ds. $(\mathcal{M}, G, \alpha)$, respectively $(\mathcal{A}, G, \beta)$, will be a W*-algebra, respectively C*-dynamical system over $G$ with a continuous, faithful and ergodic action $\alpha$, respectively $\beta$. The condition of ergodicity on $\beta$ is that the fixedpoint algebra $\mathcal{A}^\beta$ must be either C.1 or $\{0\}$, depending on whether or not $\mathcal{A}$ is unital.

By $\eta_\alpha$ we denote the set of all $x \in \mathcal{M}$ such that $\int \varphi \alpha_s(x^*x) ds < \infty$ for all $\varphi \in (\mathcal{M}_*)_+$ and $\mu_\alpha$ will stand for $\eta_\alpha^{-1}$. It is well-known that $(\mu_\alpha)_+$ consists precisely of those elements in $\mathcal{M}_+$ which are $\alpha$-integrable. More specifically, in the ergodic setting the conditions

1) $x \in (\mu_\alpha)_+$, and
2) there exists a $r_x \in \mathbb{R}_+$ so that $\int \varphi \alpha_s(x) ds = r_x \| \varphi \|$ for all $\varphi \in (\mathcal{M}_*)_+$,

are equivalent, where 2) expresses that $\int \alpha_s(x) ds = r_x \cdot 1$. By a definition of A. Connes and M. Takesaki in [2] the action $\alpha$ is called integrable when $\mu_\alpha$ is $\sigma$-weakly dense in $\mathcal{M}$. For $(\mathcal{A}, G, \beta)$ a similar definition is possible.

1.1. Definition. An $x \in \mathcal{A}_+$ is called $\beta$-integrable whenever there exists a $r_x \in \mathbb{R}_+$ so that

$$\int \varphi \beta_s(x) ds = r_x \| \varphi \|, \ \varphi \in (\mathcal{A}_*)_+.$$  

The system $(\mathcal{A}, G, \beta)$ is integrable if the set

$$\mu_\beta = \text{span} \{ x \in \mathcal{A}_+ | x \text{ is } \beta\text{-integrable} \}$$

is norm-dense in $\mathcal{A}$. 
Let us give one example.

1.2 Example. The easiest and best-known integrable, ergodic and faithful $W^*$-dynamical systems is $(L^\infty(G), G, \alpha^1)$, where $\alpha_s^{-1}(f)(t) = f(t - s)$. It contains the C*-system $(C_0(G), G, \beta^1), \beta^1 = \alpha^1|_{C_0(G)}$, which is still faithful and ergodic. Since $(C_0(G)^*)_+$ consists of all positive, bounded measures on $G$, we have for any $f \in C_c(G)$:

$$\int \varphi \beta_s^1(f) ds = \int \int f(t - s) ds d\mu_\varphi(t) = \int f(s) ds \cdot \mu_\varphi(G),$$

where $\varphi = \mu_\varphi$ in $(C_0(G)^*)_+$.

One of the main statements of [4] is that for an ergodic and faithful $W^*$-dynamical system the integrability of $\alpha$ is equivalent to the condition that for each $p \in \check{G}$ there exists a unitary operator $u_p \in \mathcal{M}$, so that

$$\alpha_s(u_p) = \langle s, p \rangle u_p, \ s \in G.$$

Example 1.2 shows that this is no longer true in the C*-case. Since $C_0(G)$ is non-unital it does not contain unitaries. This observation also holds in general.

1.3. Lemma. Let $G$ be non-compact and $(\mathcal{A}, G, \beta)$ an integrable-ergodic C*-dynamical system, then $\mathcal{A}$ is non-unital.

Proof. Suppose that $\mathcal{A}$ contains 1 and let $\varphi \in \mathcal{A}^*$ be a state. Then, for each $x \in \mathcal{A}$ with $\|x - 1\| < \frac{1}{2}$ we have $|\varphi \beta_s(x) - 1| < \frac{1}{2}$ for all $s \in G$. But then $\int \varphi \beta_s(x) ds = \infty$, so that $x \notin \mu_\beta$ and $\bar{\mu}_\beta \neq \mathcal{A}$.

Opposite to what we have in the $W^*$-setting, the map $\varepsilon : x \in \mu_\beta \rightarrow \int \beta_s(x) ds$ has its image in $\mathcal{A}^{**}$ and not in $\mathcal{A}$. In fact, using the ergodicity and Lemma 1.3, $\varepsilon(x) \in \mathcal{A}$ implies $x = 0$. On the other hand, putting $\mathcal{A}_1 = \mathcal{A} + C.1$, our definition states that $\varepsilon(x)$ is a scalar multiple of 1, which is precisely what we have in the $W^*$-case.

We conclude the section with a second, more sophisticated example, which was brought to our attention by H. H. Zettl.

1.4. Example. Let $p \in \hat{G}$ and denote $\hat{p}$ the action of $G$ on $C(T)$ defined by $(\hat{p}_s f)(v) = f((s, p)v)$. For $G = \mathbb{Z}$ and taking the crossed product $G \times \hat{p}C(T)$, we obtain all rational and irrational rotation C*-algebras, as they were introduced by M. Rieffel in [12]. More in general, for $G$ discrete, $G \times \hat{p}C(T)$ are the generalized rotation algebras, studied by M. De Brabanter and H. H. Zettl in [3].

On $G \times \hat{p}C(T)$ we have the dual action $(\hat{p})^\Lambda$ of $\hat{G}$, implemented by the
unitary representation \( q \to v_q \otimes 1 \) of \( \hat{G} \) on \( L^2(G) \otimes L^2(T) \), where \( (v_q f)(s) = \langle s, q \rangle f(s) \). There is also a second action \( \gamma \) of \( T \) on \( G \times \hat{C}(T) \), given by \( \gamma_\mu = \text{ad}(1 \otimes T_\mu) \), with \( (T_\mu \xi)(v) = \xi(\mu v), \xi \in L^2(T) \). This gives rise to a \( C^* \)-
dynamical system \( (G \times \hat{C}(T), \hat{G} \times T, \beta) \), where
\[
\beta_{(q, \mu)} = (\hat{p})_q \circ \gamma_\mu.
\]

We show that it is faithful and integrable-ergodic.

