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# RADIAL LIMITS OF FUNCTIONS OF SLOW GROWTH IN THE UNIT DISK

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**Abstract.**

There are unbounded analytic functions on the disk which exhibit arbitrarily specified asymptotic behavior on almost every ray from the origin and arbitrarily specified growth of the maximum modulus on concentric circles.

## 0. Terminology, explanation of theorems, references.

Throughout this paper $\mathsf{C}$ denotes the set of complex numbers, $\mathsf{R}$ denotes the set of real numbers, $\overline{\mathsf{R}} \equiv \mathsf{R} \cup \{-\infty,\infty\}$ is the extended real line,

$$D \equiv \{z \in \mathsf{C} : |z| < 1\}$$

is the unit disk, and

$$T \equiv \{w \in \mathsf{C} : |w| = 1\}$$

is the unit circle. We shall refer to the usual (metric) topologies on these sets and the corresponding $\sigma$-fields of Borel subsets determined by those topologies. For instance, the statement

$$»T \xrightarrow{F} \overline{\mathsf{R}} \text{ is measurable«}$$

means the domain of $F$ is $T$ and $F^{-1}(U)$ is a Borel subset of $T$ whenever $U$ is a Borel subset of $\overline{\mathsf{R}}$. *Arclength measure on the Borel subset of* $T$ is denoted by $m$ and, if $R$ is a relation on $T$, the statement

$$»R \text{ is true for almost all } w \in T«$$

means that $\{w \in T : R(w)\}$ is a Borel set and $m\{w \in T : R(w)\} = 2\pi$.

The purpose of this paper is to present

THEOREM 1. *If* $_*f_1$, $^*f_1$, $_*f_2$, $^*f_2$ *are measurable* $\overline{\mathsf{R}}$*-valued functions on* $T$ *such that*

$$_*f_1(w) \leqq {}^*f_1(w) \ \ and \ \ _*f_2(w) \leqq {}^*f_2(w), \ \ all\ w \in T,$$

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*and if*

$$[0,1) \xrightarrow{M} (0,\infty)$$

*is strictly increasing and unbounded, then there is an analytic function* $D \xrightarrow{f} \mathbf{C}$ *such that*

(1) $$|f(z)| \leqq M(|z|), \text{ all } z \in D,$$

*and*

(2) $$\varliminf_{r \uparrow 1} \operatorname{Re} f(rw) = {}_*f_1(w), \quad \varlimsup_{r \uparrow 1} \operatorname{Re} f(rw) = {}^*f_1(w),$$

$$\varliminf_{r \uparrow 1} \operatorname{Im} f(rw) = {}_*f_2(w), \quad \varlimsup_{r \uparrow 1} \operatorname{Im} f(rw) = {}^*f_2(w), \text{ almost all } w \in T.$$

To facilitate the proofs and explanations we use the following terminology. A function $[0,1) \xrightarrow{M} (0,\infty)$ is called a *growth rate* if $M$ is *strictly increasing* and *unbounded*. When we say

»*f is of growth M*«

we mean $D \xrightarrow{f} \mathbf{C}$, $f$ is analytic, $M$ is a growth rate, and $|f(z)| \leqq M(|z|)$, all $z \in D$. On occasion we employ the notation

$$M(r,|f|) \equiv \sup\{|f(rw)| : w \in T\}, \quad 0 \leqq r < 1$$

with reference to a function $D \xrightarrow{f} \mathbf{C}$.

According to Theorem 1 there are unbounded analytic functions, of any specified growth, which exhibit any consistently specified asymptotic behavior on almost every ray. It is the possibility of specifying *growth* in addition to asymptotic behavior which is the *new* aspect of Theorem 1. Indeed, by the well known constructions of Bagemihl and Seidel [2], asymptotic behavior (alone) can be specified on a far broader class of paths to the boundary than that class comprised of rays from the origin.

The consequences of Theorem 1 should be compared to those of the following result of Dahlberg [4], Corollary 1, p. 302].

