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ON LIMIT SETS OF GEOMETRICALLY FINITE KLEINIAN GROUPS

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A. Introduction.

Our main idea in this paper can be expressed by saying that if we look at the limit set $L(G)$ of a geometrically finite Kleinian group G of $\bar{\mathsf{R}}^n = \mathsf{R}^n \cup \{\infty\}$ (cf. Section B), then the landscapes we see vary in a compact set, regardless of where we are or what the scale is.

We now formulate this more precisely. Given such a group G, there is a family $\mathscr{F}$ of subsets of $\bar{\mathsf{R}}^n$ which is compact in a natural topology (see (A1)) with the following property. Let $x \in L(G) \cap \mathsf{R}^n$ and let $t \in (0, d(L(G))]$ $(t < \infty)$. Let α be a similarity of R^{n+1} preserving R^n such that

$$\alpha(x,t) = e_{n+1} = (0,\ldots,0,1).$$

Then $\alpha(L(G)) \in \mathscr{F}$ (here we set $\alpha(\infty) = \infty$). We give this in Theorem C2 in a more general and precise form.

Furthermore, if the group G does not contain parabolic fixed points of rank n (cf. Section B), then every $F \in \mathscr{F}$ is nowhere dense in $\bar{\mathsf{R}}^n$. This means that in this case $L(G)$ is somehow uniformly nowhere dense. As a consequence (Lemma D) we can find an integer q such that if Q is an n-cube of R^n and if we divide Q into q^n equal subcubes, then at least one of these subcubes does not touch $L(G)$.

It follows that the Hausdorff dimension of $L(G)$ is less than n (Theorem D). This latter result is, however, valid even if G has parabolic elements of rank n, but then new methods are necessary, cf. [7].

We then apply these ideas to study the shape of components of $\bar{\mathsf{R}}^n \setminus L(G)$. We show that under certain circumstances such a component cannot be very thin in comparison with its diameter, however we transform it with Möbius transformations (cf. Theorem E).

Finally, we remark that the situation is especially simple for such groups which do not contain parabolic elements, the existence of such elements causes essential complications. We have briefly outlined the situation in this case in the Introduction of [7].

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DEFINITIONS AND NOTATIONS. The $(n+1)$-dimensional hyperbolic space is

$$H^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathsf{R}^{n+1} : x_{n+1} > 0\}$$

and

$$\bar{H}^{n+1} = H^{n+1} \cup \bar{\mathsf{R}}^n.$$

The *closed* euclidean ball of R^p with center x and radius r is $B^p(x,r)$; we set

$$B^p(r) = B^p(0,r) \text{ and } B^p = B^p(1).$$

The standard basis of R^{n+1} is $e_1, \ldots, e_{n+1}$.

We use in this paper several different metrics. The *hyperbolic metric* of H^{n+1} is denoted by d and is defined by the element of length $|dx|/x_{n+1}$, $x = (x_1, \ldots, x_{n+1})$. The *euclidean* distance of two points $x, y \in \mathsf{R}^{n+1}$ is $|x-y|$. A third metric q is obtained by choosing a Möbius transformation h mapping $\bar{H}^{n+1}$ onto B^{n+1} such that $h(e_{n+1}) = 0$ and setting

$$q(x,y) = |h(x) - h(y)|;$$

q is the *spherical metric* of $\bar{H}^{n+1}$. It is independent of the choice of h and obviously Möbius transformations of $\bar{H}^{n+1}$ which fix e_{n+1} preserve q.

The diameter of a set and the distance of a point from a set in a given metric d are denoted $d(A)$ and $d(x,A)$, respectively. We use this notation also for the euclidean metric. If confusion is possible, we say whether we mean the hyperbolic or the euclidean metric. If $A \subset \bar{\mathsf{R}}^n$ and $\infty \in A$, we regard $d(A)$ also defined and set $d(A) = \infty$.

Now we give the fourth metric used in this paper. It is defined in the family $\mathscr{C}^n$ of closed and non-empty subsets of $\bar{\mathsf{R}}^n$. If $X, Y \in \mathscr{C}^n$, we set

$$\rho(X,Y) = \sup\{q(x,Y), q(y,X) : x \in X, y \in Y\}. \tag{A1}$$

This is the *Hausdorff metric* of $\mathscr{C}^n$ and we topologize $\mathscr{C}^n$ by means of it.

We extend here an affine homeomorphism α of R^m to $\bar{\mathsf{R}}^m$ by $\alpha(\infty) = \infty$. A *similarity* is a homeomorphism that multiplies euclidean distances by a positive constant. Again, we can extend similarities to the point ∞ by the above rule and thus we can speak of a similarity of $\bar{H}^{n+1}$.

Closure and boundary are denoted by cl and ∂, respectively, and are taken in $\bar{\mathsf{R}}^{n+1}$.

The orthogonal group of R^p is denoted by $O(p)$.

