MATH. SCAND. 126 (2020), 617–638

STABLE HOMOTOPY, 1-DIMENSIONAL NCCW COMPLEXES, AND PROPERTY (H)

QINGNAN AN, GEORGE A. ELLIOTT, ZHICHAO LIU, and YUANHANG ZHANG

Abstract

In this paper, we show that the homomorphisms between two unital one-dimensional NCCW complexes with the same KK-class are stably homotopic, that is, after adding on a common homomorphism (with finite dimensional image), they are homotopic. As a consequence, any one-dimensional NCCW complex has the Property (H).

1. Introduction

DEFINITION 1.1. Let A, B be C*-algebras and ψ_1, ψ_2 be homomorphisms from A to B. We say ψ_1, ψ_2 are *stably homotopic*, if there exists a homomorphism η from A to $M_r(B)$ for some integer r such that

$$\psi_1 \oplus \eta \sim_h \psi_2 \oplus \eta,$$

i.e., $\psi_1 \oplus \eta$ and $\psi_2 \oplus \eta$ are homotopic as homomorphisms from A to $M_{r+1}(B)$. In particular, stably homotopic homomorphisms induce the same KK-class.

Recall that the homotopy classes of homomorphisms from A to $B \otimes \mathscr{K}$ form an abelian semigroup $[A, B \otimes \mathscr{K}]$. Note that there is a natural map $\gamma\colon [A, B \otimes \mathscr{K}] \to \mathrm{KK}(A, B)$, which is not always injective. In fact, from the construction of the Grothendieck group of $[A, B \otimes \mathscr{K}]$, we know that, as mentioned above, two homomorphisms, which are not homotopic with each other but stably homotopic with each other, will induce the same KK-class. So a reasonable question is, conversely, if two homomorphisms induce the same KK-class, are they stably homotopic? Further, if they are stably homotopic but not homotopic, what is the obstruction?

We will consider these questions in the class $\mathscr{C}$ of sub-homogeneous algebras. Making such questions clear, on one hand, will be helpful for understanding the structure of the KK-group which is significant for classification theory of C*-algebras, and on the other hand, will lead to a proof about the Property (H) (Theorem 4.8) which is quite important in the uniqueness theorem

Received 18 June 2019. Accepted 13 April 2020.
DOI: https://doi.org/10.7146/math.scand.a-121092

(see Lemma 7.5 of [2] as an application for a special case). (Property (H) was introduced by Dadarlat in [4], and used in the classification of C*-algebras, especially in uniqueness part; see [4] and [5].)

The following is the class of algebras we are concerned with:

DEFINITION 1.2 (The class $\mathscr{C}$). Let F_1 and F_2 be finite dimensional C*-algebras and let $\varphi_0, \varphi_1\colon F_1 \to F_2$ be unital homomorphisms. Set

$$\begin{aligned} A &= A(F_1, F_2, \varphi_0, \varphi_1) \\ &= \{\, (f, a) \in C([0,1], F_2) \oplus F_1 : f(0) = \varphi_0(a) \text{ and } f(1) = \varphi_1(a) \,\}. \end{aligned}$$

Denote by $\mathscr{C}$ the class of all *unital* such C*-algebras.

The C*-algebras constructed in this way have been studied by Elliott and Thomsen [9] (see also [8] and [12]), and are sometimes called Elliott-Thomsen algebras or one-dimensional non-commutative finite CW complexes (NCCW).

We begin with three examples with different types of obstructions, which exist for homomorphisms but vanish in the stable sense.

EXAMPLE 1.3. Let $A = A(F_1, F_2, \varphi_0, \varphi_1)$, where $F_1 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, $F_2 = M_2(\mathbb{C})$, and

$$\varphi_0(a \oplus b \oplus c) = \begin{pmatrix} a & \\ & c \end{pmatrix}, \quad \varphi_1(a \oplus b \oplus c) = \begin{pmatrix} b & \\ & c \end{pmatrix}.$$

Define three homomorphisms δ_1, δ_2, and $\delta_3\colon A \to \mathbb{C}$:

$$\delta_1(f, a \oplus b \oplus c) = a, \quad (f, a \oplus b \oplus c) \in A,$$

$$\delta_2(f, a \oplus b \oplus c) = b, \quad (f, a \oplus b \oplus c) \in A,$$

and

$$\delta_3(f, a \oplus b \oplus c) = c, \quad (f, a \oplus b \oplus c) \in A.$$

Note that we have $\mathrm{Hom}(A, \mathbb{C}) = \{0, \delta_1, \delta_2, \delta_3\}$ (all the homomorphisms from A to $\mathbb{C}$). Suppose that

$$\delta_1 \sim_h \delta_2,$$

and denote the homotopy path in $\mathrm{Hom}(A, \mathbb{C})$ by Φ_θ, $\theta \in [0,1]$. Then $\{\Phi_\theta(g) \mid \theta \in [0,1]\}$, is a connected subset of $\mathbb{C}$, for any $g \in A$. But with

$$g_0 = \left(\begin{pmatrix} t & \\ & 0 \end{pmatrix}, \ 0 \oplus 1 \oplus 0 \right) \in A,$$

we have $\{\Phi_\theta(g_0) \mid \theta \in [0,1]\} = \{0, 1\}$, which is a contradiction.

So

$$\delta_1 \nsim_h \delta_2, \quad \text{but} \quad \delta_1 \oplus \delta_3 \sim_h \delta_2 \oplus \delta_3.$$

The stable homotopy path can be chosen as $H_\theta(f, a\oplus b\oplus c) = f(\theta), \theta \in [0,1]$. In particular,

$$\mathrm{KK}(\delta_1) = \mathrm{KK}(\delta_2).$$

EXAMPLE 1.4. Let $F_1 = \mathbb{C}$, $F_2 = \mathbb{C} \oplus \mathbb{C}$, and $\varphi_0 = \varphi_1 \colon F_1 \ni a \mapsto a \oplus a \in F_2$. Set $A = A(F_1, F_2, \varphi_0, \varphi_1) \cong C(X)$; in fact, X is the topological space "figure 8" and $B = \widetilde{C_0(0,1)} \cong C(S^1)$.

Construct homomorphisms ψ_1, ψ_2 from A to B,

$$\psi_1(f_1 \oplus f_2, a) = \begin{cases} f_1(4t), & \text{if } 0 \le t \le \frac{1}{4}, \\ f_2(4t-1), & \text{if } \frac{1}{4} \le t \le \frac{1}{2}, \\ f_1(3-4t), & \text{if } \frac{1}{2} \le t \le \frac{3}{4}, \\ f_2(4t-3), & \text{if } \frac{3}{4} \le t \le 1, \end{cases}$$

and

$$\psi_2(f_1 \oplus f_2, a) = \begin{cases} f_2(2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ f_2(2t-1), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that the fundamental group of X is the free group $\mathbb{F}_2$ with two generators $\{x, y\}$. By the Gelfand transform, the homomorphisms ψ_1, ψ_2 induce two continuous maps ψ_1', ψ_2' from S^1 to X and

$$[\psi_1'] = xyx^{-1}y \quad \text{and} \quad [\psi_2'] = y^2.$$

By the Gelfand transform, we know that $\psi_1 \nsim_h \psi_2$. But we have $\psi_1 \oplus \delta \sim_h \psi_2 \oplus \delta$, where

$$\delta(f_1 \oplus f_2, a) = a.$$

(A concrete path can be obtained by applying the trick in Remark 4.2 to the points $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.) In particular, $\mathrm{KK}(\psi_1) = \mathrm{KK}(\psi_2)$.