To see this, take \( \xi \in C(T) \) and \( f \in L^1(G) \), then the operators \( \pi(\xi) \lambda_f \), with
\[
(\pi(\xi)(g \otimes \xi))(t, v) = (\hat{p}_+(\xi))(v) \cdot g(t) \cdot \xi(v)
\]
and
\[
(\lambda_f(g \otimes \xi))(t, v) = \int f(s)g(t - s)ds \cdot \xi(v),
\]
are norm-dense in \( G \times \hat{C}(T) \). Denoting \( \xi''(v) = \xi(\mu v) \), one verifies that
\[
\beta_{(q, \mu)}(\pi(\xi) \lambda_f) = \pi(\xi'') \lambda_{\langle ., q \rangle f},
\]
from which the faithfulness of \( \beta \) is obtained.

To prove the ergodicity, first observe that \( s \to (v \to \langle s, p \rangle v) \) determines a continuous homomorphism of \( G \) in the group of all isometries on \( T \). So, by [6; Proposition 3.3], \( (C(T), G, p) \) is an almost periodic system in the sense of [6; Definition 3.2], and from [6; Theorem 4.8] it follows that \( G \times \hat{C}(T) \) is contained in
\[
C^*\{m_f \lambda_q | f, g \in C^b(G)\} \otimes C(T).
\]

Thus,
\[
G \times \hat{C}(T) \subset \mathcal{B}(L^2(G)) \otimes C(T).
\]

Next, let \( x \) be a fixed point for \( \beta \) in \( G \times \hat{C}(T) \). To any \( \varphi \in C(T)^* \) we can associate a bounded operator
\[
(1 \otimes \varphi) : \mathcal{B}(L^2(G)) \otimes C(T) \to \mathcal{B}(L^2(G))
\]
defined by
\[
(1 \otimes \varphi)(x \otimes \xi) = x \cdot \varphi(\xi), \ x \in \mathcal{B}(L^2(G)), \ \xi \in C(T).
\]

For \( q \in \hat{G} \) we then have
\[
(1 \otimes \varphi)(x) = (1 \otimes \varphi)(\beta_{(q, 1)}(x))
\]
\[
= (1 \otimes \varphi)((\text{adv}_q \otimes 1)(x))
\]
\[
= \text{adv}_q((1 \otimes \varphi)(x)),
\]
and since \( \{ v_q \mid q \in G \}^* = L^\infty(G) \), there exists a \( f_\varphi \in L^\infty(G) \) so that \( (1 \otimes \varphi)(x) = m_{f_\varphi} \).

By a similar argument we also obtain that for each \( \psi \in \mathcal{B}(L^2(G))^* \) there is a \( \lambda_\psi \in \mathbb{C} \) satisfying \( (\psi \otimes 1)(x) = \lambda_\psi \). Combining the 2 relations, we get

\[
\psi(m_{f_\varphi}) = (\psi \otimes \varphi)(x) = \lambda_\psi \cdot \varphi(1),
\]

so that the operator \( \varphi(1)^{-1} m_{f_\varphi} \) is obviously independent of \( \varphi \). In addition we conclude from the above that

\[
(\psi \otimes \varphi)(\varphi(1)^{-1} m_{f_\varphi} \otimes 1) = (\psi \otimes \varphi)(x),
\]

which, by weak*-density of \( \mathcal{B}(L^2(G))^* \otimes C(T)^* \) in \( (\mathcal{B}(L^2(G)) \otimes C(T))^* \), means that \( x = \varphi(1)^{-1} m_{f_\varphi} \otimes 1 \). Finally, using [7; Theorem 4.10] and \( x \in G \times \mathbb{R} C(T) \), we get that \( f_\varphi \) must be translation-invariant. Thus, \( f_\varphi \in C.1 \) and \( \beta \) is ergodic.

The major problem, however, is the integrability. Let \( \varphi \in (G \times \mathbb{R} C(T)) \), then by [11; Proposition 7.6.8] there exists a norm-continuous, bounded function \( \phi : G \rightarrow C(T)^* \) so that for \( n \in \mathbb{N}_0, s_1, s_2, \ldots, s_n \in G \) and \( \xi_1, \xi_2, \ldots, \xi_n \in C(T) \)

\[
(1) \quad \sum_{i,j}^n \phi(s_i - s_j)(\bar{\xi}_j \xi_i) \geq 0,
\]

and, for \( \xi \in C(T) \) and \( f \in L^1(G) \)

\[
\varphi(\pi(\xi)\lambda_f) = \int \phi(s)(\xi) \cdot f(s) \, ds.
\]

Furthermore, \( \phi(s) \in C(T)^* \), so that for some bounded Radon-measure \( m_{\varphi,s} \) on \( T \)

\[
\phi(s)(\xi) = \int_s \xi(v) \, dm_{\varphi,s}(v).
\]

Now let \( K^1(G) \) denote the set of all \( L^1(G) \)-functions such that \( f \) has compact support and take \( f \in K^1(G) \). Then

\[
\int_G \int_T \hat{\varphi}(\beta_{(q,\mu)}(\pi(\xi)\lambda_f)) \, dq \, d\mu
\]

\[
= \int_G \int_T \int_G \int_T \bar{\xi}(uv) \langle s, q \rangle f(s) \, dm_{\varphi,s}(v) \, ds \, d\mu \, dq
\]

\[
= \int_T \xi(\mu) \, d\mu \int_G \int_G \int_T \bar{m}_{\varphi,s}(T) \langle s, q \rangle f(s) \, ds \, dq.
\]

Using (1), with \( \xi_i \) equal to a constant function \( v \rightarrow \lambda_i \), we get

\[
\sum_{i,j}^n \int_T \lambda_j \lambda_i \, dm_{\varphi,s_i - s_j}(v) \geq 0.
\]

So, since \( s \rightarrow m_{\varphi,s}(T) \equiv \phi(s)(1) \) is clearly continuous and bounded, it is also
positive-definite. Bochner’s theorem may be applied, so that there exists a positive, bounded Radon-measure \( m_\phi \) on \( \hat{G} \), with \( \phi(\cdot)(1) = m_\phi^\lambda \). Then, by Fubini’s theorem, [8; Theorem 31.27] and the inversion theorem, we get

\[
\int_{G} \int_{G} m_\phi (\pi) \langle s, q \rangle f(s) ds dq \\
= \int_{G} (\phi(\cdot)(1) \cdot f)^\Lambda (-q) dq \\
= \int_{G} \int_{G} f(q - r) dm_\phi (r) dq \\
= \int_{G} f(q) dq \cdot \phi(0)(1).
\]