B. Möbius groups.

We denote the group of Möbius transformations of $\bar{\mathsf{R}}^n$ by Möb(n); it contains also orientation reversing elements. Every $g \in$ Möb(n) can be extended to a unique Möbius transformation of $\bar{H}^{n+1}$; we do not distinguish between g and its extension to $\bar{H}^{n+1}$. Every $g \in \text{Möb}(n)$, $g \neq \text{id}$, can be classified as either as *elliptic*, *parabolic* or *loxodromic*, cf. [1, 2.2] or [6, 1C].

Subgroups of Möb(n) are called *Möbius groups* and such a group G is *Kleinian* if it acts discontinuously somewhere in $\bar{\mathsf{R}}^n$ and the group is *geometrically finite* if the action of G in H^{n+1} has a finite-sided hyperbolic fundamental polyhedron D such that $D \cap g(D) \neq \varnothing$ for only finitely many $g \in G$; such a polyhedron D is said to be a *fundamental polyhedron of finite type* for G. For a more precise definition see [6,1B].

The limit set of a Möbius group G is denoted by $L(G)$ and it can be defined by

$$L(G) = \bar{\mathsf{R}}^n \cap \text{cl}\ Gz,$$

where $z \in H^{n+1}$ is an arbitrarily chosen point. The group G is *elementary* if $L(G)$ contains at most two points.

The *hyperbolic convex hull* H_G of $L(G)$ is defined to be the smallest closed and convex (with respect to the hyperbolic geometry) subset of H^{n+1} such that

$$\text{cl}\ \ H_G \supset L(G),$$

which is well-defined unless $L(G) = \{\text{a point}\}$ in which case we set $H_G = \varnothing$. We define for $m \geqq 0$,

$$H_G^m = \{z \in H^{n+1} : d(z,H_G) \leqq m\};$$

here $d(z,H_G) = \infty$ if $H_G = \varnothing$.

A point $v \in \bar{\mathsf{R}}^n$ is a parabolic fixpoint of G if there is parabolic $g \in G$ such that $g(v) = v$. The set of such points is denoted by $P(G)$. Then $P(G) \subset L(G)$. The *stabilizer*

$$G_v = \{g \in G : g(v) = v\}$$

of $v \in P(G)$ contains an abelian subgroup H of finite index. Then the rank $k \in [1,n]$ of H depends only on G_v and v and is called the *rank* of v (cf. [6,2B]).

A *cusp neighbourhood in* $\bar{H}^{n+1}$ of a point $v \in P(G)$ of rank k is a G_v-invariant set $U \subset \bar{H}^{n+1} \setminus L(G)$ which is of the form

$$U = h((H^{n+1} \cup \mathsf{R}^n) \setminus \mathsf{R}^k \times B^{n+1-k})$$

for some $h \in \text{Möb}(n)$ and such that $g(U) \cap U = \varnothing$ for $g \in G \setminus G_v$. A cusp neighbourhood of v in $\bar{\mathsf{R}}^n$ is a set of the form $V = U \cap \bar{\mathsf{R}}^n$, when U is a cusp neighbourhood of v in $\bar{H}^{n+1}$. Thus the definition depends also on G but we call them simply "cusp neighbourhoods", the group meant being clear from the context.

It is not true always that every $v \in P(G)$ has cusp neighbourhoods. However, if G is geometrically finite, this is so. We summarize what we need to know of cusp neighbourhoods in the following

THEOREM B. *Let G be a geometrically finite group of $\bar{R}^n$ and let D be a fundamental polyhedron of finite type for G. Then for any $m \geqq 0$, $V = \mathrm{cl}\ (D \cap H_G^m) \cap \bar{R}^n$ is a finite set of parabolic fixpoints of G and $GV = P(G)$ unless $L(G) =$ a point.*

Furthermore, if U_v is a cusp neighbourhood in H^{n+1} for $v \in V$, then $(D \cap H_G^m) \setminus (\bigcup_{v \in V} U_v)$ is compact and every compact subset of $D \cap H_G^m$ is contained in a set of this form.

PROOF. Otherwise this follows from [6, Theorem 2.4] but to show that $GV \supset P(G)$ if $L(G) \neq$ a point, we need an additional argument. We can assume that $L(G)$ contains at least two points. Pick $v \in P(G)$; we must show that $v \in GV$. We use [6, Theorem 2.4]. This theorem implies that we can assume that $v \in L_D = \mathrm{cl}\ D \cap L(G)$ which is a finite subset of $P(G)$ and that each $u \in L_D$ has a cusp neighbourhood U_u such that if $x \in U_u \cap D$, then the hyperbolic ray $R(x,u)$ joining x and u is in D. Furthermore, by making U_v small enough, we can assume that $g(U_v) \cap D \neq \varnothing$ for $g \in G$ if and only if $g(v) \in L_D$ and that then $g(U_v) \subset U_{g(v)}$. Pick now $x \in \partial U_v \cap H_G$; since $L(G)$ contains at least two points there is such x. Pick $g \in G$ such that $g(x) \in D$; then $g(v) \in L_D$ and hence $R(g(x),g(v)) \subset D$ and consequently

$$g(v) \in \mathrm{cl}(D \cap H_G) \cap \bar{R}^n \subset V.$$

We have shown that $v \in GV$.