EXAMPLE 1.5. Set $A = M_2(\mathbb{C})$, $B = M_2(C(S^1)) \cong A(M_2(\mathbb{C}), M_2(\mathbb{C}), \mathrm{id}, \mathrm{id}) \in \mathscr{C}$, and

$$u = \begin{pmatrix} z & \\ & 1 \end{pmatrix} \in U(B).$$

Consider the natural embedding map ι from A to B, $\iota(a) = a$. Set $\psi_1 = \iota$ and $\psi_2 = u \cdot \iota \cdot u^*$. We point out that

$$\mathrm{KK}(\psi_1) = \mathrm{KK}(\psi_2) = 1 \in \mathbb{Z} \cong \mathrm{KK}(A, B).$$

Note that if the winding number corresponding to a unitary U in B is $2n$, then there exists a unitary path H_t in B with $H_0 = U$ and

$$H_1 = \begin{pmatrix} z^n & \\ & z^n \end{pmatrix}$$

such that

$$U \cdot \iota \cdot U^* \sim_h H_1 \cdot \iota \cdot H_1^* = \iota \quad \text{by} \quad H_t \cdot \iota \cdot H_t^*.$$

But we cannot apply the above trick to u, whose winding number is 1.

In fact, every unital homomorphism from A to B corresponds to a continuous map from S^1 to $\mathrm{Aut}(M_2(\mathbb{C}))$, and the fundamental group of the space of homomorphisms is the same as the fundamental group of $\mathrm{Aut}(M_2(\mathbb{C}))$. As

$$\pi_1(\mathrm{Aut}(M_2(\mathbb{C}))) = \pi_1(U(2)/\{\lambda \cdot 1_2 : \lambda \in \mathbb{C}, \|\lambda\| = 1\}) \cong \mathbb{Z}/2\mathbb{Z},$$

we have $[\psi_1] = 2\mathbb{Z}$ and $[\psi_2] = 1 + 2\mathbb{Z}$.

However, we still have

$$[u \oplus u^*] = [1_B \oplus 1_B] \in U(M_2(B))/U_0(M_2(B)),$$

i.e., there exists a unitary path V_t in $M_2(B)$ with $V_0 = u \oplus u^*$ and $V_1 = 1_B \oplus 1_B$. Then $V_t(\psi_1 \oplus 0)V_t^*$ gives

$$\psi_1 \oplus (u^* \cdot 0 \cdot u) = \psi_1 \oplus 0 \sim_h \psi_2 \oplus (u^* \cdot 0 \cdot u),$$

where 0 is the 0 homomorphism from A to B.

In fact, Example 1.3 shows we need more points to push the spectrum; and Example 1.4 shows that we need more points to make the homotopy class of homomorphisms become coarser ($\mathrm{KK}(C(X), C(S^1))$ is a commutative group, while the fundamental group of X is $\mathbb{F}_2$); Example 1.5 shows that the fact that the whole homotopy class of a unitary u does not commute with a given homomorphism may form an obstruction, but it will vanish in the stable sense. From the tips of these examples, we pass to the general sub-homogeneous class $\mathscr{C}$ and state our main result:

THEOREM 1.6. *Let $A, B \in \mathscr{C}$ (Definition 1.2), and let ψ_1, ψ_2 be homomorphisms from A to B. Then ψ_1 and ψ_2 are stably homotopic if, and only if,*

$$\mathrm{KK}(\psi_1) = \mathrm{KK}(\psi_2).$$

In addition, the homomorphism η we add can be always chosen to be a homomorphism with finite dimensional image.

This paper is organized as follows. In §2 below, we collect some basic notions and tools which we will use later. In §3, we show that each homomorphism between two Elliott-Thomsen algebras is homotopic to a homomorphism with m-standard form for some positive integer m. In §4, we prove our main result, and conclude with the consequence that every unital one-dimensional NCCW complex has property (H).

2. Preliminaries

We first list some basic notions and tools we will use later, which also help us to understand the examples more clearly.

2.1 ([10, §3]). For $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathscr{C}$ with $\mathrm{K}_0(F_1) = \mathbb{Z}^p$ and $K_0(F_2) = \mathbb{Z}^\ell$, consider the short exact sequence

$$0 \to SF_2 \xrightarrow{\iota} A \xrightarrow{\pi} F_1 \to 0,$$

where $SF_2 = C_0(0,1) \otimes F_2$ is the suspension of F_2, ι is the embedding map, and $\pi(f,a) = a$, $(f,a) \in A$. Then one has the (degenerate) six-term exact sequence

$$0 \to \mathrm{K}_0(A) \xrightarrow{\pi_*} \mathrm{K}_0(F_1) \xrightarrow{\partial} \mathrm{K}_0(F_2) \xrightarrow{\iota_*} \mathrm{K}_1(A) \to 0,$$

where $\partial = \alpha - \beta$, and α, β are the matrices (with entries $\alpha_{ij}, \beta_{ij} \in \mathbb{N}$, $i = 1, \dots, \ell$, $j = 1, \dots, p$) corresponding to the maps $\mathrm{K}_0(\varphi_0), \mathrm{K}_0(\varphi_1) \colon \mathrm{K}_0(F_1) \to K_0(F_2)$, respectively. Hence,

$$\mathrm{K}_0(A) = \ker(\alpha - \beta) \subset \mathbb{Z}^p, \quad \mathrm{K}_1(A) = \mathbb{Z}^\ell / \mathrm{Im}(\alpha - \beta),$$

and

$$\mathrm{K}_0^+(A) = \ker(\alpha - \beta) \cap \mathrm{K}_0^+(F_1).$$

2.2. Throughout this paper, when talking about $\mathrm{KK}(A, B)$ with $A, B \in \mathscr{C}$, we shall assume the notational convention that

$$A = A(F_1, F_2, \varphi_0, \varphi_1), \quad B = B(F_1', F_2', \varphi_0', \varphi_1'),$$

with

$$F_1 = \bigoplus_{i=1}^{p} M_{k_i}(\mathbb{C}), \quad F_2 = \bigoplus_{j=1}^{\ell} M_{h_j}(\mathbb{C}),$$

and

$$F_1' = \bigoplus_{i'=1}^{p'} M_{k'_{i'}}(\mathbb{C}), \quad F_2' = \bigoplus_{j'=1}^{\ell'} M_{h'_{j'}}(\mathbb{C}).$$

And we will use α, β, α', and β' to denote the matrices induced by φ_{0*}, φ_{1*}, φ'_{0*} and φ'_{1*}, respectively.

The following lemma comes from Remarks 2.1 and 2.2 in [1].

LEMMA 2.3. *Let $A, B \in \mathscr{C}$ and let $\phi\colon A \to B$ be a homomorphism. Then ϕ is homotopic to a new homomorphism $\psi\colon A \to B$ with $Sp(\pi_e \circ \psi) \subset Sp(F_1)$, where π_e is the natural map $\pi_e\colon B \to F'_2$.*

2.4 ([1]). Let $A, B \in \mathscr{C}$. Denote by $C(A, B)$ the set of all the commutative diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \lambda_{0*}} & & \Big\downarrow{\scriptstyle \lambda_0} & & \Big\downarrow{\scriptstyle \lambda_1} & & \Big\downarrow{\scriptstyle \lambda_{1*}} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0, \end{array}$$

and by $M(A, B)$ the subset of $C(A, B)$ of all the commutative diagrams

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \mu_0} & & \Big\downarrow{\scriptstyle \mu_1} & & \Big\downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0 \end{array}$$

such that there exists $\mu \in \mathrm{Hom}(\mathrm{K}_1(SF_2), \mathrm{K}_0(F'_1))$ satisfying $\mu_0 = \mu\circ(\alpha-\beta)$, $\mu_1 = (\alpha' - \beta') \circ \mu$. Since such a diagram is completely determined by μ, and so we may denote it by λ_μ.

