

THE BLOCK SCHUR PRODUCT IS A HADAMARD PRODUCT

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(Dedicated to the memory of Richard V. Kadison)

Abstract

Given two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ with entries in $B(H)$ for some Hilbert space H , their *block Schur product* is the $n \times n$ matrix $A \square B := (a_{ij}b_{ij})$. Given two continuous functions f and g on the torus with Fourier coefficients (f_n) and (g_n) their convolution product $f \star g$ has Fourier coefficients $(f_n g_n)$. Based on this, the *Schur product* on scalar matrices is also known as the *Hadamard product*.

We show that for a C^* -algebra \mathcal{A} , and a discrete group G with an action α_g of G on \mathcal{A} by $*$ -automorphisms, the reduced crossed product C^* -algebra $C_r^*(\mathcal{A}, \alpha, G)$ possesses a natural generalization of the convolution product, which we suggest should be named *the Hadamard product*.

We show that this product has a natural Stinespring representation and we lift some known results on block Schur products to this setting, but we also show that the block Schur product is a special case of the Hadamard product in a crossed product algebra.

1. Introduction

Based on an interest in properties of spectral triples, which is a basic object in Connes' *noncommutative geometry*, we have been studying the block Schur product – the entry-wise product on infinite matrices over $B(H)$ – for some time [3], [4], [5]. The leading expert in this field, Roger A. Horn has always used the term *Hadamard product* and he explains in [8, pp. 92–93] and [9, pp. 302–303] the reason why. I will quote Halmos' answer to Horn, when Horn asked Halmos, why he used the term *Hadamard product*. Halmos answered *because Johnny said so*. To those who may not know, I can tell, that P. R. Halmos was John von Neumann's assistant in Princeton. This is of course a perfectly good reason, but for a person coming from outside, this seemed a bit unfair towards Schur, who actually studied this product in details and published the fundamental article on this product on bilinear forms in [15]. Hadamard's product [7] is based on a product of complex functions represented by Laurent series, and it is obtained via element-wise products of the coefficients. Schur's

product is really based on a study of properties of matrices, so I found that in my studies, Schur's approach seemed to fit me best.

The present article studies the reduced crossed product $\mathcal{C} := C_r^*(\mathcal{A}, \alpha, G)$ of a C^* -algebra \mathcal{A} by an action α of a discrete group G , a construction you may find described in detail in [10, Definition 13.1.1]. In this version of the reduced crossed product, the C^* -algebra \mathcal{A} acts on a Hilbert space H , there is an injective $*$ -representation Ψ of \mathcal{A} on the Hilbert space $H \otimes \ell_2(G)$ and a unitary representation $g \rightarrow L_g$ of G on the same Hilbert space. The reduced crossed product \mathcal{C} is the C^* -algebra generated by all the operators $\Psi(A)$ and L_g . There exists a faithful conditional expectation π of \mathcal{C} onto $\Psi(\mathcal{A})$ which satisfies $\pi(L_g \Psi(A)) = \delta(e, g) \Psi(A)$. The mapping π makes it possible to describe elements in \mathcal{C} via generalized Fourier series over G with coefficients in $\Psi(\mathcal{A})$ such as

$$\mathcal{C} \ni X : \quad X \sim \sum_{g \in G} L_g X_g \text{ and } X_g := \pi(L_g^* X), \quad X_g \in \Psi(\mathcal{A}).$$

It is known, that this sum is convergent in a topology, which depends on π , and we will show in §2 that this convergence result may be viewed as a direct generalization of the classical Parseval identity. We study a product \star on \mathcal{C} which is usually called the convolution product and it is defined at the level of the generalized Fourier series via the formula

$$\forall X \sim \sum_{g \in G} L_g X_g, \quad Y \sim \sum_{g \in G} L_g Y_g : \quad X \star Y \sim \sum_{g \in G} L_g X_g Y_g.$$

In this article we show that this product has a natural *Stinespring representation* as a *completely bounded bilinear operator* in the sense of [6]. On the other hand the product is a straight forward generalization of the product Hadamard studied in [7]. Since the convolution operation is defined via an integral over the dual group, the convolution does not always exist in the setting of a general discrete group, so we think it is reasonable to call this product the Hadamard product. Later on we show that the block Schur product is a special case of this product. In the end, it turns out that the results we have obtained for block Schur products during the last couple of years [4], [5] do extend to this Hadamard product in a reduced crossed product C^* -algebra.