Note that in particular it follows that the set $P(G)/G$ is finite for all geometrically finite G.

C. The limit set.

In this section G denotes a fixed non-elementary Kleinian group.

If $x \in H^{n+1}$ and $X \subset H^{n+1}$, we set

(C0) $$M_x = \{h \in \text{Möb}(n) \colon h(x) = e_{n+1}\} \text{ and } M_X = \bigcup_{x \in X} M_x.$$

Thus $M_{e_{n+1}}$ is a subgroup of Möb(n). It is isomorphic as a topological group to the orthogonal group of R^{n+1} and hence it is compact. It follows now easily that M_x and M_X are compact whenever $x \in H^{n+1}$ and $X \subset H^{n+1}$

is compact. We record the following obvious relation for later reference. If $h \in \mathrm{Möb}(n)$, then

(C1) $$M_{h(X)} = M_X \circ h^{-1}.$$

We now come to the central idea of this paper. Define

$$\mathscr{L}(X) = \{h(L(G)) : h \in M_X\}.$$

This represents the different possible views that we can get when we look at $L(G)$ from a point $x \in X$. We need to define $\mathscr{L}(x)$ also if x is a parabolic fixpoint of G; then each element of $\mathscr{L}(x)$ is a limit in $\mathscr{C}^n$, when we approach x from H^{n+1}. Here we must take account of the fact that we can approach x obliquely. Therefore we set if $m \geqq 0$ and $x \in P(G)$ has rank k,

$$\mathscr{L}^m(x) = \{h(\overline{\mathsf{R}}^k) : h \in \mathrm{Möb}\,(n) \text{ and } d(e_{n+1}, h(e_{n+1})) \leqq m\}.$$

If now $X \subset H^{n+1} \cup P(G)$ and $m \geqq 0$, we set

$$\mathscr{L}^m_X = \mathscr{L}(X) \cup \bigcup \{\mathscr{L}^m(x) : \ x \in X \cap P(G)\}.$$

We may denote also $\mathscr{L}^m(X,G)$, $\mathscr{L}(X,G)$, etc. if the group used in the definition of these sets is not clear.

In view of (C1), the following little lemma is obvious.

Lemma C1. *Let $X \subset H^{n+1} \cup P(G)$ and let $m \geqq 0$. Then*

(a) $\mathscr{L}^m(GX) = \mathscr{L}^m(X)$ *and*

(b) $\mathscr{L}^m(X,G) = \mathscr{L}^m(h(X), hGh^{-1})$ *for* $h \in \mathrm{Möb}\,(n)$.

It is also obvious that

Lemma C2. *Let $X \subset H^{n+1}$ be compact. Then*

$$\mathscr{L}(GX) = \mathscr{L}(X)$$

is compact. More precisely, M_X is a compact set of Möbius transformations such that if $h \in M_{GX}$, then there are $g \in G$ and $h' \in G_X$ such that $h = h'g$ and hence

(C2) $$h(L(G)) = h'(L(G)).$$

The next lemma is more difficult. In it we approach a parabolic fixpoint in H^m_G and show that in the limit $L(G)$ looks like a k-sphere.

Lemma C3. *Let v be a parabolic fixpoint of rank k of a non-elementary Kleinian group G such that v has cusp neighbourhoods in $\overline{H}^{n+1}$. Let $m \geqq 0$.*

Then, given $\varepsilon > 0$, there are cusp neighbourhoods V and W of v in $\bar{H}^{n+1}$ such that if $z \in H_G^m \cap V$ and $g \in$ Möb (n) *satisfies $g(z) = e_{n+1}$, then*

$$\text{(C3)} \qquad \rho(g(L(G)), h(\bar{\mathsf{R}}^k)) \leqq \varepsilon$$

for some $h \in$ Möb (n) *such that $d(e_{n+1}, h(e_{n+1})) \leqq m$. Furthermore, using the spherical metric q, we have if $z \in W$*

$$\text{(C4)} \qquad g(V) \supset \{x \in \bar{H}^{n+1} : q(x, h(\bar{\mathsf{R}}^k)) \geqq \varepsilon\}.$$

PROOF. By auxiliary Möbius transformations we can transform the situation in such a way that $v = \infty$ and that R^k is G_v-invariant and that R^k/G_v is compact when

$$G_v = \{g \in G : g(v) = v\},$$

cf. [6, Theorem 2.1]. Thus by [6, (2.1)], every $g_0 \in G_v$ is of the form

$$\text{(C5)} \qquad g_0(x,y,t) = (\alpha(x).\ \beta(y), t)$$

if $(x,y) \in \mathsf{R}^k \times \mathsf{R}^{n-k}$ and $t \geqq 0$, and where $\alpha \in$ Möb(k) is parabolic and $\beta \in O(n-k)$.