2.5. For two commutative diagrams $\lambda_I, \lambda_{II} \in C(A, B)$,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \lambda_{I0*}} & & \Big\downarrow{\scriptstyle \lambda_{I0}} & & \Big\downarrow{\scriptstyle \lambda_{I1}} & & \Big\downarrow{\scriptstyle \lambda_{I1*}} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0, \end{array}$$

and

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \lambda_{II0*}} & & \Big\downarrow{\scriptstyle \lambda_{II0}} & & \Big\downarrow{\scriptstyle \lambda_{II1}} & & \Big\downarrow{\scriptstyle \lambda_{II1*}} & & \\ 0 & \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow & 0, \end{array}$$

define the sum of λ_I and λ_{II} as

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "\pi_*"] \arrow[d, "\lambda_{I0*}+\lambda_{II0*}"'] & \mathrm{K}_0(F_1) \arrow[r, "\alpha-\beta"] \arrow[d, "\lambda_{I0}+\lambda_{II0}"'] & \mathrm{K}_1(SF_2) \arrow[r, "\iota_*"] \arrow[d, "\lambda_{I1}+\lambda_{II1}"'] & \mathrm{K}_1(A) \arrow[r] \arrow[d, "\lambda_{I1*}+\lambda_{II1*}"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "\pi'_*"'] & \mathrm{K}_0(F'_1) \arrow[r, "\alpha'-\beta'"'] & \mathrm{K}_1(SF'_2) \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0.
\end{tikzcd}$$

Note that $\lambda_I + \lambda_{II} \in C(A, B)$. And we point out that there exist a zero element and inverse elements for the addition as one would expect.

Then $C(A, B)$ is an Abelian group, and $M(A, B)$ is a subgroup of $C(A, B)$.

THEOREM 2.6 (Theorem 2.13 of [1]). *Let $A, B \in \mathscr{C}$. Then we have a natural isomorphism of groups*

$$\mathrm{KK}(A, B) \cong C(A, B)/M(A, B).$$

LEMMA 2.7 (Lemma 2.5 of [1]). *Let $A, B \in \mathscr{C}$ be minimal. Let λ be a commutative diagram,*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "\pi_*"] \arrow[d, "\lambda_{0*}"'] & \mathrm{K}_0(F_1) \arrow[r, "\alpha-\beta"] \arrow[d, "\lambda_0"'] & \mathrm{K}_1(SF_2) \arrow[r, "\iota_*"] \arrow[d, "\lambda_1"'] & \mathrm{K}_1(A) \arrow[r] \arrow[d, "\lambda_{1*}"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "\pi'_*"'] & \mathrm{K}_0(F'_1) \arrow[r, "\alpha'-\beta'"'] & \mathrm{K}_1(SF'_2) \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0,
\end{tikzcd}$$

where the map λ_0 is positive. Let

$$\tau = \bigoplus_{j'=1}^{\ell'} \tau_{j'} \colon A \to C([0,1], F'_2)$$

be a homomorphism such that $Sp(\pi'_0 \circ \tau)$, $Sp(\pi'_1 \circ \tau) \subset Sp(F_1)$ and

$$\mathrm{K}_0(\pi'_0 \circ \tau) = \alpha' \circ \lambda_0 \quad \textit{and} \quad \mathrm{K}_0(\pi'_1 \circ \tau) = \beta' \circ \lambda_0,$$

where π'_0 and π'_1 are the point evaluations at 0 *and* 1*, respectively. Then there exists a unitary $u \in C([0,1], F_2)$ such that $\mathrm{Ad}\, u \circ \tau$ gives a homomorphism from A to B.*

3. Perturbation and standard form

DEFINITION 3.1. If $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathscr{C}$, we shall say that a homomorphism $\phi \colon A \to M_r(C[0,1])$ has *standard form* if:

(i) ϕ has the expression

$$\phi_t(f) = U(t)^* \begin{pmatrix} f(s_1(t)) & & & \\ & f(s_2(t)) & & \\ & & \ddots & \\ & & & f(s_k(t)) \end{pmatrix} U(t), \quad \forall f \in A, \quad (*)$$

where $U \in M_r(C[0,1])$ and $\{s_i(t)\}_{i=1}^k \subset C([0,1], Sp(F_1)) \cup C([0,1], Sp(SF_2))$;

(ii) each of $\{s_i(t)\}_{i=1}^k$ has one of the following basic forms:

$$\{\theta_1, \ldots, \theta_p, (t,1), \ldots, (t,\ell), (1-t,1), \ldots, (1-t,\ell)\},$$

where θ_j is the jth point in $Sp(F_1)$ and (t,m) denotes the point t on the mth copy of the interval (0, 1) in $Sp(SF_2)$.

DEFINITION 3.2. Let $n \in \mathbb{N}^+$, $A \in \mathscr{C}$ with $Sp(A) = Sp(F_1) \cup \bigsqcup_{i=1}^{\ell} (0,1)_i$. Set $I_i^0 = \left(0, \frac{1}{n}\right]_i, \ldots, I_i^r = \left[\frac{r-1}{n}, \frac{r}{n}\right]_i, \ldots, I_i^{n-1} = \left[\frac{n-1}{n}, 1\right)_i$ for $i = 1, 2, \ldots, \ell$. Then $Sp(A) = Sp(F_1) \cup \bigsqcup_{i=1}^{\ell} \bigcup_{r=1}^{n} I_i^r$. We shall call this an *n-partition* of $Sp(A)$ and refer to $I_1^1, \ldots, I_i^r, \ldots, I_\ell^n$ as dividing intervals.

DEFINITION 3.3. Let $A, B \in \mathscr{C}$ and let ϕ be a homomorphism from A to B. We shall say ϕ has *n-standard form* if after identifying each dividing interval of $Sp(B)$ with [0, 1], ϕ has standard form on each dividing interval.

EXAMPLE 3.4. Choose A as in Example 1.3. Define $\phi\colon A \to M_3(C[0,1])$ as follows:

$$\phi(f, a \oplus b \oplus c) = \begin{cases} \begin{pmatrix} a & \\ & f(2t) \end{pmatrix}, & \text{if } 0 \le t \le \frac{1}{2}; \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f(2t-1) & \\ & b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then ϕ has 2-standard form. Note that the relative unitary $U(t)$ is not continuous at 1/2.

$$U(t) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{if } 0 \le t \le \frac{1}{2}; \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

In fact, for any 2-standard form of ϕ, the relative unitary $V(t)$ can't be continuous at $1/2$. Otherwise, we will have

$$V\left(\frac{1}{2}\right)\phi_- V\left(\frac{1}{2}\right)^* = V\left(\frac{1}{2}\right)\phi_+ V\left(\frac{1}{2}\right)^*$$

implies

$$\phi_- = \phi_+,$$

where $\phi_- = \operatorname{diag}\{a, f(1)\}$ or $\operatorname{diag}\{f(1), a\}$ and $\phi_+ = \operatorname{diag}\{b, f(0)\}$ or $\operatorname{diag}\{f(0), b\}$. This is impossible.

The following result is essentially contained in Lemma 3.5 of [11].