As mentioned above, we show that the Hadamard product – seen as a bilinear operator on $\mathcal{C} \times \mathcal{C}$ – has a natural Stinespring representation

$$X \star Y = V^* \rho(X) F \rho(Y) V,$$

where V is an isometry, ρ a representation of \mathcal{C} and F a self-adjoint unitary. It is worth remarking, that the conditional expectation π of \mathcal{C} onto $\Psi(\mathcal{A})$ is

given via the same representation ρ and the same isometry V , which are used above and the completely positive projection π has the form

$$\pi(X) = V^* \rho(X) V.$$

This decomposition of the bilinear operator \star has some similarities with the description of a bilinear form on \mathbb{C}^n via a scalar matrix, and in this analogy we can see that the Hadamard product is a symmetric operator. It is not positive, since F is a non trivial self-adjoint unitary, and then we get

$$\forall X \in \mathcal{C} : \quad -\pi(X^* X) \leq X^* \star X \leq \pi(X^* X).$$

Since the Stinespring decompositions of both the Hadamard product \star and the conditional expectation π are based on the same representation ρ and the same outer multipliers V^* and V , it is easy to obtain the following identity

$$\begin{aligned} \forall X, Y \in \mathcal{C} \exists S(X, Y) \in B(H \otimes \ell_2(G)), \quad \|S(X, Y)\| \leq 1 : \\ X \star Y = \pi(X X^*)^{1/2} S(X, Y) \pi(Y^* Y)^{1/2}, \end{aligned}$$

and from here follows immediately that Livshits' inequality [11] extends to the Hadamard product as

$$\forall X, Y \in \mathcal{C} : \quad \|X \star Y\| \leq \|\pi(X X^*)^{1/2}\| \|\pi(Y^* Y)^{1/2}\|,$$

and this is a generalization of the classical inequality from Fourier analysis

$$\|g \star h\|_\infty \leq \|g\|_2 \|h\|_2.$$

In the proofs we use some existing theory on the reduced crossed product of a C^* -algebra by a discrete group, and in §2 we present the results we need in a way which demonstrates that they actually form a quite natural extension of Parseval's identity.

2. Extension of Parseval's identity

The content of this section is not new and it is closely connected to the fundamental work [18] by Zeller-Meier on crossed products. We are in need of the results presented here to show our main results in §3, and in the process of writing this down, we realized that the statements may be known by many people, so with the help of Erik Bédos, we found the needed results in the articles [1], [2] by Bédos and Conti and the article [14] by Rørdam and Sierakowski. Nevertheless we have included our own proofs of the propositions 2.2 and 2.3 because we think that they illustrate our point of view, namely that the results form a clear extension of the classical results named

Parseval's identity and the nearest point property. We think that this aspect is most easily demonstrated, when the generalized Fourier series are written in the form $X \sim \sum_g L_g X_g$ rather than $X \sim \sum_g X_g L_g$, which is used in most articles. This is nothing but a notational difference, which can not be detected in classical Fourier analysis.

We will consider a C^* -algebra \mathcal{A} acting on a Hilbert space H and an action, by $*$ -automorphisms, α_g on \mathcal{A} by a discrete group G . We will use the notation of [10, Definition 13.1.1] and define the reduced left crossed product

$$\mathcal{C} := C_r^*(\mathcal{A}, \alpha_g, G)$$

as the C^* -algebra generated by the operators on $H \otimes \ell_2(G)$ given by

$$\Psi(A) := \sum_{g \in G} \alpha_g^{-1}(A) \otimes E_g, \quad A \in \mathcal{A}, \quad L_g := I \otimes \ell_g, \quad g \in G.$$

It is well known that there is a faithful conditional expectation π of \mathcal{C} onto $\Psi(\mathcal{A})$ which satisfies

$$\pi(L_g \Psi(A)) = \delta(g, e) \Psi(A).$$

It is worth remarking, that by [1, Proposition 3.1], which is based on [18], any faithful covariant representation of the C^* -dynamical system $(\mathcal{A}, \alpha_g, G)$ which has a faithful conditional expectation from the crossed product C^* -algebra onto its copy of \mathcal{A} will be an isomorphic copy of the reduced crossed product we look at here. Since our starting point is the concretely given C^* -algebra \mathcal{A} acting on H we are studying the left regular reduced crossed product, but we have chosen a specific representation of it. Before we present the results we will like to mention the text books [13], [17] by G. K. Pedersen and D. Williams, respectively, on crossed product C^* -algebras.

In [10] the authors study von Neumann algebras, whereas we study C^* -algebras, and we want to emphasize that there is a difference. In particular Proposition 2.3 does not hold in a von Neumann algebraic setting, and you may look in [12] by Mercer, who presents examples of *non convergence* in a crossed product of a von Neumann algebras by a discrete group.