Next, by positive-definiteness of \( \phi(\cdot)(1) \) again, \( \phi(0)(1) \geq 0 \), so that [11; p. 258] gives \( \|\phi(0)(1)\| = \|\phi(0)\| = \|\phi\| \). Therefore we may conclude that

\[
\int_{G} \int_{T} \phi(\beta_{(t,u)}(\pi(\xi)\lambda_f)) dq d\mu \\
= \int_{T} \xi(\mu) d\mu \cdot \int_{G} f(q) dq \cdot \|\phi\|
\]

and since the operators \( \pi(\xi)\lambda_f, f \in K^1(G) \), are norm-dense, \( \beta \) is integrable.

2. The classification theorem.

Let \((\mathcal{A}, G, \beta)\) be a faithful, integrable-ergodic C*-dynamical system, then \(\tau: \mathcal{A} \to [0, \infty], x \to \int \varphi(\beta_s(x)) ds\), where \(\varphi\) is a state on \(\mathcal{A}\), defines a faithful, lower semi-continuous weight on \(\mathcal{A}\). The lower semi-continuity of \(\tau\) follows from the existence of a net \(\{\tau_i\}_{i \in I}\) of continuous functions \(\tau_i: \mathcal{A} \to [0, + \infty]\),

\[
x \to \int_{K_i} \varphi(\beta_s(x)) ds,
\]

where \(K_i\) is compact in \(G\), such that \(\tau = \sup_{i \in I} \tau_i\).

Denote \(\eta_\tau = \{x \in \mathcal{A} | \tau(x^*x) < \infty\}\) and let \(\mathcal{H}_\tau\) be the completion of the pre-Hilbert space \(\eta_\tau\) for the norm arising from the inner product \(\langle \xi_x, \xi_y \rangle = \tau(y^* \cdot x), x, y \in \eta_\tau\). We then get a faithful, non-degenerate \(*\)-representation of \(\mathcal{A}\) on \(\mathcal{H}_\tau\), given by \(\pi_\tau(x) \xi_y = \xi_{xy}\).

Next, let \(\mathcal{M}\) be the \(\sigma\)-weak completion of \(\pi_\tau(\mathcal{A})\) in \(\mathcal{B}(\mathcal{H})\). We can see that \(\beta\) is unitary implemented on \(\mathcal{H}_\tau\) by \(U\), where \(U \xi_x = \xi_{\beta(x)}\), so that the action of \(G\) on \(\mathcal{A}\) can be extended to \(\mathcal{M}\). The extended action \(\alpha = \text{ad} U\) is continuous, since \(\tau\) is lower semi-continuous. So, we obtain a W*-dynamical system \((\mathcal{M}, G, \alpha)\).

2.1. Lemma. \((\mathcal{M}, G, \alpha)\) is an integrable, ergodic and faithful W*-dynamical system.

Proof. The faithfulness of \(\text{ad} U\) is clear, while the integrability follows from the integrability of \(\beta\). To prove the ergodicity, let \(x_0\) be a fixed point in \(\mathcal{M}\) and take \(a\) and \(b\) integrable in \(\mathcal{M}\), denoting
\[ a_0 = \int_G U_s a U_s^* ds, \quad b_0 = \int_G U_s b U_s^* ds. \]

Then,
\[ \int_G U_t (\int_G U_s a U_s^* x_0 b ds) U_t^* dt = a_0 x_0 b_0. \]

Thus, if
\[ \mathcal{R} = \{ \int_G U_s x U_s^* ds \mid x \text{ integrable in } \mathcal{M} \}'' = C.1 \]
and \( y \in \mathcal{R}' \), \( y a_0 x_0 b_0 = a_0 x_0 b_0 y \), so that \( a_0 y x_0 b_0 = ax_0 yb_0 \). But since \( a_0, b_0 \in C.1 \), we have \( yx_0 = x_0 y \) or \( x_0 \in \mathcal{R}'' = C.1 \).

In [10], for a compact group \( G \), D. Olesen, G. K. Pedersen, and M. Takesaki prove that \((\mathcal{A}, \beta)\) can be reconstructed from \((\mathcal{M}, \alpha)\) by taking \((\mathcal{M}^c, \alpha|_{\mathcal{M}^c})\), where
\[ \mathcal{M}^c = \{ x \in \mathcal{M} \mid s \rightarrow \alpha_s(x) \text{ is norm-continuous} \}. \]

In the locally compact case, however, \( \mathcal{M}^c \) can no longer be used for this purpose, as we can see from the abelian \( G \)-system \((L^\infty(G), G, \alpha^1)\). Since \((L^\infty(G))^c = C^b(G)\), this algebra is unital, contradicting Lemma 1.3.

What we will prove here is that \( \mathcal{A} = \mathcal{M}^{ic} \), with \( \mathcal{M}^{ic} = (\mathcal{M}^c \cap \mu) - \| \cdot \| \). To do this we want to use the classification theorem 2.7 of [4], for integrable, ergodic and faithful \( W^* \)-dynamical systems. Unfortunately, this theorem was formulated for \( W^* \)-systems with a separable predual, and although this is a very natural condition to impose on von Neumann algebra's, the equivalent condition on an underlying \( C^* \)-subalgebra is completely unacceptable. This problem is solved in the following lemma.

2.2 Lemma. Let \((\mathcal{M}, G, \alpha)\) be a \( W^* \)-dynamical system, with \( \alpha \) integrable, ergodic and faithful, and \( G \) second countable, then \( \mathcal{M}_\alpha \) is separable.

Proof. First observe that with the technique of [4; Lemma 1.13] we can construct a continuous cross-section \( p \rightarrow u_p \) from a neighbourhood of any point in \( \hat{G} \) into the group \( G_\alpha \), of unitary eigenoperators for \( \alpha \). Using the second countability of \( \hat{G} \), these can be linked together, so that we obtain a Borel-measurable cross-section \( p \rightarrow u_p \) of \( \hat{G} \) onto \( G_\alpha \), which is continuous at 0. Also, for any \( x \in \mu_\alpha \), the map
\[ p \in \hat{G} \rightarrow \hat{x}(p) = \int \langle s, p \rangle^{-1} \alpha_s(x) ds \]
is continuous.