A cusp neighbourhood of v in $\bar{H}^{n+1}$ contains cusp neighbourhoods of the form

$$V_s = (H^{n+1} \cup \mathsf{R}^n) \setminus \mathsf{R}^k \times B^{n+1-k}(s),$$

$s > 0$, as follows from the fact that cusp neighbourhoods of v are G_v-invariant and from [6, Theorem 2.1 (a)]. Hence it suffices to show that if $z \in H_G^m \cap V_s$, then (C3) and (C4) are true for some $\varepsilon = \varepsilon_s > 0$ such that $\varepsilon_s \to 0$ as $s \to \infty$.

We first show this in case $m = 0$. Now, some V_r is a cusp neighbourhood of v in $\bar{H}^{n+1}$. Then

$$A = \mathsf{R}^k \times B^{n-k}(r) \supset L(G) \cap \mathsf{R}^n,$$

and thus
$$H_G \subset \mathsf{R}^k \times B^{n-k}(r) \times (0,\infty).$$

Since R^k/G is compact ([6, Theorem 2.1]), we can choose here r so big that (cf. (C5))

$$G_v(B^k(r) \times B^{n-k}(r)) = A.$$

We fix now such an r and consider $s \geqq r$ so that $V_s \subset V_r$.

We can replace z and g by $h(z)$ and gh^{-1}, respectively, where $h \in G_v$. Thus we can assume that z is of the form

$$z = (u, w, t),$$

where $(u,w) \in B^k(r) \times B^{n-k}(r)$, $t \geqq 0$, and $t \to \infty$ as $s \to \infty$. Let v be a similarity of $\bar{H}^{n+1}$ such that $v(z) = e_{n+1}$. Then $g = \beta v$ for some $\beta \in \mathrm{Möb}(n)$ fixing e_{n+1} and we can assume that $\beta = \mathrm{id}$ since such elements preserve ρ and q. We can choose v to be of the form

$$v(x) = e_{n+1} + (x - z)/t = x/t + a \tag{C6}$$

where $a \in \mathrm{R}^n$ and $|a| \to 0$ as $t \to \infty$.

Now if $s \geqq s_0$, s_0 fixed,

$$v(V_s) \supset \{x \in \bar{H}^{n+1} : x \neq \infty \text{ and } d(x, \mathrm{R}^k) \geqq s_0/t + a\}.$$

This implies (C4) for $h = \mathrm{id}$ if s is big enough. Furthermore, since G is non-elementary, $L(G) \neq \{\infty\}$ and hence there is $c \geqq 0$ such that $d(u, L(G)) \leqq c$ and $d(w, \mathrm{R}^k) \leqq c$ for $e \in \mathrm{R}^k$ and $w \in L(G)$. Since $q(x,y) \leqq c_0|x - y|$ for some constant c_0 if $x,y \leqq \mathrm{R}^n$, (C6) now implies that

$$\begin{aligned}\rho(v(L(G)), \bar{\mathrm{R}}^k) &\leqq \rho(v(L(G)), v(\mathrm{R}^k)) + \rho(v(\bar{\mathrm{R}}^k), \bar{\mathrm{R}}^k)\\ &\leqq c_0 c/t + c_0|a| \to 0 \text{ as } t \to \infty.\end{aligned}$$

This implies the lemma if $m = 0$. Let then $m > 0$ and suppose that $z \in H_G^m \cup V_s$. Let $z' \in H_G$ be a point such that $d(z,z') = d(z, H_G)$. Then $g = h'g'$ for some $h', g' \in \mathrm{Möb}(n)$ such that $g'(z') = e_{n+1}$ and $d(e_{n+1}, h'(e_{n+1})) \leqq m$. It is easy to see that $z' \in H_G \cup V_{s'}$ where $s' \to \infty$ as $s \to \infty$. Since in addition

$$\{h \in \mathrm{Möb}(n) : d(e_{n+1}, h(e_{n+1})) \leqq m\}$$

is compact, we can infer that the lemma is true also for $m > 0$.

We now combine these lemmas and consider compact sets $X \subset H_G^m \cup P(G)$, $m \geqq 0$.

LEMMA C4. *Let $m \geqq 0$ and let $X \subset H_G^m \cup P(G)$ be a compact set such that if $v \in P(G) \cap X$ and if V is a cusp neighbourhood of v in $\bar{H}^{n+1}$, then v has an (ordinary) neighbourhood U in $\bar{H}^{n+1}$ such that*

$$X \cap U \subset X \cap V. \tag{C7}$$

Then $\mathscr{L}^m(X)$ is compact.