The classical Fourier theory is a special case of the left regular crossed product of a discrete C^* -dynamical system. In that case the algebra \mathcal{A} is just the scalars \mathbb{C} , the group is \mathbb{Z} and the action is trivial. The Hilbert space is $L^2(\mathbb{T})$ and the unitaries L_n are the multiplication operators M_{z^n} which multiply an $L^2(\mathbb{T})$ function by z^n . The C^* -algebra \mathcal{C} then consists of the multiplication operators M_f , where f is a continuous function on \mathbb{T} . The conditional expectation π is

a state in this case. It is given by

$$\pi(M_f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta,$$

and we find that the n 'th Fourier coefficient of f is given by $f_n = \pi(L_n^* M_f)$, so we will use the following terminology.

DEFINITION 2.1. For an X in \mathcal{C} and a g in G we define the coefficient X_g of X as $X_g := \pi(L_g^* X)$.

FOURIER COEFFICIENTS. The operators X_g we defined above are denoted *Fourier coefficients* in analogy with the classical case. The main difference here is that we have chosen to have the coefficients to the right of the unitaries L_g instead of the traditional $f \sim \sum_{n \in \mathbb{Z}} f_n z^n$ for continuous functions on \mathbb{T} and $X \sim \sum_{g \in G} X_g L_g$ for operators in a crossed product C^* -algebra. Hence the Fourier series for an operator X in \mathcal{C} becomes $X \sim \sum_{g \in G} L_g X_g$. We have several reasons for this choice, the first is based on the convention that for a bounded operator T on a Hilbert space one uses the symbol $|T|$ for the operator $(T^* T)^{1/2}$ and the polar decomposition is written $T = V|T|$. This implies, from our point of view, the nicest analogies for the generalized Parseval identity and the nearest point result are obtained when the coefficients X_g are placed to the right of the unitaries L_g . Another reason may be found in §4 where we describe how $M_n(B(H))$ may be viewed as a crossed product of the diagonal algebra by the cyclic group C_n . The most natural choice for the unitary generator of the copy of C_n is the forward shift, say S , and then – again according to our judgement – the most natural description of an element X in $M_n(B(H))$ as an element in the crossed product C^* -algebra is given in the form

$$X = \sum_{k=0}^{n-1} S^k D_k,$$

with diagonal operators D_k . This change of order between coefficients and unitaries has the effect that the formulae for the product and adjoint inside a crossed product change a bit, and we will make an explicit computation of this for elements in \mathcal{C}_0 which is the linear span of the operators of the form $L_g X_g$. Let X and Y in \mathcal{C}_0 be given as finite sums $X = \sum_g L_g X_g$ and $Y = \sum_g L_g Y_g$ then

$$\begin{aligned} XY &= \sum_g \sum_h L_g X_g L_h Y_h = \sum_g \sum_h L_{gh} \alpha_{h^{-1}}(X_g) Y_h \\ &= \sum_k L_k \left(\sum_h \alpha_{h^{-1}}(X_{kh^{-1}}) Y_h \right), \end{aligned}$$

and

$$X^* = \sum_g X_g^* L_g^* = \sum_g L_g^* \alpha_g(X_g^*) = \sum_k L_k \alpha_{k^{-1}}(X_{k^{-1}}^*).$$

Then by continuity we see that for elements X and Y in \mathcal{C} the coefficients in products and adjoints are given by the formulas

$$(XY)_g = \sum_{h \in G} \alpha_{h^{-1}}(X_{gh^{-1}}) Y_h \quad \text{and} \quad (X^*)_g = \alpha_{g^{-1}}(X_{g^{-1}}^*). \quad (2.1)$$

It should be remarked that by Proposition 2.3 it follows that the sum in (2.1) converges in the C^* -norm.

Staying in the case of the continuous functions on the circle, but focussing on the conditional expectation π , we can reformulate Parseval's identity for Fourier series of square integrable 2π -periodic functions to obtain

PARSEVAL'S IDENTITY.

$$\pi(M_f^* M_f) = \|f\|_2^2 = \sum_{n \in \mathbb{Z}} |f_n|^2 = \sum_{n \in \mathbb{Z}} \pi(L_n^* M_f)^* \pi(L_n^* M_f),$$

NEAREST POINT THEOREM.

$$\forall K \subseteq \mathbb{Z} \quad \forall g \in \text{span}(\{z^k : k \in K\}) :$$

$$\begin{aligned} & \pi \left(\left(M_f - \sum_{h \in K} f_h M_{z^h} \right)^* \left(M_f - \sum_{k \in K} f_k M_{z^k} \right) \right) \\ & \leq \pi \left((M_f - M_g)^* (M_f - M_g) \right), \end{aligned}$$

and we will show that these formulae hold in our setting too. We start by defining the analogue to the square integrable norm.

PROPOSITION 2.2. *The expression*

$$\forall X \in \mathcal{C} : \|X\|_\pi := \|\pi(X^* X)\|^{1/2}$$

defines a norm on \mathcal{C} .