LEMMA 3.5. *Let $A \in \mathscr{C}$, $F \subset A$ be a finite subset, and $\delta > 0$. There exist a finite subset G and $\varepsilon > 0$ such that if $\psi, \varphi: A \to M_r(\mathbb{C})$ are homomorphisms, for some integer r, with*

$$\|\psi(g) - \varphi(g)\| < \varepsilon, \quad \forall\, g \in G,$$

then there is a homomorphism $\phi: A \to M_r(C[0,1])$ such that $\phi_0 = \psi$, $\phi_1 = \varphi$, and

$$\|\phi_t(f) - \psi(f)\| < \delta, \quad \forall\, f \in F,\ t \in [0,1].$$

Moreover, ϕ may be chosen to be homotopic to a homomorphism with 3-standard form.

PROOF. The first part is included in [11, Lemma 3.5] and the homomorphism ϕ has the expression $(*)$ (condition (i) in Definition 3.1) on $\left[0, \frac{1}{3}\right]$, $\left[\frac{1}{3}, \frac{2}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. For the second part, consider the interval $\left[0, \frac{1}{3}\right]$; after identifying $\left[0, \frac{1}{3}\right]$ with $[0,1]$, for each k, we require that $s_k(t)$ satisfy one of the following conditions:

(1) $s_k(t) \in \{\theta_1, \ldots, \theta_p\}$;

(2) $\operatorname{Im} s_k(t) \subset (0,1)_{i_k}$ for some $i_k \in \{1, 2, \ldots, \ell\}$;

(3) $s_k(t) \in (0,1)_{i_k}$, $(0 \le t < 1)$ and $s_k(1) = 0_{i_k}$ for some i_k;

(4) $s_k(t) \in (0,1)_{i_k}$, $(0 \le t < 1)$ and $s_k(1) = 1_{i_k}$ for some i_k;

(5) $s_k(t) \in (0,1)_{i_k}$, $(0 < t \le 1)$ and $s_k(0) = 0_{i_k}$ for some i_k;

(6) $s_k(t) \in (0,1)_{i_k}$, $(0 < t \le 1)$ and $s_k(0) = 1_{i_k}$ for some i_k;

(7) $s_k(t) \in (0,1)_{i_k}$, $(0 < t < 1)$ and $s_k(0) = s_k(1) = 0_{i_k}$ for some i_k;

(8) $s_k(t) \in (0,1)_{i_k}$, $(0 < t < 1)$ and $s_k(0) = s_k(1) = 1_{i_k}$ for some i_k;

(9) $s_k(t) \in (0,1)_{i_k}$, $(0 < t < 1)$ and $s_k(0) = 0_{i_k}$, $s_k(1) = 1_{i_k}$ for some i_k;

(10) $s_k(t) \in (0,1)_{i_k}$, $(0 < t < 1)$ and $s_k(0) = 1_{i_k}$, $s_k(1) = 0_{i_k}$ for some i_k.

Then we use the following basic techniques:

(a) if $s_k(t)$ satisfies (2), (3), (5) or (7), then $s_k(t)$ is homotopic to $\widetilde{s_k}(t) = 0_{i_k}$;

(b) if $s_k(t)$ satisfies (4), (6) or (8), then $s_k(t)$ is homotopic to $\widetilde{s_k}(t) = 1_{i_k}$;

(c) if $s_k(t)$ satisfies (9), then $s_k(t)$ is homotopic to $\widetilde{s_k}(t) = (t, i_k)$;

(d) if $s_k(t)$ satisfies (10), then $s_k(t)$ is homotopic to $\widetilde{s_k}(t) = (1-t, i_k)$.

Now we get a new homomorphism $\phi^{(1)}$ which has standard form on $\left[0, \frac{1}{3}\right]$. Note that after the homotopy step on the interval $\left[0, \frac{1}{3}\right]$, $\phi^{(1)}$ still has the expression $(*)$ on the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$. Then for each k, $s_k(t)$ still satisfies one of the ten conditions above, so we repeat the same techniques for $\left[\frac{1}{3}, \frac{2}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. In this way, we obtain a homomorphism $\phi^{(3)}$ with 3-standard form. This shows that ϕ is homotopic to a homomorphism of 3-standard form.

The following lemma is [3, Corollary 4.3].

LEMMA 3.6. *Let A be a semiprojective* C^**-algebra generated by a finite or countable set $\mathscr{G} = \{x_1, x_2, \ldots\}$ with $\lim_{j\to\infty} \|x_j\| = 0$ if $\mathscr{G}$ is infinite. Then there is a $\delta > 0$ such that, whenever B is a* C^**-algebra, and ϕ_0 and ϕ_1 are homomorphisms from A to B with $\|\phi_0(x_j) - \phi_1(x_j)\| < \delta$ for all j, then ϕ_0 and ϕ_1 are homotopic.* (*The number δ depends on A and the set $\mathscr{G}$ of generators, but not on B or the maps ϕ_0, ϕ_1.*)

THEOREM 3.7. *Let $A, B \in \mathscr{C}$. Then any homomorphism from A to B is homotopic to a homomorphism of m-standard form for some m.*

PROOF. Note that A is semiprojective and finitely generated by [6]. With $\mathscr{G}$ a finite set generating A, and with $\delta > 0$ as in Lemma 3.6, apply Lemma 3.5 for $A, \mathscr{G}, \delta/2$ and $M_k(\mathbb{C})$, with k large enough that $B \subset M_k(C[0,1])$, to obtain a finite subset $G \subset A$ and $\varepsilon > 0$.

Let $\phi: A \to B$ be a homomorphism. By Lemma 2.3, we may assume that $Sp(\pi_e \circ \phi) \subset Sp(F_1)$. Since $G \cup \mathscr{G}$ is finite, then there exist $n \in \mathbb{N}^+$ and an n-partition of $Sp(B)$, such that for any x, y in the same dividing interval, we have

$$\|\phi_x(g) - \phi_y(g)\| < \min\left\{\varepsilon, \frac{\delta}{2}\right\}, \quad \forall\, g \in G \cup \mathscr{G}.$$

Consider each dividing interval $I_i^r \subset Sp(B)$, and let $x = (r/n, i)$, $y = ((r+1)/n, i)$. By Lemma 3.5, there exists a homomorphism $\psi|_{I_i^r}: A \to M_k(C(I_i^r))$ such that

$$\psi_{(r/n,i)} = \phi_{(r/n,i)}, \quad \psi_{((r+1)/n,i)} = \phi_{((r+1)/n,i)}$$

and

$$\|\psi_t(g) - \phi_{(r/n,i)}(g)\| < \frac{\delta}{2}, \quad \forall\, g \in \mathscr{G},\ t \in I_i^r.$$

The $\psi|_{I_i^r}$ fit together to define a single homomorphism $\psi\colon A \to B$ and $Sp(\pi_e \circ \psi) \subset Sp(F_1)$.

It is clear that

$$\|\phi|_{I_i^r}(g) - \psi|_{I_i^r}(g)\| < \delta, \quad \forall g \in \mathcal{G},$$

and it follows that

$$\|\phi(g) - \psi(g)\| < \delta, \quad \forall g \in \mathcal{G}.$$

By the conclusion of Lemma 3.6, ϕ is homotopic to ψ.

Consider the $3n$-partition of $Sp(B)$; ψ has the expression $(*)$ on each dividing interval $J_1^1, \dots, J_i^r, \dots, J_\ell^{3n}$. Beginning with J_1^1, using the basic homotopic techniques in Lemma 3.5, ψ is homotopic to a homomorphism $\psi^{(1)}$ which has standard form on J_1^1. Step by step, we can obtain that ψ is homotopic to $\psi^{(3n)}$ and $\psi^{(3n)}$ has $3n$-standard form on all the dividing intervals. Then ϕ is homotopic to a homomorphism of $3n$-standard form.