Next, let \( \mathcal{M} \) be faithfully represented on a Hilbert space \( \mathcal{H} \) and take \( \xi \in \mathcal{H}, \| \xi \| = 1 \). By the continuity-conditions on the maps \( u \), and \( \hat{x}(\cdot) \) and, again, the second countability of \( \hat{G} \), it is not hard to see that \( \{ u_p \xi \mid p \in \hat{G} \} \) and \( \{ \hat{x}(p)\xi \mid p \in \hat{G} \} \) generate separable sub-Hilbert spaces \( \mathcal{H}_\xi^u \) and \( \mathcal{H}_\xi^x \) of \( \mathcal{H} \).
Since $\alpha_\delta(\hat{x}(p)) = \langle s, p \rangle \hat{x}(p)$, $\alpha$ is ergodic and $\| \hat{x}(p) \| \leq |\rho(x)|$ (see [4; Lemma 1.13]), we can define a bounded, scalar function $f_\xi$ on $\hat{G}$ by $f_\xi(p) = u_\xi^* \hat{x}(p)$. With $F$ we denote the map $x \in \mu_\alpha \rightarrow f_x$. Observe that by the separability of $\mathcal{H}_\xi$ and $\mathcal{H}_\xi^*$, there exists a countable, orthonormal set $\{\xi_i\}_{i \in \mathbb{N}_0}$ in $\mathcal{H}$, so that for all $p \in \hat{G}$

$$f_\xi(p) = \sum_{i=1}^\infty \langle \hat{x}(p) \xi_i, \xi_i, u_p \xi_i \rangle.$$

Therefore, $f_\xi \in L^\infty(\hat{G})$.

We now extend $\{\xi_i\}_{i \in \mathbb{N}_0}$ to a total orthonormal basis $\{\xi_j\}_{j \in J}$ of $\mathcal{H}$. By arguments similar to the ones of [4; Lemma 1.10] and using the monotone convergence theorem for nets of lower semi-continuous functions as it was formulated in [1; Proposition 5], we have

$$\| x \|_\xi^2 = \int_{\hat{G}} \langle \alpha_\delta(x^* x) \xi_i, \xi_i \rangle ds \leq \| f_x \|_{L^2}^2,$$

which shows that $F : \mu_\alpha \rightarrow L^2(\hat{G}) \cap L^\infty(\hat{G})$, $x \rightarrow f_x$, is an isometry of a dense part of the Hilbert space $\mathcal{H}_\rho$ associated to the left Hilbert algebra $\eta_\alpha \cap \eta_\alpha^*$ into $L^2(\hat{G})$. Thus, $\dim \mathcal{H}_\rho \leq \dim L^2(\hat{G}) = \chi_0$, since $\hat{G}$ is second countable.

So, by [4; Theorem 2.7], $(\mathcal{M}, G, \alpha)$ is covariantly isomorphic to some $(\hat{G} \times_\omega C, G, \text{ad} v)$, where $\omega \in Z_2^b(\hat{G}, T)$, the group of the Borel-measurable 2-cocycles of $\hat{G}$ in $T$, and $(v_s f)(p) = \langle s, p \rangle f \in L^2(\hat{G})$. If the 2-cocycle is trivial, we get the abelian case $(\mathcal{M}(\hat{G}), G, \text{ad} v)$, where $\mathcal{M}(\hat{G})$ is the group von Neumann algebra of $\hat{G}$. Obviously, this system is covariantly isomorphic to $(L^\infty(G), G, \alpha^1)$ of Example 1.2 and under the same isomorphism, $(C_0(G), G, \beta^1)$ corresponds with $(C^*_r(\hat{G}), G, \text{ad} v)$, where $C^*_r(\hat{G})$ is the reduced group $C^*$-algebra of $\hat{G}$.

This can easily be generalized towards the non-abelian case. First, from [5], we recall some facts on the twisted group algebras $L^1_\omega(\hat{G})$.

Let $\omega \in Z_2^b(\hat{G}, T)$ and $f, g \in L^1(\hat{G})$, then the function $f \ast_\omega g$ and $f \ast_\omega$, $\hat{G} \rightarrow C$, defined by
\[ (f \ast_\omega g)(p) = \int f(p - q)\omega(p - q,q)g(p)\,dq, \]

and

\[ f\ast\omega(p) = \omega(p, -p)^{-1} f(-p), \]

are both in \( L^1(\hat{G}) \). One easily verifies that \( L^1_\omega(\hat{G}) = (L^1(\hat{G}), \ast_\omega, \ast^\omega) \) is an involutive Banach algebra.

Next recall that \( T \times_\omega \hat{G} \) denotes the locally compact group of all \((\mu, p), \mu \in T \) and \( p \in \hat{G} \), with multiplication defined by

\[ (\mu, p) \cdot (v, q) = (\mu \cdot v \cdot \omega(p, q), p + q), \mu, v \in T, \ p, q \in \hat{G}. \]

C. M. Edwards and J. T. Lewis then consider 2 maps, which we will denote \( \Lambda \) and \( \Omega \), linking \( L^1_\omega(\hat{G}) \) to \( L^1(T \times_\omega \hat{G}) \). The map \( \Lambda : L^1_\omega(\hat{G}) \to L^1(T \times_\omega \hat{G}) \) is defined by

\[ (\Lambda f)(\mu, p) = \mu f(p), \ f \in L^1_\omega(\hat{G}), \]

while \( \Omega : L^1(T \times_\omega \hat{G}) \to L^1_\omega(\hat{G}) \) is given by

\[ (\Omega F)(p) = \int F(\mu, p)^{-1} d\mu, \ F \in L^1(T \times_\omega \hat{G}). \]

The following statements summarize the results given in [5; Lemma 3.1 and Lemma 3.2].

2.3. LEMMA. \( \Lambda \) is an isometric \(*\)-isomorphism from \( L^1_\omega(\hat{G}) \) onto a closed 2-sided ideal of \( L^1(T \times_\omega \hat{G}) \). \( \Omega \) is a norm non-increasing \(*\)-homomorphism from \( L^1(T \times_\omega \hat{G}) \) onto \( L^1_\omega(\hat{G}) \).