PROOF. By Stinespring's theorem [16] there exists a Hilbert space K a representation ρ of \mathcal{C} on K and a contraction V in $B(H \otimes \ell_2(G), K)$ such that for any X in \mathcal{C} we have $\pi(X) = V^* \rho(X) V$. From here we get via the C^* -algebraic norm identity, $\|Z^* Z\| = \|Z\|^2$ that

$$\|X\|_\pi = \|V^* \rho(X)^* \rho(X) V\|^{1/2} = \|\rho(X) V\|.$$

With this identity in mind it follows easily that $\|\cdot\|_\pi$ is a seminorm and since π is faithful in this construction it follows that $\|\cdot\|_\pi$ is a norm.

We can now extend Parseval's identity and the nearest point result.

PROPOSITION 2.3. *Let X, Y be in \mathcal{C} , K a finite subset of G and $(Z_k)_{k \in K}$ elements in $\Psi(\mathcal{A})$ then*

$$\begin{aligned} \pi \left(\left(X - \sum_{h \in K} L_h Z_h \right)^* \left(X - \sum_{k \in K} L_k Z_k \right) \right) \\ \geq \pi \left(\left(X - \sum_{h \in K} L_h X_h \right)^* \left(X - \sum_{k \in K} L_k X_k \right) \right). \end{aligned}$$

- (i) *The series $\sum_{g \in G} L_g X_g$ converges in the norm $\|\cdot\|_\pi$ to X .*
- (ii) *The series $\sum_{g \in G} X_g^* X_g$ converges in the operator norm to $\pi(X^* X)$.*
- (iii) *For $\hat{X} := X - \sum_{g \in K} L_g X_g$.*

$$\|\hat{X}\|_\pi = \left\| \sum_{g \notin K} X_g^* X_g \right\|^{1/2} = \inf_{Y \in \text{span}(\{L_k \Psi(A_k) : k \in K, A_k \in \mathcal{A}\})} \|X - Y\|_\pi.$$

- (iv) *The series $\sum_{g \in G} \alpha_g (X_g X_g^*)$ converges in the operator norm to $\pi(X X^*)$.*
- (v) *The series $\sum_{h \in G} \alpha_{h^{-1}} (X_{gh^{-1}}) Y_h$ from (2.1) converges in the operator norm.*

PROOF. Remember that π is an $\Psi(\mathcal{A})$ -bimodular map, so we have

$$\begin{aligned} \pi \left(\left(X - \sum_{h \in K} L_h Y_h \right)^* \left(X - \sum_{k \in K} L_k Y_k \right) \right) \\ = \pi(X^* X) - \sum_{k \in K} X_k^* Y_k - \sum_{h \in K} Y_h^* X_h + \sum_{h \in K} Y_h^* Y_h \\ = \pi(X^* X) - \sum_{g \in K} X_g^* X_g + \sum_{k \in K} (X_k - Y_k)^* (X_k - Y_k) \\ \geq \pi(X^* X) - \sum_{g \in K} X_g^* X_g \\ = \pi \left(\left(X - \sum_{h \in K} L_h X_h \right)^* \left(X - \sum_{k \in K} L_k X_k \right) \right) \\ = \pi(\hat{X}^* \hat{X}) \geq 0, \end{aligned}$$

where the second last equality sign again is based on the $\Psi(\mathcal{A})$ modularity of π . By the definition of \mathcal{C} there exists to any $\varepsilon > 0$ a finite subset L of G and for any g in L an operator Y_g in $\Psi(\mathcal{A})$ such that $\|X - \sum_{g \in L} L_g Y_g\| < \varepsilon$, and

we may with the help of the inequality above estimate that for any finite subset K containing L

$$\begin{aligned}\varepsilon^2 I_{B(H \otimes \ell_2(G))} &\geq \pi \left(\left(X - \sum_{h \in L} L_h Y_h \right)^* \left(X - \sum_{k \in L} L_k Y_k \right) \right) \\ &\geq \pi \left(\left(X - \sum_{h \in L} L_h X_h \right)^* \left(X - \sum_{k \in L} L_k X_k \right) \right) \\ &\geq \pi \left(\left(X - \sum_{h \in K} L_h X_h \right)^* \left(X - \sum_{k \in K} L_k X_k \right) \right) \geq 0,\end{aligned}$$

so the convergence of the series $\sum_g L_g X_g$ in the $\|\cdot\|_\pi$ norm follows immediately, and item (i) is proved. The norm convergence of the series $\sum_g X_g^* X_g$ follows since the inequalities above imply that there exists a finite subset L such that for any finite subset K of G which contains L

$$\varepsilon^2 I_{B(H \otimes \ell_2(G))} \geq \pi(X^* X) - \sum_{g \in K} X_g^* X_g \geq 0,$$