4. Stable homotopy

At the beginning of this section, we list one more example and write down some homotopy paths, the idea of which also works for Example 1.4.

Example 4.1. Suppose that $A = C[0,1]$, $B = C(S^1)$, and ψ is a 2-standard form homomorphism from A to B:

$$\psi(f) = \begin{cases} f(2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ f(2-2t), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Denote by η the 1-standard form homomorphism from A to B defined by

$$\eta(f) = f(1).$$

Note that the homomorphism $\psi \oplus \eta$ is homotopic to a 3-standard form homomorphism ρ,

$$\rho(f) = \begin{cases} \begin{pmatrix} f(3t) & \\ & f(1) \end{pmatrix}, & \text{if } 0 \le t \le \frac{1}{3}, \\ \begin{pmatrix} f(1) & \\ & f(1) \end{pmatrix}, & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \begin{pmatrix} f(3-3t) & \\ & f(1) \end{pmatrix}, & \text{if } \frac{2}{3} \le t \le 1, \end{cases}$$

which can also be written as

$$\rho(f) = \begin{cases} v(t)\begin{pmatrix} f(3t) & \\ & f(1) \end{pmatrix} v(t)^*, & \text{if } 0 \le t \le \frac{1}{3}, \\ v(t)\begin{pmatrix} f(1) & \\ & f(1) \end{pmatrix} v(t)^*, & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ v(t)\begin{pmatrix} f(1) & \\ & f(3-3t) \end{pmatrix} v(t)^*, & \text{if } \frac{2}{3} \le t \le 1, \end{cases}$$

where

$$v(t) = \begin{cases} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, & \text{if } 0 \le t \le \frac{1}{3}, \\ \begin{pmatrix} \cos\left(\frac{\pi}{2}\cdot(3t-1)\right) & \sin\left(\frac{\pi}{2}\cdot(3t-1)\right) \\ -\sin\left(\frac{\pi}{2}\cdot(3t-1)\right) & \cos\left(\frac{\pi}{2}\cdot(3t-1)\right) \end{pmatrix}, & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

From this we can see that ρ is homotopic to the 1-standard form homomorphism ρ',

$$\rho'(f) = v(t)\begin{pmatrix} f(t) & \\ & f(1-t) \end{pmatrix} v(t)^*.$$

The construction above can be generalized as follows:

REMARK 4.2. Suppose that $A, B \in \mathscr{C}$, $\psi\colon A \to B$ is a homomorphism, and ι' is the natural embedding map from B to $F_2' \otimes C[0,1]$.

For $t_0 \in (0,1)$, denote by η_{t_0} the homomorphism from A to $F_2' \otimes C[0,1]$ defined by

$$\eta_{t_0}(f) = \psi(f)(t_0) \in F_2' \otimes C[0,1], \quad \forall f \in A.$$

Note that $\iota' \circ \psi \sim_h \rho$, where

$$\rho(f)(t) = \begin{cases} \psi(f)(3t_0 \cdot t), & \text{if } 0 \le t \le \frac{1}{3}, \\ \psi(f)(t_0), & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \psi(f)(3(1-t_0)t + 3t_0 - 2), & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

Also,

$$\rho \oplus \eta_0 = V(t)\rho' V(t)^*,$$

where

$$\rho'(f)(t) = \begin{cases} \begin{pmatrix} \psi(f)(3t_0 \cdot t) & \\ & \psi(f)(t_0) \end{pmatrix}, & \text{if } 0 \le t \le \frac{1}{3}, \\ \begin{pmatrix} \psi(f)(t_0) & \\ & \psi(f)(t_0) \end{pmatrix}, & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \begin{pmatrix} \psi(f)(t_0) & \\ & \psi(f)(3(1-t_0)t + 3t_0 - 2) \end{pmatrix}, & \text{if } \frac{2}{3} \le t \le 1, \end{cases}$$

$$V(t) = \begin{cases} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \otimes 1_{F_2' \otimes C[0,1]}, & \text{if } 0 \le t \le \frac{1}{3}, \\ \begin{pmatrix} \cos\left(\frac{\pi}{2} \cdot (3t-1)\right) & \sin\left(\frac{\pi}{2} \cdot (3t-1)\right) \\ -\sin\left(\frac{\pi}{2} \cdot (3t-1)\right) & \cos\left(\frac{\pi}{2} \cdot (3t-1)\right) \end{pmatrix} & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \quad \otimes 1_{F_2' \otimes C[0,1]}, & \\ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \otimes 1_{F_2' \otimes C[0,1]}, & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

We also have $\rho' \sim_h \rho''$, where

$$\rho''(f)(t) = \begin{pmatrix} \psi(f)\left(\frac{t}{t_0}\right) & \\ & \psi(f)\left(\frac{t-t_0}{1-t_0}\right) \end{pmatrix}.$$

That is, we have

$$\phi \oplus \eta_0 \sim_h V(t)\rho'' V(t)^*.$$

Furthermore, the homotopy path H_θ with $\theta \in [0, 1]$, $H_0 = \phi \oplus \eta_0$ and $H_1 = V(t)\rho'' V(t)^*$, can be chosen to satisfy

$$\pi_0' \circ H_\theta(f) = \begin{pmatrix} \psi(f)(0) & \\ & \psi(f)(t_0) \end{pmatrix}$$

and

$$\pi_1' \circ H_\theta(f) = \begin{pmatrix} \psi(f)(1) & \\ & \psi(f)(t_0) \end{pmatrix},$$

where π_0', π_1' are the point evaluations of $M_2(F_2 \otimes C[0, 1])$ at 0 and 1.

As the homomorphism $\eta_{t_0} \otimes 1_B$ has the form

$$u\left(\bigoplus^r \eta_{t_0}\right)u^*, \quad \text{for some } u \in C([0, 1], M_r(F_2')),$$

where $r = \sum_{j'=1}^{\ell'} h'_{j'}$, for the homomorphism

$$\psi \oplus (\eta_{t_0} \otimes 1_B) = (1_B \oplus u)\left(\psi \oplus \bigoplus^{r} \eta_{t_0}\right)(1_B \oplus u^*),$$

we can apply the same construction to $\psi \oplus \bigoplus^r \eta_{t_0}$ by regarding the first copy of η_{t_0} in the diagonal as η_{t_0}. We will then have a homotopy path from A to $M_r(B)$.

The idea of the following lemma comes from Example 1.4, Example 4.1, and Remark 4.2.

LEMMA 4.3. *Let $A, B \in \mathscr{C}$. Given any m-standard form homomorphism ψ from A to B, there exists a homomorphism η from A to $M_r(B)$, for some integer r, such that $\psi \oplus \eta$ is homotopic to a homomorphism with 1-standard form.*

PROOF. Consider the homomorphism

$$\eta := \bigoplus_{k=1}^{m-1} \psi_{k/m} \otimes 1_B, \quad A \to M_r(B),$$

where $r = (m-1)\sum_{j'=1}^{\ell'} h'_{j'}$, and $\psi_{k/m}\colon A \to F'_2$ is the point evaluation of B at k/m composed with ψ. Applying the trick we describe in Remark 4.2 $m-1$ times to the points $\{k/m\}_{k=1}^{m-1}$, we obtain that $\psi \oplus \eta$ is homotopic to a homomorphism of 1-standard form.

Recall that for an m-standard form, the unitary is just piecewise continuous (Example 3.4). But when we consider the 1-standard form, the unitary is a continuous element in $F'_2 \otimes C[0, 1]$.