THEOREM. *Suppose* $D \xrightarrow{f} \mathbf{C}$ *is analytic,* $T \xrightarrow{f^*} \mathbf{C}$ *is measurable.*

(1) $$\varlimsup_{r \uparrow 1} |f(rw)| < \infty, \text{ all } w \in T,$$

(2) $$\lim_{r \uparrow 1} (1-r)^2 \log M(r,|f|) = 0,$$

(3) $$\lim_{r \uparrow 1} f(rw) = f^*(w), \text{ almost all } w \in T,$$

*and*

(4) $\int_T |f^*(w)|\,dm(w) < \infty.$

*Then*

$$\sup_{0 \leqq r < 1} \int_T |f(rw)|\,dm(w) < \infty;$$

*that is, $f \in H^1$.*

To draw the comparison consider a fixed measurable function $T \xrightarrow{f^*} \mathrm{C}$ for which $\int_T |f^*(w)|\,dm(w) < \infty$ but which does *not* coincide almost everywhere with the radial limits of an analytic function of class $H^1$ (for instance, stipulate that $0 < m\{w \in T : f^*(w) = 0\} < 2\pi$). Then, by Dahlberg, there is *no* analytic function possessing *all three* properties (1), (2), (3) with reference to the present function $f^*$. However, by Theorem 1, there *is* an analytic function $f$ meeting the two conditions (2), (3) of Dahlberg (for, in Theorem 1, we may set ${}_*f_1 = {}^*f_1 = \mathrm{Re}\, f^*$, ${}_*f_2 = {}^*f_2 = \mathrm{Im}\, f^*$, and $M(r) = (1-r)^{-1}$). Consequently, condition (1) of Dahlberg *must* be violated by any such function $f$ (that is, its restriction to at least one ray must be unbounded).

Also, with the same function $f^*$ in mind, we may ask if there is an analytic function meeting the two conditions (1), (3) of Dahlberg. We have not answered this question; but Dahlberg has given a relevant example on page 302 of [4].

Bagemihl and Seidel exhibited the utility of their construction through two applications [2, Theorems 4 and 5]. We present similar applications making use of the new features of the present construction.

The first corollary of Theorem 1 pertains to the various hypotheses which imply existence and uniqueness of solutions of boundary value problems (of the second and third kind) for Laplace's equation.

COROLLARY 1. *Specify $\alpha \in (0,1)$ and two measurable functions*

$$T \xrightarrow{N,A} \mathrm{R}.$$

*Then there exists a function $\bar{D} \xrightarrow{H} \mathrm{R}$ such that*

(1) *$H$ is harmonic in $D$,*

(2) $\sup\{|H(z) - H(\zeta)|\,|z-\zeta|^{-\alpha} : z \in \bar{D},\ z \neq \zeta\} < \infty,$

(3) $\lim_{r \uparrow 1} \dfrac{\partial H}{\partial r}(rw) = N(w)$ *and* $\lim_{r \uparrow 1} \dfrac{\partial H}{\partial \theta}(rw) = A(w)$

*almost all $w \in T$.*

PROOF. Construct $D \xrightarrow{f} \mathbf{C}$, as in Theorem 1, such that

(4) $\quad |f(z)| \leqq (1-|z|)^{\alpha-1}$, all $z \in D$,

and

(5) $\quad \lim_{r \uparrow 1} f(rw) = (N(w)\operatorname{Re} w - A(w)\operatorname{Im} w) - i(N(w)\operatorname{Im} w + A(w)\operatorname{Re} w)$,

almost all $w \in T$. In turn, construct $H$ such that

$$\frac{\partial H}{\partial x} - i\frac{\partial H}{\partial y} = f.$$

Then (1) is immediate and, by well known theorems of Hardy and Littlewood [5, Theorems 40, 41], $H$ has an extension to $\bar{D}$ which satisfies (2). Moreover, by (5)

$$\frac{\partial H}{\partial r}(rw) = \frac{\partial H}{\partial x}(rw)\operatorname{Re} w + \frac{\partial H}{\partial y}(rw)\operatorname{Im} w \to N(w)$$

and

$$\frac{\partial H}{\partial \theta}(rw) = \frac{\partial H}{\partial x}(rw)(-\operatorname{Im} rw) + \frac{\partial H}{\partial y}(rw)(\operatorname{Re} rw) \to A(w),$$

as $r \to 1$, almost all $w \in T$.

We are unable to extend Corollary 1 to the case in which $N$ and $A$ take values in $\bar{\mathbf{R}}$.