In particular, if D is a fundamental polyhedron of finite type for G and if $X \subset \mathrm{cl}(D \cup H_G^m)$ is compact, then $X \subset H_G^m \cup P(G)$, and satisfies (C7) *and hence $\mathscr{L}^m(X)$ is compact.*

PROOF. Let $F_i \in \mathscr{L}^m(X)$. We must show that there is $F \in \mathscr{L}^m(X)$ such that by passing to a subsequence we obtain that

(C8) $$F_i \to F$$

in the Hausdorff metric. Now $F_i \in \mathscr{L}^m(x_i)$ for some $x_i \in X$. If all but a finite number of x_i's are in $P(G)$, then (C8) is clear. Hence we can assume that $x_i \in H^{n+1}$. Then we obtain by passing to a subsequence that $x_i \to x \in X$. If now $x \in H^{n+1}$, then it is again clear by the definition of $\mathscr{L}(x_i)$ and $\mathscr{L}(x)$ that we obtain (C8). If $x \in P(G)$, then (C7) and Lemma C3 imply (C8).

This proves the first paragraph. The second follows then from Theorem B.

In particular, it follows that if D is a fundamental polyhedron of finite type for G, then $\mathscr{L}^m(\mathrm{cl}(D \cap H_G^m))$ is compact. Since

$$\mathscr{L}^m(\mathrm{cl}(D \cap H_G^m)) = \mathscr{L}^m(H_G^m \cup P(G))$$

by Lemma C1 and Theorem B, we have

THEOREM C1. *Let G be a geometrically finite, non-elementary Kleinian group of $\overline{\mathrm{R}}^n$. Then*

$$\mathscr{L}^m(H_G^m \cup P(G))$$

is compact for every $m \geqq 0$.

This theorem has a couple of corollaries of which the first gives a uniformity property for the limit set of G.

COROLLARY C1. *Let G be as in Theorem C1. Then there is $c > 0$ such that for any $h \in \mathrm{M\ddot{o}b}(n)$ and $x,y \in h(L(G)) \cap \mathrm{R}^n$, there is $z \in h(L(G)) \cap \mathrm{R}^n$ such that*

(C9) $$c|y-x| \leqq |z-x| \leqq |y-x|/2.$$

PROOF. We can assume that $x \neq y$. Then $w = ((x+y)/2, |x-y|/2)$ is on the hyperbolic line joining x and y and hence $w \in h(H_G)$. Let α be a similarity of $\overline{H}^{n+1}$ such that $\alpha(w) = e_{n+1}$, $\alpha(x) = e_1$, and that $\alpha(y) = -e_1$. Then

$$\alpha h(L(G)) \in \mathscr{L}^0(H_G \cup P(G)) = \mathscr{L}(H_G) \cup \mathscr{L}^0(P(G))$$

which is a compact subset of $\mathscr{C}^n$ by Theorem C1. Since G is non-elementary, there is for every $F \in \mathscr{L}(H_G)$ such that $e_1, -e_1 \in F$ a point $z_F \in F$ with $1 > |z_F - e_1| > 0$; if

$$e_1, -e_1 \in F \in \mathscr{L}^0(P(G)),$$

then this is also true since F is now connected. By compactness of

$$\{F \in \mathscr{L}_G : e_1, -e_1 \in F\},$$

we can now find a number $c \in (0, \frac{1}{2})$ such that these points can be always chosen in such a way that $|z_F - e_1| \in [2c, 1]$. Setting $F = \alpha h(L(G))$ and $z = \alpha^{-1}(z_F)$, we get that (C9) is true for this c and this z.

This corollary in turn has the following consequence. Suppse that $x, y \in h(L(G)) \cap \mathsf{R}^n$ and $x \neq y$. Then we can find a sequence of points $y_0 = y, y_1, y_2, \ldots \in h(L(G)) \cap \mathsf{R}^n$ such that

$$|y_i - x| / |y_{i+1} - x| \in [c, \tfrac{1}{2}].$$

Thus if $t \in (0, |x-y|]$, there is y_i such that $t/|y_i - x| \in [c, 1]$. Let L_i be the hyperbolic line joining x and y_i. Then $L_i \in h(H_G)$ and hence

$$d((x,t), h(H_G)) \leqq d((x,t), L_i) \leqq m$$

for some $m \geqq 0$ depending only on G and we have

COROLLARY C2. *Let G and h be as above. Then there is $m \geqq 0$, depending only on G, with the property that if $t \in (0, \infty)$, $t \leqq d(h(L(G)) \cap \mathsf{R}^n)$, and $x \in h(L(G)) \cap \mathsf{R}^n$, then $(x,t) \in h(H_G^m)$.*

Now we finally obtain the theorem to which we have been aiming at.

THEOREM C2. *Let G be a geometrically finite, non-elementary Kleinian group of $\overline{\mathsf{R}}^n$. Let $m \geqq 0$ be as in Corollary C2. Let $h \in$ Möb(n), $x \in h(L(G)) \cap \mathsf{R}^n$ and $t \leqq (0, \infty)$, $t \leqq d(h(L(G))$, and let $g \in$ Möb(n) be such that $g(x,t) = e_{n+1}$. Then*

(C10) $$gh(L(G)) \in \mathscr{L}_G^m = \mathscr{L}^m(H_G^m \cup P(G))$$

which is a compact family of closed subsets of $\overline{\mathsf{R}}^n$.