We list the basic homotopy lemma from functional calculus:

PROPOSITION 4.4. *Let $D \subset M_n(\mathbb{C})$ be a sub C*-algebra. Let v be a unitary in $M_n(\mathbb{C})$ with*

$$vdv^* = d, \quad \text{for all } d \in D.$$

Then there exists a unitary path V_t, $t \in [0, 1]$, in $M_n(\mathbb{C})$, with $V_0 = v$, $V_1 = 1_n$, inducing the relation

$$v \sim_h 1_n,$$

such that

$$V_t d V_t^* = d, \quad \text{for all } d \in D \text{ and } t \in [0, 1].$$

PROOF. The commutant of D in $M_n(\mathbb{C})$ is a von Neumann algebra containing v and I_n, which is isomorphic to a direct sum of matrix algebras. We denote it by D', and then we can pick the homotopy path $V_t \in U(D') \subset U(M_n(\mathbb{C}))$, $t \in [0, 1]$, satisfying our goal.

Suggested by Example 1.5, we have

LEMMA 4.5. *Let $A, B \in \mathscr{C}$, ψ_1, ψ_2 be two 1-standard form homomorphisms from A to B, inducing the same $\lambda \in C(A, B)$,*

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{K}_0(A) & \xrightarrow{\pi_*} & \mathrm{K}_0(F_1) & \xrightarrow{\alpha-\beta} & \mathrm{K}_1(SF_2) & \xrightarrow{\iota_*} & \mathrm{K}_1(A) & \longrightarrow 0 \\
& \Big\downarrow{\scriptstyle \lambda_{0*}} & & \Big\downarrow{\scriptstyle \lambda_0} & & \Big\downarrow{\scriptstyle \lambda_1} & & \Big\downarrow{\scriptstyle \lambda_{1*}} & \\
0 \longrightarrow & \mathrm{K}_0(B) & \xrightarrow[\pi'_*]{} & \mathrm{K}_0(F'_1) & \xrightarrow[\alpha'-\beta']{} & \mathrm{K}_1(SF'_2) & \xrightarrow[\iota'_*]{} & \mathrm{K}_1(B) & \longrightarrow 0,
\end{array}$$

Then there is a homomorphism ζ with finite dimensional image from A to $M_r(B)$ such that

$$\psi_1 \oplus \zeta \sim_h \psi_2 \oplus \zeta.$$

PROOF. As the 1-standard form homomorphisms ψ_1, ψ_2 induce the same diagram, we have that for each j'th block of B, the differences of the multiplicities of $f_j(t)$ and $f_j(1-t)$ in ψ_1, ψ_2 are the same, which is equal to the (j', j)th entry of λ_1. On one hand, from Example 4.1, we know that

$$\rho' \sim_h \psi \oplus \eta \sim_h \xi,$$

where

$$\xi(f) = \begin{pmatrix} f(0) & \\ & f(1) \end{pmatrix}.$$

On the other hand, using the same trick as in Example 4.1 and Remark 4.2, we have

$$\psi_1 \sim_h \widetilde{\psi}_1 = u(t)\chi u(t)^* \quad \text{for some } u(t) \in U(F'_2 \otimes C[0, 1])$$

and

$$\psi_2 \sim_h \widetilde{\psi}_2 = v(t)\chi v(t)^* \quad \text{for some } v(t) \in U(F'_2 \otimes C[0, 1]).$$

(Here, we can require both $u(t)\chi u(t)^*$ and $v(t)\chi v(t)^*$ to have 1-standard form with χ satisfying that at each j'th block of B, at least one of the multiplicities of $f_j(t)$ and $f_j(1-t)$ is 0.)

We hope that $u(t)v(t)^*$ is a unitary in B. This is not always true, but we claim that $u(t)v(t)^*$ is homotopic to a unitary in B. This is because there exist

$\zeta_1, \zeta_2\colon F_1 \to F_1'$ such that we have the following commutative diagrams:

$$\begin{tikzcd} A \arrow[r, "u(t)\chi u(t)^*"] \arrow[dr, "\pi"'] & B \arrow[r, "\pi'"] & F_1' \\ & F_1 \arrow[ur, "\zeta_1"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "v(t)\chi v(t)^*"] \arrow[dr, "\pi"'] & B \arrow[r, "\pi'"] & F_1' \\ & F_1 \arrow[ur, "\zeta_2"'] & \end{tikzcd} .$$

As ζ_1, ζ_2 induce the same map λ_0 between $\mathrm{K}_0(F_1)$ and $\mathrm{K}_0(F_1')$, there exists a unitary $W \in F_1'$ such that

$$W\zeta_2 W^* = \zeta_1,$$
$$\varphi_0'(W)(\pi_0 \circ \widetilde{\psi}_2)\varphi_0'(W)^* = u(0)v(0)^*(\pi_0 \circ \widetilde{\psi}_2)(u(0)v(0)^*)^*, \quad \text{and}$$
$$\varphi_1'(W)(\pi_1 \circ \widetilde{\psi}_2)\varphi_1'(W)^* = u(1)v(1)^*(\pi_1 \circ \widetilde{\psi}_2)(u(1)v(1)^*)^*.$$

Applying Proposition 4.4, we have

$$\varphi_0'(W)^*u(0)v(0)^* \sim_h 1_{F_2'}, \quad \text{by } V_s,\ s \in [0,1],$$

and

$$\varphi_1'(W)^*u(1)v(1)^* \sim_h 1_{F_2'}, \quad \text{by } \widetilde{V}_s,\ s \in [0,1].$$

Note that

$$u(t)v^*(t) \sim_h w(t),$$

where

$$w(t) = \begin{cases} \varphi_0'(W) \cdot V_{(1-3t)}, & \text{if } 0 \le t \le \frac{1}{3}, \\ u(3t-1)v^*(3t-1), & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \varphi_1'(W) \cdot V_{(3t-2)}, & \text{if } \frac{2}{3} \le t \le 1, \end{cases} \quad \in U(B),$$

and this gives

$$\widetilde{\psi}_1 = u(t)v(t)^*\widetilde{\psi}_2 v(t)u(t)^* \sim_h w(t)\widetilde{\psi}_2 w(t)^*.$$

Using the trick from Example 1.5, we deduce that

$$(w(t)\widetilde{\psi}_2 w(t)^*) \oplus (w(t)^* 0 w(t)) \sim_h \widetilde{\psi}_2 \oplus (w(t)^* 0 w(t)).$$

Example 1.3 shows that two homomorphisms with the same KK-class may induce different diagrams in $C(A, B)$, and in that example we need to add enough point evaluations and push the spectrum point from 0 to 1. Suggested by this example, we will need:

LEMMA 4.6. *Let ψ_1, ψ_2 two 1-standard form homomorphisms from A to B inducing the same* KK*-class. Then there is a homomorphism ζ with finite dimensional image from A to $M_r(B)$, for some integer r such that*

$$\psi_1 \oplus \zeta \sim_h \psi_2 \oplus \zeta.$$

PROOF. ψ_1, ψ_2 are two 1-standard form homomorphisms inducing $\lambda_1, \lambda_2 \in C(A, B)$. If $\lambda_1 = \lambda_2$, then by Lemma 4.5, the proof is finished.

As $\mathrm{KK}(\psi_1) = \mathrm{KK}(\psi_2)$, if $\lambda_1 \neq \lambda_2$, by Theorem 2.6, then there exists a map $\mu\colon \mathrm{K}_1(SF_2) \to \mathrm{K}_0(F_1')$ inducing an element $\lambda_\mu \in M(A, B)$ such that

$$\lambda_1 + \lambda_\mu = \lambda_2.$$

So, as suggested by Lemma 4.5, we wish to show that there exists a homomorphism η, with finite dimensional image and inducing $\xi \in C(A, B)$, such that $\psi_1 \oplus \eta$ is homotopic to a 1-standard form homomorphism inducing the diagram $\lambda_2 + \xi$, which is also the diagram induced by $\psi_2 \oplus \eta$. And we only need to show that this is true when μ is any matrix unit, since the stably homotopic relation is an equivalence relation. (μ is a finite sum of matrix units.)