In the terminology of the theory of trigonometric series, the second corollary implies that any measurable function on $[0,2\pi)$, with values in the number system $\bar{\mathbf{R}} + i\bar{\mathbf{R}}$, agrees almost everywhere with the Abel sum of trigonometric series of positive type with unbounded coefficients of specified growth.

COROLLARY 2. *Specify four measurable functions* ${}_*f_1$, ${}^*f_1$, ${}_*f_2$, ${}^*f_2$ *as in Theorem* 1 *and specify a strictly increasing, unbounded sequence* $(C_n)_0^\infty$ *from* $(0,\infty)$. *Then there is an analytic function* $D \xrightarrow{f} \mathbf{C}$ *with Taylor expansion* $f(z) = \sum_{n=0}^\infty a_n z^n$ *such that*

(1) $\quad |a_n| \leqq C_n$, *all* $n \in \{0, 1, 2, \ldots\}$,

*and*

(2) $\quad$ *f has property* (2) *in the conclusion of Theorem* 1.

PROOF. Since the sequence $(C_n)_0^\infty$ is strictly increasing we may construct a corresponding *increasing* sequence $(r_n)_0^\infty$ from $(0, 1)$ such that

$$C_0 < C_1 r_1 < C_2 r_2^2 < C_3 r_3^3 < \dots$$

and

$$\tfrac{1}{2} C_n < C_n r_n^n, \text{ all } n \in \{0, 1, 2, \dots\}.$$

Since $(C_n)_0^\infty$ is unbounded and $(r_n)_0^\infty$ is increasing, there is a growth rate $[0,1) \xrightarrow{M} (0, \infty)$ such that

$$M(r_n) = C_n r_n^n, \text{ all } n \in \{0, 1, 2, \dots\}.$$

Construct $f(z) = \sum_{n=0}^{\infty} a_n z^n$ as in Theorem 1 with data $M, {}_*f_1, {}^*f_1, {}_*f_2, {}^*f_2$. Since

$$|a_n| r_n^n = \left| \frac{1}{2\pi} \int_T f(r_n w)(\bar{w})^n dm(w) \right| \leqq M(r_n) = C_n r_n^n$$

for all $n \in \{0, 1, 2, \dots\}$ the proof is complete.

We do not know if it is possible to arbitrarily specify almost everywhere the radial limits of a function with *bounded* Taylor coefficients. This is a typical question in the theory of representation of functions through generalized summation of trigonometric series. A number of similar open questions appear in the survey [7].

We turn now to describe the proof of Theorem 1 and to establish a lemma (Lemma 0) which is utilized at several points in the text of the proof. For the sake of completeness we also reproduce the special theorems of Carathéodory and Keldyš which support Lemma 0. Besides Lemma 0, the only reference cited in the proof of Theorem 1 is the theorem of Barth and Schneider which is reproduced in the next paragraph.

The proof of Theorem 1 is set out as a sequence of lemmas (sections 1 through 8) in which functions exhibiting successively more general asymptotic behavior on the disk are constructed. This process is initiated by modifying the functions appearing in the following statement.

THEOREM. *(Barth and Schneider, [3, page 4]). If $M$ is a growth rate there is an (analytic) function $D \xrightarrow{g} \mathbf{C}$ of growth $M$ such that*

$$\lim_{r \uparrow 1} \operatorname{Re} g(rw) = +\infty, \textit{ almost all } w \in T.$$

Functions of more general asymptotic behavior are obtained by modifying and combining functions previously constructed. To modify a function on the disk we *compose* it with auxiliary analytic functions constructed with special purposes in mind. Three such *auxiliary* functions appear in the following.