More precisely, the following is true. Let $v_1 \ldots, v_p \in P(G)$ be parabolic fixpoints of G such that every $v \in P(G)$ is conjugate to exactly one v_i. Then, given $\varepsilon > 0$, there are a compact subset M_ε of Möb(n) *and cusp neighbourhoods V_i of v_i in $\overline{H}^{n+1}$ such that either*

(C11) $$gh(L(G)) = g'(L(G))$$

for some $g' \in M_\varepsilon$, or there are $g_0 \in G$ and $i \leqq p$ such that $(x,t) \in hg_0(V_i)$ and then

(C12) $$\rho(gh(L(G)), \beta(\overline{\mathsf{R}}^k)) \leqq \varepsilon$$

for some $\beta \in$ Möb(n) such that $d(e_{n+1}, \beta(e_{n+1})) \leqq m$, where k is the rank of v_i, and now in addition

(C13) $$ghg_0(V_i) \supset \{x \in \bar{H}^{n+1} : q(x, \beta(\bar{\mathsf{R}}^k)) \geqq \varepsilon\}.$$

REMARK. In particular, if G does not contain parabolic elements, then there is a compact set $M \subset$ Möb(n) such that (C11) is always true for some $g' \in M$.

PROOF. Note that by Corollary C2, $(x,t) \in h(H_G^m)$. Hence $gh(L(G)) \in \mathscr{L}_G^m$ which is compact by Theorem C1. This implies the first paragraph.

To get the second, pick for every v_i cusp neighbourhoods V_i and W_i in $\bar{H}^{n+1}$ such that Lemma C3 is true for $v = v_i$, $W = W_i$ and $V = V_i$. Let

$$H = H_G^m \setminus (GW_1 \cup \ldots \cup GW_p).$$

Then H/G is compact (Theorem B) and hence there is compact $X \subset H^{n+1}$ such that $GX \supset H$. Now Lemma C3 and (C2) imply that the second paragraph is true with $M_\varepsilon = M_X$, M_X as in (C0).

D. The cube lemma and the Hausdorff dimension.

We will now apply the results of the preceding section and show that if the geometrically finite Kleinian group G does not have parabolic elements of rank n, then the limit set, which in any case is nowhere dense in $\bar{\mathsf{R}}^n$, is somehow uniformly nowhere dense. We formulate this as a lemma on subdivision of cubes. As a corollary we have then that the Hausdorff dimension of the limit set is less than n in this case.

We introduce the following notation for n-cubes Q of R^n. We let s_Q be the sidelength of Q, z_Q its center and we denote $\bar{z}_Q = (z_Q, s_Q) \in H^{n+1}$. A *horoball* B at $v \in \bar{\mathsf{R}}^n$ is an open $(n+1)$-ball $B \subset H^{n+1}$ such that ∂B is tangent to $\bar{\mathsf{R}}^n$ at v. If v is a parabolic fixpoint of rank n of a geometrically finite group G, then cusp neighbourhoods of v are horoballs at v (and horoballs at v contain cusp neighbourhoods). Hence we can in the following lemma use horoballs instead of cusp neighbourhoods.

LEMMA D. *Let G be a geometrically finite Kleinian group of $\bar{\mathsf{R}}^n$. Let $v_1, \ldots, v_p$ be parabolic fixpoints of G of rank n such that every $v \in P(G)$ of rank n is conjugate under G to some v_i. Let B_i be a horoball at v_i for $i \leqq p$. Then there is an integer $q > 1$ with the following property.*

Let $h \in$ Möb(n) and let Q be an n-cube of R^n such that $\bar{z}_Q \in hg(B_i)$ for no $g \in G$ and $i \leqq p$. Then, if we divide Q into q^n equal subcubes, $Q' \cap h(L(G)) = \varnothing$ for at least one subcube Q'.

PROOF. Obviously we can assume that G is non-elementary. We can also assume that

$$d(h(L(G)) \cap Q) \geqq s_Q/2,$$

otherwise the lemma is true with $q = \frac{1}{4}$.

Pick then $x \in h(L(G)) \cap Q$ and let $t = s_Q/2 \leqq d(h(L(G)))$. Thus $(x,t) \in h(H_G^m)$, m as in Corollary C2. Now the hyperbolic distance

$$d((x,t), \bar{z}_Q) \leqq \sqrt{n} + \log 2$$

and hence $\bar{z}_Q \in h(H_G^{m'})$, where $m' = m + \sqrt{n} + \log 2$.