We shall deal with the case $\mu = e_{11}$; the other cases are similar. Then, $\lambda_\mu \in M(A, B)$ is the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "\pi_*"] \arrow[d, "0"'] & \mathrm{K}_0(F_1) \arrow[r, "\alpha-\beta"] \arrow[d, "e_{11}(\alpha-\beta)"'] & \mathrm{K}_1(SF_2) \arrow[r, "\iota_*"] \arrow[d, "(\alpha'-\beta')e_{11}"'] & \mathrm{K}_1(A) \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "\pi'_*"'] & \mathrm{K}_0(F_1') \arrow[r, "\alpha'-\beta'"'] & \mathrm{K}_1(SF_2') \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0.
\end{tikzcd}$$

Let κ be the following diagram in $C(A, B)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(A) \arrow[r, "\pi_*"] \arrow[d, "\kappa_{0*}"'] & \mathrm{K}_0(F_1) \arrow[r, "\alpha-\beta"] \arrow[d, "\kappa_0"'] & \mathrm{K}_1(SF_2) \arrow[r, "\iota_*"] \arrow[d, "0"'] & \mathrm{K}_1(A) \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & \mathrm{K}_0(B) \arrow[r, "\pi'_*"'] & \mathrm{K}_0(F_1') \arrow[r, "\alpha'-\beta'"'] & \mathrm{K}_1(SF_2') \arrow[r, "\iota'_*"'] & \mathrm{K}_1(B) \arrow[r] & 0,
\end{tikzcd}$$

where

$$\kappa_0 = \begin{pmatrix} k_1' & k_1' & \cdots & k_1' \\ k_2' & k_2' & \cdots & k_2' \\ \vdots & \vdots & \ddots & \vdots \\ k_p' & k_p' & \cdots & k_p' \end{pmatrix}_{p' \times p}.$$

Recall that $(k_1', k_2', \dots, k_p') = [1_B]$ is the class of unit in $\mathrm{K}_0(B)$.

There exists a large enough integer c ($ck_1' \geq \|e_{11}\beta\|_\infty$, and the following $\Delta_{j'i} \geq 0$, for all $j' = 1, 2, \ldots, \ell'$, $i = 1, 2, \ldots, p$) that we have the following 1-standard form homomorphism η_0 inducing $c\kappa + \lambda_\mu$.

For any $j' \in \{1, 2, \ldots, \ell'\}$, define a homomorphism from A to the algebra $M_r(M_{h'_{j'}}(C[0,1]))$ (for some integer r):

$$A \ni (f, a) \xmapsto{\phi_{j'}} g_{j'} \in M_r(M_{h'_{j'}}(C[0,1]))$$

with

$$\begin{aligned} g_{j'}(t) = \operatorname{diag}\{&\underbrace{f(t,1), f(t,1), \ldots, f(t,1)}_{\alpha_{j'1}}\} \\ &\oplus \operatorname{diag}\{\underbrace{f(1-t,1), f(1-t,1), \ldots, f(1-t,1)}_{\beta_{j'1}}\} \\ &\oplus \bigoplus_{i=1}^{p} \operatorname{diag}\{\underbrace{a(\theta_i), a(\theta_i), \ldots, a(\theta_i)}_{\Delta_{j'i}}\}, \end{aligned}$$

where

$$\Delta_{j'i} = (\alpha' \circ (e_{11}(\alpha - \beta) + c\kappa_0))_{j'i} - (\alpha_{11} \cdot \alpha'_{j'1} + \beta_{ji} \cdot \beta'_{j'1}).$$

Then, with

$$\phi = \bigoplus_{j'=1}^{\ell'} \phi_{j'},$$

by Lemma 2.7 we have a unitary u such that $\eta_0 = u\phi u^*$ is a homomorphism from A to $M_r(B)$ inducing the commutative diagram $c\kappa + \lambda_\mu$.

Furthermore, we can choose the above u which satisfies that uH_su^* is also a homomorphism for every $s \in [0,1]$ with

$$Sp(\theta_{i'} \circ H_s) \cap \bigsqcup_{j=1}^{\ell} (0,1)_j = \emptyset, \quad i' = 2, 3, \ldots, p', \ s \in (0,1),$$

where H_s is a homotopy path of homomorphisms from A to $M_r(F_2 \otimes C[0,1])$. Write

$$H_s(f) = \bigoplus_{j'=1}^{\ell'} \phi_{s,j'},$$

where

$$A \ni (f, a) \xmapsto{\phi_{s,j'}} g_{s,j'} \in M_r(M_{h'_{j'}}(C[0,1]))$$

with

$$g_{s,j'}(t) = \mathrm{diag}\{\underbrace{f(st,1), f(st,1), \dots, f(st,1)}_{\alpha_{j'1}}\}$$
$$\oplus\, \mathrm{diag}\{\underbrace{f(1-st,1), f(1-st,1), \dots, f(1-st,1)}_{\beta_{j'1}}\}$$
$$\oplus \bigoplus_{i=1}^{p} \mathrm{diag}\{\underbrace{a(\theta_i), a(\theta_i), \dots, a(\theta_i)}_{\Delta_{j'i}}\}.$$

Then, uH_su^* induces a homotopy

$$\eta_0 \sim_h \eta,$$

where $\eta = uH_1u^*$ is a homomorphism inducing the diagram $c\kappa$ with finite dimensional image. (One can imagine for comparison the simple example that the identity map from $C[0, 1]$ to $C[0, 1]$ is homotopic to the point evaluation at 1.)

Now we have

$$\psi_1 \oplus \eta \sim_h \psi_1 \oplus \eta_0,$$

and $\psi_1 \oplus \eta_0$ induces $\lambda_1 + c\kappa + \lambda_\mu = \lambda_2 + c\kappa$.

For the general $\mu = \mu_{i'j}e_{i'j}$, we repeat the above construction $\sum_{j=1}^{\ell}\sum_{i'=1}^{p'} |\mu_{i'j}|$ times. The conclusion of the lemma follows with ζ the sum of $\sum_{j=1}^{\ell}\sum_{i'=1}^{p'} |\mu_{i'j}|$ such homomorphisms with finite dimensional images.

Proof of Theorem 1.6. We only need to show that two homomorphisms ψ_1, ψ_2 inducing the same KK-class are stably homotopic. By Theorem 3.7, we have the two homomorphisms ψ_1, ψ_2 homotopic to ψ_1', ψ_2', which are of m_1- and m_2-standard form, respectively. By Lemma 4.3, there exists a homomorphism η_1 such that the homomorphisms

$$\psi_1' \oplus \eta_1, \psi_2' \oplus \eta_1$$

are homotopic to two 1-standard form homomorphisms ψ_1'', ψ_2'', respectively. And at last, by Lemma 4.6, there exists a homomorphism η_2 such that

$$\psi_1'' \oplus \eta_2 \sim_h \psi_2'' \oplus \eta_2.$$

That is,

$$\psi_1 \oplus \eta_1 \oplus \eta_2 \sim_h \psi_2 \oplus \eta_1 \oplus \eta_2,$$

as desired.

The property (H) of C*-algebras plays an important role in the classification theory. We shall use stable homotopy to prove that any unital one-dimensional NCCW complex has this property.