LEMMA 0. (1) *There is an entire function G such that*

$$\lim_{\operatorname{Re} z \uparrow \infty} \operatorname{Re} G(z) = +\infty \text{ and } \lim_{\operatorname{Re} z \uparrow \infty} \operatorname{Im} G(z) = 0.$$

(2) *There is an entire function G such that*

$$\lim_{\substack{x \uparrow \infty \\ -\frac{3}{2} \leq y \leq -\frac{1}{2}}} G(x+iy) = 0, \quad \lim_{\substack{x \uparrow \infty \\ \frac{1}{2} \leq y \leq \frac{3}{2}}} \operatorname{Re} G(x+iy) = +\infty$$

*and*

$$\lim_{\substack{x \uparrow \infty \\ \frac{1}{2} \leq y \leq \frac{3}{2}}} \operatorname{Im} G(x+iy) = 0.$$

(3) *If* $\underline{m}$ *and* $\bar{m}$ *satisfy* $-\infty \leq \underline{m} \leq \bar{m} \leq \infty$ *there is an entire function with the following property: If* $(0,1) \xrightarrow{\gamma} \mathbf{C}$ *is a continuous function such that*

$$|\operatorname{Im} \gamma(t)| \leq 1; \textit{ all } t \in (0,1), \textit{ and } \lim_{t \uparrow 1} \operatorname{Re} \gamma(t) = +\infty$$

*then*

$$\varliminf_{t \uparrow 1} \operatorname{Im} G(\gamma(t)) = \underline{m}, \quad \varlimsup_{t \uparrow 1} \operatorname{Im} G(\gamma(t)) = \bar{m},$$

$$\lim_{t \uparrow 1} \operatorname{Re} G(\gamma(t)) = 1, \textit{ and } \lim_{\substack{\operatorname{Re} z \downarrow -\infty \\ |\operatorname{Im} z| \leq 1}} G(z) = 0.$$

We cite the required theorems of Carathéodory and Keldyš before the proof of Lemma 0.

Our reference for the theorem of Carathéodory is section 4–6 of [1]. To present this theorem let $d$ denote the *spherical metric* on $\mathbf{C} \cup \{\infty\} \equiv \hat{\mathbf{C}}$ and let $\Omega$ denote a connected and simply-connected open subset of $\mathbf{C}$. A continuous injection $(0,1) \xrightarrow{\sigma} \Omega$ is called a *crosscut* of $\Omega$ if $d(\sigma(t), \hat{\mathbf{C}} - \Omega) \to 0$ as $|t(t-1)| \downarrow 0$. A sequence $(w_n)_1^\infty$ is called a *fundamental sequence in* $\Omega$ if given any $\varepsilon > 0$ there is a crosscut $\sigma$ of $\Omega$ such that the spherical diameter of

$$\sigma^* \equiv \{\sigma(t) : 0 < t < 1)$$

is at most $\varepsilon$ and such that $w_n$ and $w_1$ lie in different connected components of $\Omega - \sigma^*$ for all $n$ sufficiently large.

THEOREM. (Carathéodory). *Let* $\Omega_1$ *and* $\Omega_2$ *be connected and simply connected open subsets of* $\mathbf{C}$ *and let* $\Omega_1 \xrightarrow{f} \Omega_2$ *be an analytic bijection. Then*

*$(w_n)_1^\infty$ is a fundamental sequence in $\Omega_1$ if and only if $(f(w_n))_1^\infty$ is a fundamental sequence in $\Omega_2$.*

The Theorems A and B which appear below are weakened versions of theorems of Keldyš which appear in the article [6] of Mergelyan.

DEFINITION ([6, page 326, (B)]). Suppose $E$ is a closed connected subset of $\mathbf{C}$ and there is a continuous, strictly increasing, unbounded function $[0,\infty) \xrightarrow{r} (0,\infty)$ such that to every $z \in \mathbf{C} - E$ there corresponds a continuous injection

$$[0,\infty) \xrightarrow{\gamma} \{\zeta \in \mathbf{C} - E : |\zeta| > r(|z|)\}$$

with $\gamma(0) = z$ and

$$\lim_{t \uparrow \infty} |\gamma(t)| = \infty.$$

Then write »$E$ has property B«.