and item (ii) is proved. By item (ii) it follows that the sum $\sum_{g \notin K} X_g^* X_g$ is norm convergent with sum $\pi(\hat{X}^* \hat{X})$, so the first equality sign in item (iii) follows, and the second equality follows from the computations made in the beginning of the proof of this proposition. We know that \mathcal{C} is a C^* -algebra so X^* is in \mathcal{C} with coefficients given as in (2.1), so the statement in item (iv) follows from the one in item (ii), but applied to X^* . In order to prove item (v) we will fix a g in G and show that the series is a Cauchy series in the C^* -norm. For a given $\varepsilon > 0$ there exists by the items (ii) and (iv) finite subsets K, L of G such that $\|\sum_{f \notin K} Y_f^* Y_f\| \leq \varepsilon$ and $\|\sum_{f \notin L} \alpha_f(X_f X_f^*)\| \leq \varepsilon$ then for any 2 finite subsets M, N of G which both contains $K \cup \{\ell^{-1}g : \ell \in L\}$ and any pair of unit vectors ξ, γ in $H \otimes \ell_2(G)$ we have

$$\begin{aligned}&\left| \left\langle \left(\sum_{h \in M} \alpha_{h^{-1}}(X_{gh^{-1}}) Y_h - \sum_{h \in N} \alpha_{h^{-1}}(X_{gh^{-1}}) Y_h \right) \xi, \gamma \right\rangle \right| \\ &\leq \sum_{h \notin \{K \cup L^{-1}g\}} \|Y_h \xi\| \|\alpha_{h^{-1}}(X_{gh^{-1}}^*) \gamma\| \\ &\leq \left(\sum_{h \notin K} \|Y_h \xi\|^2 \right)^{1/2} \left(\sum_{h \notin L^{-1}g} \|\alpha_{h^{-1}}(X_{gh^{-1}}^*) \gamma\|^2 \right)^{1/2} \\ &= \left(\left\langle \left(\sum_{h \notin K} Y_h^* Y_h \right) \xi, \xi \right\rangle \right)^{1/2} \left(\left\langle \alpha_{g^{-1}} \left(\sum_{h \notin L^{-1}g} \alpha_{gh^{-1}}(X_{gh^{-1}} X_{gh^{-1}}^*) \right) \gamma, \gamma \right\rangle \right)^{1/2} \leq \varepsilon,\end{aligned}$$

so

$$\left\| \sum_{h \in M} \alpha_{h^{-1}}(X_{gh^{-1}})Y_h - \sum_{h \in N} \alpha_{h^{-1}}(X_{gh^{-1}})Y_h \right\| \leq \varepsilon$$

and item (v) and the proposition follows.

3. The Hadamard product in $C_r^*(\mathcal{A}, \alpha, G)$

We may now formulate and prove the main result of this article, and we will use the projection π from \mathcal{C} onto $\Psi(\mathcal{A})$ in the formulation of the theorem. We will also use the term *non degenerate* for the C^* -algebra \mathcal{A} to express that $\text{span}(\mathcal{A}H)$ is dense in H . Since the construction below is very concrete and not an abstract existence result, we prefer to define these concretely given items now, and then let them be part of the formulation of the theorem.

DEFINITION 3.1. Let K be the Hilbert space given by

$$K := \ell_2(G) \otimes H \otimes \ell_2(G),$$

$(x_g)_{g \in G}$ the standard orthonormal basis for $\ell_2(G)$ and $\rho: B(H \otimes \ell_2(G)) \rightarrow B(K)$ the representation given by the amplification $\rho(X) := I_{B(\ell_2(G))} \otimes X$. The isometry $V: H \otimes \ell_2(G) \rightarrow K$ is given on elementary tensors by

$$V(\xi \otimes x_g) := x_g \otimes \xi \otimes x_g.$$

The self-adjoint unitary F on K is given on elementary tensors by

$$F(x_g \otimes \xi \otimes x_h) := x_h \otimes \xi \otimes x_g.$$

THEOREM 3.2. Let H be a Hilbert space, \mathcal{A} a non degenerate C^* -algebra on H , G a discrete group and $g \rightarrow \alpha_g$ a homomorphism of G into the group of $*$ -automorphisms of \mathcal{A} . The faithful completely positive projection $\pi: \mathcal{C} \rightarrow \Psi(\mathcal{A})$ is given by $\pi(X) = V^* \rho(X) V$ and the expression $X \star Y$ given on $\mathcal{C} \times \mathcal{C}$ by

$$X \star Y := V^* \rho(X) F \rho(Y) V$$

defines a completely contractive associative product on \mathcal{C} which satisfies the following statements.