DEFINITION 4.7 ([4]). A C*-algebra is said to have *property* (*H*) if, for any finite $F \subset A$ and $\varepsilon > 0$, there exist a homomorphism $\rho\colon A \to M_{r-1}(A)$ and a homomorphism $\mu\colon A \to M_r(A)$ with finite dimensional image such that

$$\|f \oplus \rho(f) - \mu(f)\| < \varepsilon, \quad \forall f \in F.$$

THEOREM 4.8. *Every unital one-dimensional NCCW complex has property* (*H*).

PROOF. By Theorem 2.6, the map id: $A \to A$ induces the following commutative diagram, λ:

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r, "\pi_*"] \arrow[d, "\lambda_{0*}"'] & K_0(F_1) \arrow[r, "\alpha-\beta"] \arrow[d, "\lambda_0"'] & K_1(SF_2) \arrow[r, "\iota_*"] \arrow[d, "\lambda_1"'] & K_1(A) \arrow[r] \arrow[d, "\lambda_{1*}"'] & 0 \\ 0 \arrow[r] & K_0(A) \arrow[r, "\pi_*"'] & K_0(F_1) \arrow[r, "\alpha-\beta"'] & K_1(SF_2) \arrow[r, "\iota_*"'] & K_1(A) \arrow[r] & 0, \end{tikzcd}$$

where $\lambda_0 = I_{p\times p}$, $\lambda_1 = I_{\ell\times\ell}$.

Construct a commutative diagram $\lambda' \in C(A, A)$ as follows:

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r, "\pi_*"] \arrow[d, "\lambda'_{0*}"'] & K_0(F_1) \arrow[r, "\alpha-\beta"] \arrow[d, "\lambda'_0"'] & K_1(SF_2) \arrow[r, "\iota_*"] \arrow[d, "\lambda'_1"'] & K_1(A) \arrow[r] \arrow[d, "\lambda'_{1*}"'] & 0 \\ 0 \arrow[r] & K_0(A) \arrow[r, "\pi_*"'] & K_0(F_1) \arrow[r, "\alpha-\beta"'] & K_1(SF_2) \arrow[r, "\iota_*"'] & K_1(A) \arrow[r] & 0, \end{tikzcd}$$

with

$$\lambda'_0 = L \begin{pmatrix} k_1 & k_1 & \cdots & k_1 \\ k_2 & k_2 & \cdots & k_2 \\ \vdots & \vdots & \ddots & \vdots \\ k_p & k_p & \cdots & k_p \end{pmatrix} - \lambda_0 \quad \textit{and} \quad \lambda'_1 = -\lambda_1,$$

where $L \in \mathbb{N}$ is large enough. Consider the sum $\zeta = \lambda + \lambda'$:

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r, "\pi_*"] \arrow[d, "\lambda_{0*}+\lambda'_{0*}"'] & K_0(F_1) \arrow[r, "\alpha-\beta"] \arrow[d, "\lambda_0+\lambda'_0"'] & K_1(SF_2) \arrow[r, "\iota_*"] \arrow[d, "\lambda_1+\lambda'_1"'] & K_1(A) \arrow[r] \arrow[d, "\lambda_{1*}+\lambda'_{1*}"'] & 0 \\ 0 \arrow[r] & K_0(A) \arrow[r, "\pi_*"'] & K_0(F_1) \arrow[r, "\alpha-\beta"'] & K_1(SF_2) \arrow[r, "\iota_*"'] & K_1(A) \arrow[r] & 0, \end{tikzcd}$$

Then we can construct a homomorphism $\phi\colon A \to M_m(A)$ of 1-standard form inducing the diagram λ' for some m (just as in the construction in the proof of Lemma 4.6; see also [1, Lemma 3.6]) and a homomorphism ψ from A to $M_{m+1}(A)$ with finite dimensional image inducing the diagram ζ. Then by Theorem 2.6,

$$\mathrm{KK}(\mathrm{id} \oplus \phi) = \mathrm{KK}(\psi).$$

Hence by Theorem 1.6, there exists a homomorphism $\eta: A \to M_n(A)$ with finite dimensional image such that

$$\mathrm{id} \oplus \phi \oplus \eta \sim_h \psi \oplus \eta.$$

Set $s = m + n + 1$, $\sigma = \mathrm{id} \oplus \phi \oplus \eta$, $t = \sum_{j=1}^{\ell} h_j$, and for any $x \in (0, 1)$, set

$$\mu_x = \bigoplus_{j=1}^{\ell} \mathrm{diag}\{\underbrace{\sigma_{(x,j)}, \sigma_{(x,j)}, \ldots, \sigma_{(x,j)}}_{t}\}.$$

Then there exists a unitary u such that $u \cdot \mu_x \cdot u^*$ gives a homomorphism from A to $M_{st}(A)$.

Choose $x_0, x_1, x_2, \ldots$ to be a dense set in $(0, 1)$. Let D be a C^*-algebra constructed as an inductive limit $D = \lim(D_i, \nu_{i,j})$, where $D_i = M_{(s+st)^i}(A)$, $i \geq 0$ and $\nu_{i,i+1}(f) = \sigma(f) \oplus \mu_{x_i}(f)$. ($\sigma$ extends to $M_k(A)$ for any k. Here we still denote it by σ.) A similar construction can be found in [5, Lemma 6.1]. The perturbation coming from Remark 4.2 (by changing the pair $\frac{1}{3}$, $\frac{2}{3}$ into $\frac{1}{2} - \delta$, $\frac{1}{2} + \delta$, respectively, with small $\delta > 0$) shows that the inductive limit has real rank zero. Since the connecting maps always induce the zero map on K_1-groups, we have $\mathrm{K}_1(D) = 0$, and it follows from Theorem 3.1 of [7] that the inclusion $\nu_{0,\infty}$ of $D_0 = A$ into D can be approximated arbitrarily well by homomorphisms with finite dimensional range.

Then for any finite set $F \subset A$ and $\varepsilon > 0$, there exist i and a homomorphism $\mu: A \to D_i$ with finite dimensional range such that $\|\nu_{0,i}(f) - \mu(f)\| < \varepsilon$ for all $f \in F$. Since $\nu_{0,i}$ is of the form $\mathrm{id} \oplus \rho$ for some homomorphism ρ, this completes the proof.

Acknowledgements. The research of Qingnan An and Zhichao Liu was supported by the University of Toronto and NNSF of China (No.: 11920101001, 11531003), both the authors thank the Fields Institute and Hebei Normal University for their hospitality; the research of the second author was supported by the Natural Sciences and Engineering Research Council of Canada; the research of Yuanhang Zhang was partly supported by the Natural Science Foundation for Young Scientists of Jilin Province (No.: 20190103028JH) and NNSF of China (No.: 11601104, 11671167, 11201171) and the China Scholarship Council.

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SCHOOL OF MATHEMATICS AND STATISTICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN, JILIN, CHINA 130024
and
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
TORONTO, ONTARIO, CANADA M5S 2E4
E-mail: qingnanan1024@outlook.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
TORONTO, ONTARIO, CANADA M5S 2E4
E-mail: elliott@math.toronto.edu

SCHOOL OF MATHEMATICAL SCIENCES
DALIAN UNIVERSITY OF TECHNOLOGY
DALIAN, LIAONING, CHINA 116024
and
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
TORONTO, ONTARIO, CANADA M5S 2E4
E-mail: lzc.12@outlook.com

SCHOOL OF MATHEMATICS
JILIN UNIVERSITY
CHANGCHUN, JILIN, CHINA 130012
E-mail: zhangyuanhang@jlu.edu.cn