Next, let \( m_{\omega(\cdot, p)} \) and \( \lambda_p \) denote the multiplication-operator by \( \omega(\cdot, p) \) and the translationoperator by \( p \) on \( L^2(\hat{G}) \). Then, a faithful, non-degenerate representation \( \lambda^\omega \) of \( L^1_\omega(\hat{G}) \) is defined by

\[ \lambda^\omega(f) = \int f(p)\lambda_p m_{\omega(\cdot, p)} \,dp. \]

Taking the \( \sigma \)-weak completion of \( \lambda^\omega(L^1_\omega(\hat{G})) \) in \( \mathcal{B}(L^2(\hat{G})) \), we obtain the twisted group von Neumann algebra \( \mathcal{M}_\omega(\hat{G}) \) of \( \hat{G} \). If instead, we take the norm completion of it, the twisted, reduced group \( C^* \)-algebra \( C^*_{\omega}(\hat{G}) \) of \( \hat{G} \) is obtained. Of course, these notions are no different from the twisted cross-products \( C \times_\omega \hat{G} \), as they were defined in [14].

What we will show is the following. Suppose that \( p \to u_p \) is a Borel cross-section for \( (\mathcal{M}, \alpha) \). Then \( \mathcal{M} \cong \mathcal{M}_\omega(\hat{G}) \) under the natural isomorphism \( \varphi \), \( \varphi(u_p) = \lambda_p m_{\omega(\cdot, p)} \), of [4; Lemma 2.6]. We prove that \( \varphi(\mathcal{A}) = \varphi(\mathcal{M}^c) = C^*_{\omega}(\hat{G}) \), so that in particular, for each integrable-ergodic, faithful \( C^* \)-system, there exists \( \omega \in Z_b^2 \hat{G}, T \), so that \( (\mathcal{A}, \beta) \cong (C^*_{\omega}(\hat{G}), \text{ad} \nu) \).

First, one technical result.
2.4. **Lemma.** Let $x \in \mu_x$, then there exists a function $\theta_x \in L^\infty(\text{supp } \hat{x}, T)$, so that for all $f \in L^1(\hat{G})$ we have

$$\alpha_f(x) = \int h(p) u_p dp,$$

where $h = f(-\cdot) \parallel \hat{x}(-\cdot) \parallel \theta_x \in L^1(\hat{G})$.

**Proof.** Using Fubini's Theorem and the fact that both $\hat{f}$ and $x$ are integrable, we get

$$\int f(s) \alpha_s(x) ds = \int \hat{f}(p) \int \langle s, p \rangle \alpha_s(x) ds dp = \int \hat{f}(p) \hat{x}(-p) dp.$$

By ergodicity of $\alpha$, $p \rightarrow v_p = \hat{x}(p) \parallel \hat{x}(p) \parallel^{-1}$ defines a new Borel cross-section on $\text{supp } \hat{x}$. Also, on the same set, we have another Borel-measurable function $\theta_x$ defined by $\theta_x(p) = u_p v_p^*$. We get

$$\alpha_f(x) = \int \hat{f}(-p) \parallel \hat{x}(-p) \parallel \theta_x(p) u_p dp$$

and since $p \rightarrow \parallel x(-p) \parallel$ is bounded and continuous by the proof of [4; Lemma 1.13],

$$h(p) = \hat{f}(-p) \parallel \hat{x}(-p) \parallel \theta_x(p)$$

is an $L^1(\hat{G})$-function, satisfying the conditions.

2.5. **Proposition.** $\varphi(\mathcal{M}^{ic}) = C^{\ast}_{r, \omega}(\hat{G})$.

**Proof.** It is sufficient to prove that $\mathcal{M}^{ic}$ is the norm-completion of

$$\{ \int f(p) u_p dp | f \in L^1(\hat{G}) \}.$$

For the inclusion $\subseteq$, take $x \in \mathcal{M}^c \cap \mu_x$ and let $\{ f_i \}_{i \in I}$ be an approximate unit of $L^1(G)$ in the sense of [8; Theorem 33.11]. Since the functions $\{ f_i \}_{i \in I}$ have integral 1 and decreasing compact supports, and since $x \in \mathcal{M}^c$, we have

$$\parallel \int f_i(s) \alpha_s(x) ds - x \parallel \rightarrow 0.$$

On the other hand, $\hat{f}_i \in L^1(\hat{G})$ and $x \in \mu_x$, so that by Lemma 2.4 there exist functions $h_i \in L^1(\hat{G})$ such that

$$\int f_i(s) \alpha_s(x) ds = \int h_i(p) u_p dp.$$

For the second inclusion, we examine the *-algebra

$$B = \{ \int (f *_\omega g)(p) u_p dp | f, g \in L^1(\hat{G}) \cap L^2(\hat{G}) \}.$$

From [4; Lemma 1.10] it is obvious that $B \subset \mathcal{M}^{ic}$. Now, take any nest $\{ K_n \}_{n \in \mathbb{N}_0}$ of compact neighbourhoods of 0 in $\hat{G}$ with Haar-measures $\{ m(K_n) \}_{n \in \mathbb{N}_0}$, and define
\[ E_n(\mu, p) = \frac{n}{2m(K_n)} \chi_{K_n}(p) \chi_{[e^{-i\mu}, e^{i\mu}])(\mu), \]

\( \mu \in T \) and \( p \in \hat{G} \). Then, \( \{E_n\}_{n \in \mathbb{N}_0} \) is an approximate unit in \( L^1(T \times \omega, \hat{G}) \) and, using Lemma 2.3, one can check that the same holds for \( \{e_n = \Omega(E_n)\}_{n \in \mathbb{N}_0} \) in \( L^1(\hat{G}) \). For every \( f \in L^1(\hat{G}) \cap L^2(\hat{G}) \) we have

\[
\left\| \int (f * e_n - f)u_p dp \right\| = \left\| \chi( f * e_n - f ) \right\| \\
\leq \left\| f * e_n - f \right\|_1 \to 0,
\]

so that

\[
\overline{B} = \left\{ \int f(p)u_p dp | f \in L^1(\hat{G}) \cap L^2(\hat{G}) \right\} - \left\| \cdot \right\|.
\]

Finally, we prove that every faithful, I-E*-system \((\mathcal{A}, G, \beta)\) is of the form \((C^*_r(\hat{G}), G, ad \nu)\). Half the result is obtained in the following lemma.