- (i) $\forall X, Y \in \mathcal{C} : (X \star Y)^* = Y^* \star X^*$.
- (ii) $\forall X, Y \in \mathcal{C} \forall A \in \mathcal{A} : (X \star Y) \Psi(A) = X \star (Y \Psi(A))$ and $\Psi(A)(X \star Y) = (\Psi(A)X) \star Y$.
- (iii) $\forall X, Y \in \mathcal{C}, \forall g \in G : \pi(L_g^*(X \star Y)) = \pi(L_g^*X) \pi(L_g^*Y)$.
- (iv) $\forall X, Y \in \mathcal{C} \forall \xi, \gamma \in H \otimes \ell_2(G) :$
 $|(X \star Y)\xi, \gamma| \leq \|\pi(XX^*)^{1/2}\gamma\| \|\pi(Y^*Y)^{1/2}\xi\|.$

- (v) $\forall X, Y \in \mathcal{C}$: the sum $\sum_{g \in G} L_g \pi(L_g^* X) \pi(L_g^* Y)$ converges in the norm topology to $X \star Y$.
- (vi) $\forall X, Y \in \mathcal{C}$ there exists a contraction $S(X, Y)$ on $H \otimes \ell_2(G)$ such that $X \star Y = \pi(XX^*)^{1/2} S(X, Y) \pi(Y^* Y)^{1/2}$.
- (vii) $\forall X \in \mathcal{C} : -\pi(X^* X) \leq X^* \star X \leq \pi(X^* X)$.

PROOF. To see that $\pi(\cdot) = V^* \rho(\cdot) V$ we will set the stage with some easy remarks and some notation. Let \mathcal{C}_0 be the linear span of all the operators $\{L_g \Psi(A) : g \in G, A \in \mathcal{A}\}$ and let X, Y in \mathcal{C}_0 be given as finite sums $X = \sum_g L_g X_g$ and $Y = \sum_g L_g Y_g$, then the formulas in (2.1) show that \mathcal{C}_0 is a $*$ -subalgebra of \mathcal{C} , which is the norm closure of \mathcal{C}_0 . We will perform the computations on an elementary operator $X = L_g \Psi(A)$ in \mathcal{C}_0 , and we recall that $\pi(L_g \Psi(A)) = \delta(g, e) \Psi(A)$, so we may compute as follows,

$$\begin{aligned}
 & \langle V^* \rho(L_g \Psi(A)) V (\xi \otimes x_h), (\gamma \otimes x_k) \rangle \\
 &= \langle (I_{\ell_2(G)} \otimes L_g \Psi(A)) (x_h \otimes \xi \otimes x_h), (x_k \otimes \gamma \otimes x_k) \rangle \\
 &= \delta(h, k) \langle L_g (\alpha_{h^{-1}}(A) \xi \otimes x_h), (\gamma \otimes x_k) \rangle \\
 &= \delta(h, k) \langle (\alpha_{h^{-1}}(A) \xi \otimes x_{gh}), (\gamma \otimes x_k) \rangle \\
 &= \delta(h, k) \delta(g, e) \langle \alpha_{h^{-1}}(A) \xi, \gamma \rangle \\
 &= \delta(g, e) \langle \Psi(A) (\xi \otimes x_h), (\gamma \otimes x_k) \rangle \\
 &= \langle \pi(L_g \Psi(A)) (\xi \otimes x_h), (\gamma \otimes x_k) \rangle.
 \end{aligned}$$

Since $\pi(\cdot)$ and $V^* \rho(\cdot) V$ both are continuous mappings, we see that for any X in \mathcal{C} we have $\pi(X) = V^* \rho(X) V$. It is well known that π is $\Psi(\mathcal{A})$ bimodular, but we need a version of this which seems a bit different so we prove that for any A in \mathcal{A} we have

$$\forall A \in \mathcal{A} : \quad \rho(\Psi(A)) V = V \Psi(A), \quad (3.1)$$

and the proof of this follows from the following

$$\begin{aligned}
 \rho(\Psi(A)) V (\xi \otimes x_k) &= x_k \otimes \alpha_{k^{-1}}(A) \xi \otimes x_k \\
 &= V(\alpha_{k^{-1}}(A) \xi \otimes x_k) = V \Psi(A) (\xi \otimes x_k).
 \end{aligned}$$

We may then begin to show the validity of all the claims. The item (i) follows directly from the definition of the bilinear operator \star .

The item (ii) follows from the definition of \star and the equation (3.1).

With respect to (iii) we will first establish the result in \mathcal{C}_0 , by showing that for $X = L_s \Psi(A)$ and $Y = L_t \Psi(B)$ in \mathcal{C}_0 we have $X \star Y = \delta(s, t) L_s \Psi(AB)$.