Let Q_0 be a standard cube such that $\bar{z}_{Q_0} = e_{n+1}$. Let α be a similarity of $\bar{H}^{n+1}$ such that $\alpha(Q) = Q_0$. Let then D be a fundamental polyhedron of finite type for G and set

$$D' = \mathrm{cl}((D \cap H_G^{m'}) \setminus (GB_1 \cup \ldots \cup GB_p)).$$

Then Theorem B implies that D' is a compact set such that $D' \cap \bar{\mathsf{R}}^n$ is a finite set consisting of parabolic fixed points whose rank is less than n. Thus $\bar{z}_Q \in h(GD')$ and hence

$$\alpha h(L(G)) \in \mathscr{L} \doteq \mathscr{L}^{m'}(D')$$

which is compact by Lemma C4.

Since D' does not contain parabolic fixpoints of rank n, every $F \in \mathscr{L}$ is nowhere dense in $\bar{\mathsf{R}}^n$. In view of this the compactness of $\mathscr{L}$ implies that there is q such that if $F \in \mathscr{L}$, and if we divide Q_0 into q^n equal subcubes, then for at least one subcube Q', $Q' \cap F = \varnothing$. The lemma follows.

In particular, if G does not have parabolic fixed points of rank n, then there is q such that if we divide any n-cube into q^n equal subcubes, then at least one of these does not touch $L(G)$. This implies

THEOREM D. *Let G be a geometrically finite Kleinian group of $\bar{\mathsf{R}}^n$ not having parabolic fixed points of rank n. Then the Hausdorff dimension of the limit set is less than n.*

REMARKS. Actually, Theorem D is valid for *all* geometrically finite Kleinian groups [7], but this is as far as we can go by this method since, if there are parabolic fixed points of rank n,

$$\bar{\mathsf{R}}^n \in \mathscr{L}^m(H_G^m \cup P(G))$$

and $\bar{\mathsf{R}}^n$ is not nowhere dense in $\bar{\mathsf{R}}^n$. So new methods are needed to get the complete theorem.

Lemma D is the generalization of Lemma C of [7] of which we already mentioned in [7], but we consider here only the situation in $\bar{\mathsf{R}}^n$. Actually, this then implies as in the proof of Theorem C of [7] that if $C \subset H^{n+1}$ is compact and if $B_1, \ldots, B_p$ are as above, then there is q such that if Q is an n-cube of R^n such that

$$\bar{z}_Q \notin GB_1 \cup \ldots \cup GB_p,$$

then, on the subdivision of Q into q^n equal subcubes, we can find at least one subcube Q' such that

$$Q' \times [0, s_{Q'}] \cap (L(G) \cup GC) = \varnothing.$$

This result would be the exact generalization of Lemma C of [7] and it would imply as in [7, Section E] that the Poincaré series of G converges for some exponent $s < n$, provided that G does not have parabolic fixed points of rank n.

E. The shape of components of Kleinian groups.

A *component* of a Kleinian group G is a component of $\bar{\mathsf{R}}^n \setminus L(G)$. We now apply our method to study the shape of components of Kleinian groups. We can express our result by saying that, under certain circumstances, a component U of a Kleinian group cannot become arbitrarily thin however we transform it by a Möbius transformation h. We show that if B is a ball with center $x \in \mathrm{cl}\ h(U) \cap \mathsf{R}^n$ and with diameter less than $d(h(U))$, then $h(U) \cap B$ contains a smaller ball B' such that

$$d(B')/d(B) \geqq c \tag{E1}$$

for some $c > 0$ not depending on h nor on x. The idea is the same as before. We show that $g(\mathrm{cl}\ U)$ varies in a compact subset of $\mathscr{C}^n$ as g varies in a (non-compact) family of Möbius transformations.

THEOREM E. *Let G be a geometrically finite Kleinian group of $\bar{\mathsf{R}}^n$ with an invariant component U (that is $g(U) = U$ for $g \in G$). If G does not contain parabolic elements, there is $c > 0$ such that whenever $h \in \mathrm{M\ddot{o}b}(n)$ and $B = B^n(x,t)$ is an n-ball with $c \in \mathrm{cl}\ h(U) \cap \mathsf{R}^n$ and $2t \leqq d(h(U))$, then there is another ball $B' \subset B \cap h(U)$ such that* (E1) *is true.*

This remains true also if G contains parabolic elements, provided that every parabolic fixpoint v of G is of rank $k < n$ and has a cusp neighbourhood V in $\bar{\mathsf{R}}^n$ such that a component of V is contained in U.

PROOF. We can assume that G is non-elementary. Obviously, we can also assume that

$$x \in \partial h(U) \cap \mathsf{R}^n = h(L(G)) \cap \mathsf{R}^n.$$

Let α be a similarity of $\bar{H}^{n+1}$ such that $\alpha(x,t) = e_{n+1}$. If G does not contain parabolic elements, then by Theorem C2, (E2) $$\alpha h(U) = g'(U)$$

where $g' \in M$ and M is a compact set of Möbius transformations depending only on G. The validity of Theorem E in this case follows now by compactness of M.