THEOREM A. (Keldyš [6, page 337, Theorem 2.3]). *If $E = \{z \in \mathbf{C} : \operatorname{Re} z \geqq 0\}$ and $E \xrightarrow{\varphi} \mathbf{C}$ is continuous on $E$ and analytic on the interior of $E$, there is an entire function $G$ such that*

$$|G(z) - \varphi(z)| < \exp(-|z|^{1/2}), \quad \textit{all } z \in E.$$

THEOREM B. (Keldyš [6, page 338, Theorem 3.3]). *If $E \subset \{z \in \mathbf{C} : |\operatorname{Im} z| < \frac{\pi}{2}\}$, if $E$ has property* B, *and if $E \xrightarrow{\varphi} \mathbf{C}$ is continuous on $E$ and analytic at interior points of $E$, there is an entire function $G$ such that*

$$|G(z) - \varphi(z)| < \exp(-e^{\frac{|z|}{2}}), \quad \textit{all } z \in E.$$

PROOF OF LEMMA 0. As the proofs of all three statements are similar it will suffice to give the detailed proof of (3), which is the most complicated statement, and only summarize the proofs of (1) and (2).

To prove statement (3) consider two continuous functions $[0,1) \xrightarrow{\alpha,\beta} \mathbf{R}$ with the following features:

$$\alpha(0) = \beta(0) = 0; \ \alpha(u) < \beta(u), \text{ all } u \in (0,1);$$

$$(*) \ \lim_{u \uparrow 1} (\beta(u) - \alpha(u)) = 0, \ \varliminf_{u \uparrow 1} \alpha(u) = \underline{m}, \text{ and } \varlimsup_{u \uparrow 1} \alpha(u) = \bar{m}.$$

Then, set

$$\bar{\Omega}_1 = \{z \in \mathbf{C} : |\operatorname{Im} z| \leqq \tfrac{\pi}{2}\},$$

$$\bar{\Omega}_2 = \{x + iy \in \mathbf{C} : 0 \leqq x < 1 \text{ and } \alpha(x) \leqq y \leqq \beta(x)\},$$

$$\Omega_1 = \operatorname{int} \bar{\Omega}_1, \text{ and } \Omega_2 = \operatorname{int} \bar{\Omega}_2$$

(note that $\overline{\Omega}_2$ is *not* the complete closure of $\Omega_2$ in $\mathbf{C}$). A sequence $(z_n)_1^\infty$ from $\Omega_1$ is *fundamental* in $\Omega_1$ if and only if either

(a) $(z_n)_1^\infty$ converges to a point of $\overline{\Omega}_1 - \Omega_1$, or

(b) $\operatorname{Re} z_n \to +\infty$, or

(c) $\operatorname{Re} z_n \to -\infty$.

A sequence $(w_n)_1^\infty$ from $\Omega_2$ is *fundamental* in $\Omega_2$ if and only if either

$(w_n)_1^\infty$ converges to a point of $\overline{\Omega}_2 - \Omega_2$, or
$\operatorname{Re} w_n \to 1$ (*refer to condition* $(*)$).

Let $(a_n)_1^\infty$ and $(b_n)_1^\infty$ be sequences from $\Omega_2$ such that $a_n \to 0$ and $\operatorname{Re} b_n \to 1$ as $n \uparrow \infty$, and consider an arbitrary analytic bijection

$$\Omega_1 \xrightarrow{\psi} \Omega_2.$$

By the theorem of Carathéodory $(\psi^{-1}(a_n))_1^\infty$ and $(\psi^{-1}(b_n))_1^\infty$ are fundamental sequences in $\Omega_1$ and the sequence $(\psi^{-1}(a_1), \psi^{-1}(b_1), \psi^{-1}(a_2), \ldots)$, formed by alternating the terms of the previous sequences, is *not* fundamental in $\Omega_1$. Therefore $(\psi^{-1}(a_n))_1^\infty$ satisfies one of the conditions (a), (b), (c) and $(\psi^{-1}(b_n))_1^\infty$ satisfies a *different* one of these conditions. Consequently there is an analytic bijection $\Omega_1 \xrightarrow{\tau} \Omega_1$ such that

$$\operatorname{Re} \tau \circ \psi^{-1}(a_n) \to -\infty \quad \text{and} \quad \operatorname{Re} \tau \circ \psi^{-1}(b_n) \to +\infty.$$

Set $\varphi = \psi \circ \tau^{-1}$.