We do this via the following computations,

$$\begin{aligned}
& \langle (X \star Y)(\xi \otimes x_h), (\gamma \otimes x_k) \rangle \\
&= \langle V^* \rho(L_s \Psi(A)) F \rho(L_t \Psi(B)) V(\xi \otimes x_h), (\gamma \otimes x_k) \rangle \\
&= \langle F \rho(L_t \Psi(B))(x_h \otimes \xi \otimes x_h), \rho(\Psi(A^*) L_{s^{-1}})(x_k \otimes \gamma \otimes x_k) \rangle \\
&= \langle F(x_h \otimes \alpha_{h^{-1}}(B) \xi \otimes x_{th}), (x_k \otimes \alpha_{k^{-1}s}(A^*) \gamma \otimes x_{s^{-1}k}) \rangle \\
&= \langle (x_{th} \otimes \alpha_{h^{-1}}(B) \xi \otimes x_h), (x_k \otimes \alpha_{k^{-1}s}(A^*) \gamma \otimes x_{s^{-1}k}) \rangle \\
&= \delta(th, k) \delta(h, s^{-1}k) \langle a_{h^{-1}}(B) \xi, \alpha_{k^{-1}s}(A^*) \gamma \rangle \quad \text{may assume } h^{-1} = k^{-1}s \\
&= \delta(th, k) \delta(h, s^{-1}k) \langle \alpha_{h^{-1}}(AB) \xi, \gamma \rangle \\
&= \delta(s, t) \delta(sh, k) \langle a_{h^{-1}}(AB) \xi, \gamma \rangle \\
&= \langle \delta(s, t) (L_s \Psi(AB))(\xi \otimes x_h), (\gamma \otimes x_k) \rangle.
\end{aligned}$$

Hence for any pair X, Y in \mathcal{C}_0 and any g in G we have

$$\pi(L_g^*(X \star Y)) = \pi(L_g^* X) \pi(L_g^* Y),$$

and by continuity we conclude that (iii) is valid.

The inequality (iv) implies Livshits' inequality [11], which was fundamental in establishing estimates on norms of commutators in [3]. When, in [4, §3], we first realized the validity of (iv) – for Schur products – we were unaware of the fact that Horn and Johnson in [9], proof of Theorem 5.5.3, presents the same calculations as you will see below in a slightly different form. Let X, Y be a pair of operators in \mathcal{C}_0 , then item (iii) implies that $X \star Y$ is given as the finite sum

$$X \star Y = \sum_{g \in G} L_g \pi(L_g^* X) \pi(L_g^* Y) \quad (3.2)$$

which we use in the computations to come. Now let ξ, γ be in $H \otimes \ell_2(G)$, then with the aid of equation (3.2) and two different versions the Cauchy-Schwarz inequality, we get from Proposition 2.3

$$\begin{aligned}
|\langle (X \star Y) \xi, \gamma \rangle| &= \left| \sum_g \langle \pi(L_g^* Y) \xi, \pi(X^* L_g) L_g^* \gamma \rangle \right| \\
&\leq \sum_g \|\pi(L_g^* Y) \xi\| \|\pi(X^* L_g) L_g^* \gamma\| \\
&\leq \left(\sum_g \|\pi(L_g^* Y) \xi\|^2 \right)^{1/2} \left(\sum_g \|\pi(X^* L_g) L_g^* \gamma\|^2 \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&= \left(\sum_g \|Y_g \xi\|^2 \right)^{1/2} \left(\sum_g \|X_g^* L_g^* \gamma\|^2 \right)^{1/2} \\
&= \left(\left\langle \sum_g Y_g^* Y_g \xi, \xi \right\rangle \right)^{1/2} \left(\left\langle \sum_g L_g X_g X_g^* L_g^* \gamma, \gamma \right\rangle \right)^{1/2} \\
&= \left(\langle \pi(Y^* Y) \xi, \xi \rangle \right)^{1/2} \left(\langle \pi(X X^*) \gamma, \gamma \rangle \right)^{1/2} \\
&= \|\pi(Y^* Y)^{1/2} \xi\| \|\pi(X X^*)^{1/2} \gamma\| \\
&\leq \|\pi(X X^*)^{1/2}\| \|\pi(Y^* Y)^{1/2}\| \|\xi\| \|\gamma\|.
\end{aligned} \tag{3.3}$$

We know that the product \star is continuous so we may extend the inequalities above to all of \mathcal{C} , and the statement (iv) follows.