2.6. Lemma. There exists a 2-cocycle \( \omega \in Z^2_b(\hat{G}, T) \), so that \((\mathcal{A}, G, \beta)\) is covariantly isomorphic to a C*-subsystem of \((C^*_r(\hat{G}), G, ad \nu)\).

Proof. By Lemma 2.1 we have the W*-system \((\mathcal{M}, G, ad U)\) associated to \((\mathcal{A}, G, \beta)\). Obviously \(\mathcal{A} \subset \mathcal{M}^{ic}\), so that by Proposition 2.5, \(\varphi(\mathcal{A})\) is a C*-subalgebra of \(C^*_r(\hat{G})\).

It remains to show that \(\varphi : \mathcal{A} \to C^*_r(\hat{G})\) is onto, or equivalently, that for every \(f \in L^1(\hat{G})\), \(\int_{\hat{G}} f(p)u_p dp\) is in \(\mathcal{A}\).

Denote

\[ H = \{p \in \hat{G} \mid \text{there exists a compact neighbourhood } K_p \text{ of } p \text{ in } \hat{G}, \text{ so that} \]

\[ \int_{K_p} f(q)u_q dq \in \mathcal{A} \text{ for all } f \in L^1(\hat{G}) \}. \]

We will show that \(H = \hat{G}\).

2.7. Lemma. Let \(p \in \hat{G}\), for which there exist \(f \in L^1(\hat{G})\), a compact neighbourhood \(K_p\) of \(p\) in \(\hat{G}\) and \(c > 0\), so that \(|f(K_p)| \subset ]c, + \infty]\) and

\[ \int_{\hat{G}} f(q)u_q dq \in \mathcal{A}, \]

then \(p \in H\).

Proof. Let \(h \in K^1(G)\), then

\[ \int_{\hat{G}} \hat{h}(-q)f(q)u_q dq = \int_{\hat{G}} h(s)\beta_s(\int_{\hat{G}} f(q)u_q dq) ds \]

is in \(\mathcal{A}\).

Now, \(\{\hat{h} | h \in K^1(G)\}\) is dense in \(L^1(\hat{G})\), thus for each \(g \in L^1(\hat{G})\) there is a net \(\{h_i\}_{i \in I}\) in \(K^1(G)\), so that
\[ \hat{h}_i \to (f^{-1} \cdot g \cdot \chi_{K_p})(-\cdot) \text{ in } L^1(\hat{G}). \]

Then, since \(|f^{-1}| < 1/c\) on \(K_p\),
\[ \int_{K_p} \hat{h}_i(-q)f(q)u_qdq \to \int_{K_p} g(p)u_pdp \text{ in } \mathcal{A}. \]

For the following lemma, we need a Borel measurable cross-section \(p \to u_p\) which is continuous at a given element \(p_1\) of \(\hat{G}\), and for which \(q \to u_q^*\) is continuous at another point \(p_2 \neq p_1\) in \(\hat{G}\) as well. Such a map can be constructed by the same technique as we used to obtain the cross-section in the proof of Lemma 2.2.

2.8. Lemma. \(H\) is a subgroup of \(\hat{G}\).

Proof. Let \(x \in \mu_\beta \cap \mathcal{A}_0^\dagger\) and \(f \in K^1(G)\) with \(\hat{f}(0) \neq 0\). Then from Lemma 2.4 we have
\[ \int_G \hat{f}(-p)\|\hat{x}(-p)\| \theta_x(p)u_pdp = \int_G f(s)\beta_s(x)ds \in \mathcal{A}. \]
Moreover, \(x > 0\) implies \(\hat{x}(0) \neq 0\) and by continuity of \(p \to \hat{f}(p)\|\hat{x}(p)\|\) there must be a compact neighbourhood \(K_0\) of 0 and a constant \(c > 0\), such that
\[ |\hat{f}(-p)| \cdot \|\hat{x}(-p)\| > c, \text{ for all } p \in K_0. \]
Thus \(0 \in H\).

Next, if \(f \in L^1(\hat{G})\), \(K_p \subset \hat{G}\) and \(c > 0\) satisfy the conditions of Lemma 2.7 at \(p \in \hat{G}\), then the same is true for \(q \to f(-q)\omega(q, -q)^{-1}, K_{-p} = \{-q | q \in K_p\}\) and \(c\) at \(-p\). So \(-p \in H\) along with \(p\).

Proving that \(p_1 + p_2 \in H\), for \(p_1, p_2 \in H \setminus \{0\}\) is a lot harder. For each \(f, g \in L^1(\hat{G})\), we have that
\[ \int_G (\chi_{K_{p_1}} \cdot f) \ast_\omega (\chi_{K_{p_2}} \cdot g)(p)u_pdp = \int_{K_{p_1}} f(p)u_pdp \cdot \int_{K_{p_2}} g(p)u_pdp \in \mathcal{A}, \]
so that by Lemma 2.7 it will be sufficient to find \(f, g \in L^1(\hat{G})\), a compact neighbourhood \(K_{p_1 + p_2}\) of \(p_1 + p_2\) and \(c > 0\), such that
\[ |(\chi_{K_{p_1}} \cdot f) \ast_\omega (\chi_{K_{p_2}} \cdot g)(p)| > c, \text{ for } p \in K_{p_1 + p_2}. \]
First we show that there exist \(f, g \in L^1(\hat{G})\) for which \(|(\chi_{K_{p_1}} \cdot f) \ast_\omega (\chi_{K_{p_2}} \cdot g)|\) is continuous at \(p_1 + p_2\). Since \(p_1 \neq p_1 + p_2\) we know that there is a Borel measurable cross-section \(p \to v_p\), continuous at \(p_1\) and so that \(p \to v_p^*\) is continuous at \(p_1 + p_2\). Let \(\omega'(p, q) = v_pv_q^*v_{p+q}^*\), then by taking \(q\) in a small enough neighbourhood \(V_{p_2}\) of \(p_2\), \(p \to \omega'(p - q, q)\) will be continuous at \(p_1 + p_2\). Furthermore, there is a function \(h \in L^\infty(\hat{G}, T)\) satisfying
\[ \omega(p, q) = h(p)h(q)h(p + q)^{-1}\omega'(p, q). \]
We get
\[ \left| (\chi_{K_{p_1}} \cdot f) \ast_{\omega} (\chi_{K_{p_2}} \cdot g)(p) \right| = \left| \int_{G} (\chi_{K_{p_1}} \cdot f \cdot h)(p-q) \cdot (\chi_{K_{p_2}} \cdot g \cdot h)(q) \omega'(p-q,q) dq \right|. \]