Let $(z_n)_1^\infty$ be a sequence in $\Omega_1$ such that $\operatorname{Re} z_n \to -\infty$. Then $(z_1, \varphi^{-1}(a_1), z_2, \ldots)$ is a fundamental sequence in $\Omega_1$. So $(\varphi(z_1), a_1, \varphi(z_2), \ldots)$ is a fundamental sequence in $\Omega_2$ and we conclude $\varphi(z_n) \to 0$ as $n \uparrow \infty$. This proves

$$\lim_{\substack{\operatorname{Re} z \to -\infty \\ z \in \Omega_1}} \varphi(z) = 0.$$

Let $(t_n)_1^\infty$ be a sequence from $(0,1)$ such that $t_n \to 1$ as $n \uparrow \infty$. Then (refer to hypothesis of statement (3)) $(\gamma(t_1), \varphi^{-1}(b_1), \gamma(t_2), \ldots)$ is a fundamental sequence in $\Omega_1$. Thus $(\varphi(\gamma(t_1)), b_1, \varphi(\gamma(t_2)), \ldots)$ is a fundamental sequence in $\Omega_2$. This proves

$$(**) \qquad \lim_{t \uparrow 1} \operatorname{Re} \varphi(\gamma(t)) = 1.$$

Since $\varphi(\gamma(t)) \in \Omega_2$, all $t \in (0,1)$, $(*)$ and $(**)$ imply

$$\lim_{t \uparrow 1} |\operatorname{Im} \varphi(\gamma(t)) - \alpha(\operatorname{Re} \gamma(t))| = 0.$$

Therefore

$$\varliminf_{t\uparrow 1} \operatorname{Im} \varphi(\gamma(t)) = \underline{m} \text{ and } \varlimsup_{t\uparrow 1} \operatorname{Im} \varphi(\gamma(t)) = \bar{m}.$$

Finally, set $E = \{z \in \mathbf{C} : |\operatorname{Im} z| \leqq 1\}$ ($E$ has property B) and let $G$ be an entire function which approximates $\varphi|_E$ in the sense of Keldyš B. It is immediate that $G$ meets the requirements of statement (3). The proof of (3) is complete.

For the proof of (1), set

$$\Omega_1 = \{z \in \mathbf{C} : \operatorname{Re} z > -1\}$$

and

$$\Omega_2 = \{x + iy \in \mathbf{C} : 0 < x < \infty \text{ and } 0 < y < e^{-x}\}.$$

An application of the theorem of Carathéodory, similar to that in the proof of (3). shows there is an analytic bijection $\Omega_1 \xrightarrow{\varphi} \Omega_2$ such that a (fundamental) sequence in $\Omega_1$ which tends to $\infty$ corresponds, under $\varphi$, to a (fundamental) sequence in $\Omega_2$ which tends to $\infty$. Therefore, for such a function $\varphi$ we have

$$\lim_{\operatorname{Re} z \to +\infty} \operatorname{Re} \varphi(z) = +\infty \text{ and } \lim_{\operatorname{Re} z \to +\infty} \operatorname{Im} \varphi(z) = 0.$$

Set $E = \{z \in \mathbf{C} : \operatorname{Re} z \leqq 0\}$ and approximate $\varphi|_E$ by an entire function $G$ as in Keldyš A. Then $G$ has the properties specified in (1).

To prove (2) set

$$\Omega_1 = \{x + iy \in \mathbf{C} : -2 < x < \infty \text{ and } -2 < y < 2\} - [2, \infty)$$
$$\Omega_2 = \{x + iy \in \mathbf{C} : \ 0 < x < \infty \text{ and } 0 < y < xe^{-x}\}.$$

By the theorem of Carathéodory there is an analytic bijection $\Omega_1 \xrightarrow{\varphi} \Omega_2$ such that a fundamental sequence in $\Omega_1$ tending to $\infty$ through values in the upper (lower) half plane corresponds, under $\varphi$, to a fundamental sequence in $\Omega_2$ tending to $\infty$ (0), That is, the mapping $\varphi$ satisfies

$$\lim_{\substack{x \to \infty \\ y < 0,\, x+iy \in \Omega_1}} \varphi(x+iy) = 0, \qquad \lim_{\substack{x \to \infty \\ y > 0,\, x+iy \in \Omega_1}} \operatorname{Re} \varphi(x+iy) = +\infty$$

and

$$\lim_{\substack{x \to \infty \\ y > 0,\, x+iy \in \Omega_1}} \operatorname{Im} \varphi(x+iy) = 0.$$