If G contains parabolic elements, then the idea is basically as above but we must now in addition consider the situation near parabolic fixpoints. Let $v_i, \ldots, v_p$ the parabolic fixpoints of G such that every $v \in P(G)$ is conjugate in G to precisely one v_i. Fix then small $\varepsilon > 0$; we will soon see how small ε must be. Let then $m \geqq 0$, the compact set $M_\varepsilon \subset \text{Möb}(n)$ and the cusp neighbourhoods V_i of v_i be as in Theorem C2. Thus either (E2) is true for some $g' \in M_\varepsilon$ and in this case there clearly is some $c = c(G,\varepsilon)$ such that (E1) is now true with this c. If this is not the case, then there are $i \leqq p$, $g_0 \leqq G$ and $\beta \in \text{Möb}(n)$ with $d(e_{n+1}, \beta(e_{n+1})) \leqq m$ such that

(E3) $$\alpha h g_0(V_i \cap \overline{\mathsf{R}}^n) \supset \{y \in \overline{\mathsf{R}}^n : q(y, \beta(\overline{\mathsf{R}}^k)) \geqq \varepsilon\}.$$

Since the set

$$\{\beta \in \text{Möb}(n) : d(e_{n+1}, \beta(e_{n+1})) \leqq m\}$$

is compact and since at least one component of $V_i \cap \overline{\mathsf{R}}^n$ is contained in U, we see that if we have chosen small enough ε, then (E1) is true for some $c = c(G) > 0$.

Theorem E implies the corresponding result for the spherical metric q. Thus there is $c_0 > 0$, depending only on G, such that $h(U)$ contains a ball B' for which

(E4) $$q(B')/q(h(U)) \geqq c_0.$$

It follows that if V_q is the spherical n-volume, then

(E5) $$1/c_1 \leqq V_q(h(U))/q(h(U))^n \leqq c_1$$

for some $c_1 \geqq 1$ depending only on G. This has the

COROLLARY E. *Let H be a Kleinian group of $\overline{\mathsf{R}}^n$ and let U be a component of H such that U and*

$$G = \{g \in H : g(U) = U\}$$

satisfy the conditions of Theorem E. Let $U_1, U_2, \ldots$ be the components of $\overline{\mathsf{R}}^n \setminus L(H)$ conjugate in H to U. Then

(E6) $$\Sigma_i q(U_i)^n < \infty.$$

REMARKS. E1. Actually, in the following form Theorem E is valid for all geometrically finite G. Choose parabolic fixpoints $w_1, \ldots, w_s$ of G such that no cusp neighbourhood of w_i meets U and such that every $v \in P(G)$ with this property is conjugate to precisely one w_i. Fix $\varepsilon > 0$. Then there are cusp neighbourhoods W_i of w_i in $\bar{H}^{n+1}$ and $c > 0$ such that Theorem E is valid with the additional condition that $(x,t) \notin h(GW_i)$ for $i \leqq s$. If $(x,t) \in h(GW_i)$ for some i, then

$$\text{(E7)} \qquad \bar{\mathsf{R}}^n \setminus \alpha h(U) \supset \alpha h g_0(W_i) \supset \{x \in \bar{\mathsf{R}}^n : q(x, \beta(\bar{\mathsf{R}}^k)) \geqq \varepsilon\}$$

where α, β and g_0 are as in (E3) and k is the rank of W_i. The validity of (E7) follows from (E3) which is now true for $V_i = W_i$.

Thus Theorem E is valid for all geometrically finite G not containing parabolic fixpoints of rank n if we replace U by

$$U \cup \bigcup \{g(W_i) \cap \bar{\mathsf{R}}^n : g \in G, i \leqq s\}.$$

This strengthening of Theorem E would have naturally as a consequence a corresponding strengthening of the Corollary which we omit, however.

E2. If in Theorem E, $n = 2$ and $v \in \operatorname{cl} h(U)$ is a fixpoint of some accidental parabolic element of G [3, 5.5], then there is a Möbius transformation h such that $h(v) = \infty$ and

$$h(U) \subset \{(x,y) \in \mathsf{R}^2 : |y| < 1\}.$$

Thus the condition on parabolic elements is essential in Theorem E.

E3. If $n = 2$, component subgroups of geometrically finite groups are again geometrically finite (Marden [3, Corollary 6.5]). It follows that Theorem E is valid, for instance, for all geometrically finite Kleinian groups of $\bar{\mathsf{R}}^2$ not containing parabolic elements.

E4. If $n = 2$, then the series (E6) converges if the exponent 2 is replaced by 4 whenever U/G is a Riemann surface of finite type, see Maskit [4, Theorem 6] who attributes the result to Koebe. Kuroda, Mori and Takahashi [2, p. 375] have proved (E6) for a class of Kleinian groups of $\bar{\mathsf{R}}^2$. Sasaki [5] proved the convergence of (E6) for all exponents $\alpha > 2$ whenever H is a finitely generated Kleinian group of $\bar{\mathsf{R}}^2$; it converges for $\alpha = 2$ if H is geometrically finite.

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