We now take \( g \in L^1(\hat{G}) \) with \( \text{supp} \ g \subset V_{p_2} \) and \( f \in L^1(\hat{G}) \) so that \( \chi_{K_{p_1}} \cdot f \cdot h \) is continuous at \( p_1 + p_2 \).

The 2-variable function we now have under the integral is a continuous \( p \)-function at \( p_1 + p_2 \), for every \( q \), and is bounded as a \( q \)-function by a fixed integrable function, for every \( p \). So, the Dominated Convergence Theorem applies and \( \left| (\chi_{K_{p_1}} \cdot f) \ast_{\omega} (\chi_{K_{p_2}} \cdot g) \right| \) is continuous at \( p_1 + p_2 \).

It remains to show that for some \( f \) and \( g \) satisfying the conditions we already imposed, we have
\[ (\chi_{K_{p_1}} \cdot f) \ast_{\omega} (\chi_{K_{p_2}} \cdot g)(p_1 + p_2) \neq 0. \]

This can be done by taking \( f \) and \( g \) such that
\[ (\chi_{K_{p_1}} \cdot g \cdot h)(q) = \omega'(p_1 + p_2 - q,q) \]
for \( q \in \text{supp} \chi_{K_{p_2}} \cdot g \cdot h \) and \( f \cdot h > 0 \). Then,
\[ \left| (\chi_{K_{p_1}} \cdot f) \ast_{\omega} (\chi_{K_{p_2}} \cdot g)(p_1 + p_2) \right| = \int_{\text{supp} \cap K_{p_2}} (f \cdot h)(p_1 + p_2 - q) dq > 0. \]

Observe that from its definition, \( H \) is obviously open. We also have:

2.9. Lemma. \( H \) is dense in \( \hat{G} \).

Proof. We investigate the set \( \{ \beta_f(x) | \hat{f} \in L^1(G), \hat{f} \in L^1(\hat{G}) \text{ and } x \in \mu_{\beta} \} \). In the proof of Proposition 2.5 we saw that this is a dense set in \( A \). Also, in the proof of Lemma 2.4 we obtained for these elements
\[ \beta_f(x) = \int \hat{f}(-p) \parallel \hat{x}(-p) \parallel \theta_x(p) u_p dp. \]

Observe that every \( p \) in the open support of the function \( q \mapsto \hat{f}(-q) \parallel \hat{x}(-q) \parallel \theta_x(q) \) is in \( H \), since \( q \mapsto f(-q) \parallel \hat{x}(-q) \parallel \theta_x(q) \) is continuous and \( |\theta_x(q)| = 1 \) for all \( q \in \hat{G} \).

We now assume that \( H \) is not dense in \( \hat{G} \), then there exists an \( s_0 \in G \) so that \( \langle s_0, p \rangle = 1 \) for all \( p \in H \). So, by the remarks above, \( \beta_{s_0}(\beta_f(x)) = \beta_f(x) \) for every element from that dense set. But then, \( \beta_{s_0} = 1 \), which contradicts the faithfulness of \( \beta \).

Combining the Lemmas 2.8 and 2.9 with the fact that \( H \) is open, we get \( H = \hat{G} \). With this conclusion the classification theorem is within reach.

2.10. Theorem. \( (\mathcal{A}, G, \beta) \) be a C*-dynamical system with an
integrable, ergodic and faithful action $\beta$ and a second countable, abelian group $G$, then there exists a 2-cocycle $\omega \in Z^2_b(\hat{G}, T)$ so that

$$(\mathcal{A}, G, \beta) \cong (C^*_r,\omega(\hat{G}), G, \text{ad } v).$$

**Proof.** Let $f \in L^1(\hat{G})$ and take $h \in C_c(\hat{G})$, so that $\|f - h\|_1 < \varepsilon, \varepsilon > 0$. For each $p \in \text{supp } h$ we have a compact neighbourhood $K_p$, so that

$$\int_{K_p} g(q)u_q dq \in \mathcal{A}, \text{ for all } g \in L^1(\hat{G}).$$

Since $\text{supp } h$ is compact, there exist $\{p_1, p_2, \ldots, p_n\}$ in $\hat{G}$ with $\text{supp } h \subset \bigcup_{i=1}^n K_{p_i}$. Then

$$\int h(q)u_q dq = \sum_{i=1}^n \int_{K_{p_i}} h(q) \cdot \prod_{j=1}^n (1 - \chi_{p_j})(q) \cdot u_q dq,$$

so that $\int h(q)u_q dq \in \mathcal{A}$ and $\int f(q)u_q dq$ as well.

2.11. **Example.** An immediate consequence of the above theorem and Example 1.4 is that for each $p \in \hat{G}$, $G \times_p C(T)$ is isomorphic to a twisted, reduced group $C^*$-algebra on $G \times \mathbb{Z}$. The question then rises as to how the connection between a character $p \in \hat{G}$ and the associated $\omega \in Z^2_b(G \times \mathbb{Z}, T)$,

$$G \times_p C(T) \cong C^*_r,\omega(G \times \mathbb{Z}),$$

can be expressed.