Now set

$$E = \{x+iy \in \mathbf{C} : 0 \leqq x < \infty \text{ and } \tfrac{1}{2} \leqq y \leqq \tfrac{3}{2}\}$$
$$\cup \{x+iy \in \mathbf{C} : 0 \leqq x \leqq 1 \text{ and } -\tfrac{3}{2} \leqq y \leqq \tfrac{3}{2}\}$$
$$\cup \{x+iy \in \mathbf{C} : 0 \leqq x < \infty \text{ and } -\tfrac{3}{2} \leqq y \leqq -\tfrac{1}{2}\}.$$

Then $E$ has property B and $E \subset \Omega_1$. Approximate $\varphi|_E$ by an entire function $G$ as in Theorem B of Keldyš.

**1. Radial limits $\infty + i{\cdot}o$ almost everywhere.**

LEMMA 1. *If $M$ is a growth rate there exists a function $f$, of growth $M$, such that*

(1.1)

$$\lim_{r \uparrow 1} \operatorname{Re} f(rw) = +\infty \text{ and } \lim_{r \uparrow 1} \operatorname{Im} f(rw) = 0 \text{ for almost all } w \in T.$$

PROOF. Let $G$ denote an entire function with the properties specified in (1) of Lemma 0 and choose $k > 0$ so that $|G(0)| \leqq kM(0)$, Then, for each $r \in [0,1)$ there is a unique positive real number $M_1(r)$ such that

$$\max_{|z| \leqq M_1(r)} |G(z)| = kM(r).$$

The elementary properties of entire functions imply that the function

$$[0,1) \xrightarrow{M_1} (0,\infty)$$

is a growth rate.

By the theorem of Barth and Schneider there is a function $g$ of growth $M_1$ such that

$$\lim_{r \uparrow 1} \operatorname{Re} g(rw) = \infty, \text{ almost all } w \in T.$$

Set $f(z) = k^{-1}G(g(z))$, $z \in D$. Then

$$|f(z)| \leqq k^{-1} \max_{|\xi| \leqq |g(z)|} |G(\xi)| \leqq k^{-1} \max_{|\xi| \leqq M_1(|z|)} |G(\xi)| = M(|z|),$$

all $z \in D$. So $f$ is of growth $M$. Moreover, if $w \in T$ and

$$\lim_{r \uparrow 1} \operatorname{Re} g(rw) = +\infty$$

we have

$$\lim_{r \uparrow 1} \operatorname{Re} f(rw) = \lim_{\operatorname{Re}\xi \uparrow \infty} \operatorname{Re} G(\xi) = +\infty$$

and

$$\lim_{r \uparrow 1} \operatorname{Im} f(rw) = \lim_{\operatorname{Re}\xi \uparrow \infty} \operatorname{Im} G(\xi) = 0.$$

**2. Infinite and bounded values on disjoint closed arcs.**

A *closed arc* is a set of the form $\{e^{it}: a \leqq t \leqq b\}$ wherein $0 < b - a < 2\pi$ (a proper subset of $T$).

LEMMA 2. *If $I$ and $B$ are disjoint closed arcs and $M$ is a growth rate there is a function $f$, of growth $M$, satisfying*

$$\lim_{r \uparrow 1} \operatorname{Re} f(rw) = +\infty, \text{ almost all } w \in I \tag{2.1}$$

*and*

$$\sup\{|f(rw)|: 0 \leqq r < 1 \text{ and } w \in B\} < \infty, \tag{2.2}$$

PROOF. It is permissible to enlarge $B$ and rotate the resulting configuration. Thus we may assume

$$I = \{e^{it}: |t| \leqq \lambda(I)\}, \ B = \{e^{it}: |t - \pi| \leqq \lambda(B)\}$$

with $\lambda(I) > 0$, $\lambda(B) > 0$ and $\lambda(I) + \lambda(B) < \pi$.