To prove item (v) we let $X \sim \sum_g L_g X_g$ and $Y \sim \sum_g L_g Y_g$ be a pair of operators in \mathcal{C} and ε a positive real number. Then $X^* \sim \sum_g L_g^* (L_g X_g^* L_g^*)$ and by Proposition 2.3 there exists a finite subset K of G such that

$$\left\| \sum_{g \notin K} L_g X_g X_g^* L_g^* \right\| \leq \varepsilon \quad \text{and} \quad \left\| \sum_{g \notin K} Y_g^* Y_g \right\| \leq \varepsilon.$$

We may define $\widehat{X \star Y} := X \star Y - \sum_{g \in K} L_g X_g Y_g$. Let ξ, γ be a pair of vectors in $H \otimes \ell_2(G)$, then the inequalities from (3.3) first show that the sum $\sum_g \langle \pi(L_g^* Y) \xi, \pi(X^* L_g) L_g^* \gamma \rangle$ is absolutely convergent and since the inequality (3.3) has been extended to elements in \mathcal{C} , it may be reused to obtain

$$\begin{aligned}
\left| \langle \widehat{X \star Y} \xi, \gamma \rangle \right| &= \left| \sum_{g \notin K} \langle \pi(L_g^* Y) \xi, \pi(X^* L_g) L_g^* \gamma \rangle \right| \\
&\leq \left(\left\langle \sum_{g \notin K} Y_g^* Y_g \xi, \xi \right\rangle \right)^{1/2} \left(\left\langle \sum_{g \notin K} L_g X_g X_g^* L_g^* \gamma, \gamma \right\rangle \right)^{1/2} \\
&\leq \varepsilon \|\xi\| \|\gamma\|,
\end{aligned}$$

so $\|\widehat{X \star Y}\| \leq \varepsilon$, and statement (v) follows.

With respect to item (vi), we remark that for X, Y in \mathcal{C} we have defined $X \star Y := V^* \rho(X) F \rho(Y) V$. The operator $\rho(Y) V$ has a polar decomposition such that there exists a partial isometry W_Y in $B(H \otimes \ell_2(G), K)$ for which $\rho(Y) V = W_Y \pi(Y^* Y)^{1/2}$. In a similar way we may find a partial isometry W_X in $B(K, H \otimes \ell_2(G))$ such that $\pi(X X^*)^{1/2} W_X = V^* \rho(X)$, and for the contraction $S(X, Y)$ in $B(H \otimes \ell_2(G))$ which is defined by $S(X, Y) := W_X F W_Y$ we then

have

$$\begin{aligned} X \star Y &= V^* \rho(X) F \rho(Y) V \\ &= \pi(XX^*)^{1/2} W_X F W_Y \pi(Y^*Y)^{1/2} \\ &= \pi(XX^*)^{1/2} S(X, Y) \pi(Y^*Y)^{1/2}, \end{aligned}$$

and item (vi) follows.

The statement (vii) follows from (vi), since the operator $S(X^*, X)$ will be a self-adjoint contraction, and the theorem follows.

4. The block Schur product as a Hadamard product

The block Schur product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ with entries in an operator algebra \mathcal{A} is defined as the block matrix $A \square B := (a_{ij}b_{ij})$. We will take a look at a particular case of a covariant representation of a crossed product, which turns out to give us the block Schur product on $M_n(B(H))$. So let n be a natural number, H a Hilbert space and $C_n := \mathbb{Z}/n\mathbb{Z}$ denote the cyclic group of order n . Define $L := H \otimes \ell_2(C_n)$ and let the C^* -algebra \mathcal{A} on L to be given as $B(H) \otimes \ell_\infty(C_n) = \ell_\infty(C_n, B(H))$, which we will think of as the diagonal operators in $M_{C_n}(B(H))$. The action of C_n on \mathcal{A} is given as $\alpha_f(A)(g) := A(g - f)$ and a unitary representation of C_n on L implementing this group of automorphisms is given by $(L_f \Xi)(g) := \Xi(g - f)$. The C^* -algebraic crossed product \mathcal{C} generated by this covariant representation (\mathcal{A}, L_g, L) of the C^* -dynamical system $(\mathcal{A}, \alpha_g, C_n)$ is nothing but $M_{C_n}(B(H)) = B(H) \otimes M_{C_n}(\mathbb{C})$ on $H \otimes \ell_2(C_n)$. It is then well known that $M_n(B(H))$ will be isomorphic to $C_r^*(\mathcal{A}, \alpha_g, C_n)$.

The projection π of \mathcal{C} onto \mathcal{A} is in this case the diagonal projection, i.e. for $X = (x_{ij})$ in $M_{C_n}(B(K))$ we have $\pi(X)_{kl} = \delta(k, \ell)x_{kk}$. By the theory an X in \mathcal{C} may be written as

$$X = \sum_{g \in C_n} L_g X_g \quad \text{with } X_g \text{ a diagonal matrix,}$$

and the Hadamard product for $X = \sum_g L_g X_g$ and $Y = \sum_g L_g Y_g$ is given as

$$X \star Y = \sum_{g \in C_n} L_g X_g Y_g. \quad (4.1)$$

Since the operators L_g have matrices which have the element I as entries in the g' th diagonal and zero entries elsewhere, the equation (4.1) implies that this Hadamard product is nothing but the block Schur product.

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