To see this, we examine the $W^*$-system $(G \times_p L^\infty(T), \hat{G} \times T, \alpha)$, where $\alpha = \text{ad } (v \otimes T)$. Arguments similar to the ones of Example 1.4 show that $\alpha$ is faithful and ergodic. Also, $\alpha$ is integrable, since $\beta$ is integrable,

$$(G \times_p C(T))^\nu = G \times_p L^\infty(T) \text{ and } \beta = \alpha|_{G \times_p C(T)}.$$ 

Now, let $\xi_0: T \to T$, $\mu \to \mu$ and

$$u_{(s,n)} = \pi(\xi_0)^{-n} \lambda_s, \text{ s } \in G \text{ and } n \in \mathbb{Z},$$

then $u_{(s,n)}$ is a unitary eigenoperator in $G \times_p L^\infty(T)$ and

$$\alpha_{(q,u)}(u_{(s,n)}) = \langle s, q \rangle \mu^n \cdot u_{(s,n)},$$

for all $q \in \hat{G}$, $\mu \in T$. Therefore, the 2-cocycle associated to $(G \times_p L^\infty(T), \hat{G} \times T, \alpha)$ is given by

$$\omega((s,n), (t,m)) = (\langle t, p \rangle)^n,$$

and if we can show that $(G \times_p L^\infty(T))^c = G \times_p C(T)$, then by Proposition 2.5, $\omega$ is also the 2-cocycle associated to $G \times_p C(T)$.

$$G \times_p C(T) \triangleleft (G \times_p L^\infty(T))^c$$
is clear, since the operators $\pi(\xi)\lambda_f$, $\xi \in C(T)$, $f \in K^1(G)$, are $\alpha$-norm-continuous, $\beta$-integrable (thus, $\alpha$-integrable), and norm-dense in $G \times \gamma C(T)$. To prove the second inclusion, by Proposition 2.5 it will be sufficient to show that

$$\int \sum_{n \in \mathbb{Z}} h(s,n)u_{(s,n)}ds \in G \times \beta C(T),$$

for every $h \in L^1(G \times \mathbb{Z})$. Obviously, it is also enough that this holds for $h \in C_c(\hat{G} \times \mathbb{Z})$, so that it remains to show that

$$\int h(s)u_{(s,n)}ds \in G \times \beta C(T),$$

for $h \in C_c(G)$. This is the case, since

$$\int h(s)\pi(\xi_0)^{-n}\lambda_sds = \pi(\xi_0)^{-n}\lambda_h.$$

Observe that with the construction of Example 1.4 we do not obtain every twisted, reduced group C*-algebra over $G \times \mathbb{Z}$. With an easy computation one verifies that the anti-symmetric bi-characters on $G \times \mathbb{Z}$ are of the form

$$\chi((s,n),(t,m)) = \psi(s,t) \langle ms - nt, p \rangle,$$

with $\psi$ an anti-symmetric bi-character on $G$ and $p \in \hat{G}$. Therefore, by [9; p. 29],

$$H^2_b(G \times \mathbb{Z}, T) = \{\omega((s,n),(t,m)) = \omega_1(s,t)\langle t,p \rangle^n | \omega_1 \in H^2_b(G,\mathbb{Z}), p \in \hat{G}\},$$

implying that only the systems $(C^*_{\omega}(G \times \mathbb{Z}), \text{ad} \nu)$ for which $\omega_1$ is trivial can be obtained through a construction of the type $(G \times \gamma C(T), \beta)$.

Let us conclude with a remark on the unitary eigenoperators. The representation $\lambda^\omega$ of $L^1(\hat{G})$ and the projective representation $p \rightarrow \lambda_p m_{\omega(\cdot,p)}$ of $\hat{G}$ are closely related. As we know from [4], $\mathcal{M}_{\omega}(\hat{G})$ is not only the $\sigma$-weak completion of the *-algebra $\lambda^\omega(L^1_{\omega}(\hat{G}))$, but also of the linear span of the operators $\lambda_p m_{\omega(\cdot,p)}$.

In the C*-case, and for a compact group $G$, it still does not matter whether we start off with the linear span or with $\lambda^\omega(L^1_{\omega}(\hat{G}))$. In both cases the norm-completion is $C^*_{r,\omega}(\hat{G})$. For a non-compact group, however, by Lemma 1.3, an integrable $C^*_{r,\omega}(\hat{G})$ never contains any of the $\lambda_p m_{\omega(\cdot,p)}$’s. Conversely, the C*-algebra $C^*\{\lambda_p m_{\omega(\cdot,p)} | p \in \hat{G}\}$ neither contains any of the operators $\lambda^\omega(f)$, $f \in L^1(\hat{G})$, different from 0. This can easily be seen from the abelian case, where $C^*\{\lambda_p\} \cong \text{A.P.}(G)$ under the isomorphism implemented by Fourier transform, $C^*_{r,\omega}(\hat{G}) \cong C_0(G)$ under the same isomorphism and $C_0(G) \cap \text{A.P.}(G) = \{0\}$. In fact, the basic reason for $C^*\{u_p\}$ not containing $\int f(p)u_pdp$, is the lack of continuity on $p \rightarrow u_p$. 
All this seems to indicate that next to the faithful, I-E. $C^*$-$G$-systems, we get a second class of $C^*$-systems over $G$ which admits classification by means of $H^2_b(\hat{G}, T)$. Namely, the faithful and ergodic systems of the form $(C^*\{u_p | p \in \hat{G}\}, \beta)$, where $p \rightarrow u_p$ is a projective representation of $\hat{G}$, satisfying $\beta_s(u_p) = \langle s, p \rangle u_p$, for each $s \in G$ and $p \in \hat{G}$.

However, with the techniques of [6; Lemma 3.1], it is not hard to see that the action $\beta$ of $(C^*\{u_p\}, G, \beta)$ can be extended to $G_b$, the Bohr-compactification of $G$, in a continuous way by putting

$$\beta_s(u_p) = \langle \tilde{s}, p \rangle u_p, \quad \tilde{s} \in G_b \text{ and } p \in \hat{G}.$$  

This means that the systems we obtain are the $C^*$-analogue of a special case of the almost periodic $W^*$-dynamical systems described in [10]. Their special feature is that the associated pure point spectrum $Sp_d(\beta)$ for these systems is $\hat{G}$ itself. Therefore, by [10; Theorem 7.4], they admit complete classification by means of $\chi^2(\hat{G}_\text{disc}, T)$. For each of the systems, there is a 2-cocycle $\omega$ in $Z^2(\hat{G}, T)$, so that $(C^*\{u_p\}, G, \beta)$ is covariantly isomorphic to $(C^*_{r,\omega}(\hat{G}_\text{disc.}), G, \text{ad } \nu)$.

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