Choose $\mu$ such that $0 < \mu < \pi$ and $e^{i\mu} \in T - (B \cup I)$ and let $\tau \in \mathrm{R}$ satisfy $\cos \mu = \tau (\sin \mu)^2 - \frac{1}{2}$. By the well known theorem of Carathéodory on the mapping of Jordan regions (or by the theorem of his cited in section 0) there is a bijection

$$\bar{D} \xrightarrow{\psi} \{x + iy \in \mathrm{C}: x^2 + y^2 \leqq 1 \text{ and } x \geqq \tau y^2 - \tfrac{1}{2}\}$$

such that $\psi(e^{i\mu}) = e^{i\mu}$, $\psi(e^{-i\mu}) = e^{-i\mu}$, $\psi(-1) = -\frac{1}{2}$, $\psi$ is continuous, $\psi^{-1}$ is continuous, and both $\psi$ and $\psi^{-1}$ are analytic at interior points of their domains. The image of $\psi$ appears in Figure 1.

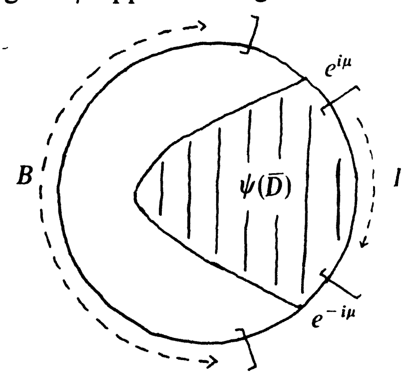


Figure 1.

Since $e^{i\mu}$ and $e^{-i\mu}$ do not lie on $B \cup I$ the reflection principle for analytic arcs implies that $\psi$ may be extended analytically to a neighborhood of $B \cup I$. The complete conclusion is as follows: *There is an open set $\Omega$ and an analytic injection $\Omega \xrightarrow{\varphi} \mathbf{C}$ such that $D \cup B \cup I \subset \Omega$ and $\varphi(z) = \psi(z)$ for all $z \in D \cup B \cup I$.*

Now we define a function $[0,1) \xrightarrow{n} (0, \infty)$:

$$n(r) = (1-r)^{-1} \max_{w \in I} \Big| \varphi(rw) - \varphi(w)|\varphi(rw)| \Big|, \quad 0 \leqq r < 1.$$

Our immediate task is to prove

$$\lim_{r \uparrow 1} n(r) = 0. \tag{2.3}$$

The proof requires Taylor's theorem. Since $\varphi$ is analytic in a neighborhood of $I$ and of unit modulus *on* $I$ there is an open set $N$ and two *uniformly bounded* functions

$$N \times N \xrightarrow{E_1, E_2} \mathbf{C}$$

such that $I \subset N \subset \Omega$, and

$$\varphi(z) = \varphi(z_0) + \varphi'(z_0)(z - z_0) + E_1(z,z_0)|z - z_0|^2$$

and

$$|\varphi(z)| = |\varphi(z_0)| + \frac{\partial|\varphi|}{\partial z}(z_0)(z - z_0) + \frac{\partial|\varphi|}{\partial \bar{z}}(z_0)\overline{(z - z_0)}$$
$$+ E_2(z,z_0)|z - z_0|^2 \text{ whenever } (z,z_0) \in N \times N.$$

By computing the derivatives of $|\varphi|^2 = \varphi\bar{\varphi}$ we conclude

$$\frac{\partial|\varphi|}{\partial z}(z) = \tfrac{1}{2}|\varphi(z)|^{-1}\varphi'(z)\overline{\varphi(z)}$$

and

$$\frac{\partial|\varphi|}{\partial \bar{z}}(z) = \tfrac{1}{2}|\varphi(z)|^{-1}\overline{\varphi'(z)}\varphi(z),$$

all $z \in N$. Therefore, if $w$ and $we^{ih}$ ($h$ real) are points on $I$, Taylor's expansion of $|\varphi|$ implies

$$w\varphi'(w)\overline{\varphi(w)}(e^{ih} - 1) + \overline{w\varphi'(w)}\varphi(w)(e^{-ih} - 1) = -E(we^{ih},w)|1 - e^{ih}|^2$$

(since $|\varphi(u)| = 1$ for all $u \in I$). If we divide both sides of this equation by $h$ and allow $h$ to pass to zero we see

$$\operatorname{Im} w\varphi'(w)\overline{\varphi(w)} = 0, \text{ all } w \in I.$$

Consequently, the Taylor expansions of $\varphi$ and $|\varphi|$ lead to the statement that (add and subtract $\varphi(w)$)

$$\varphi(rw) - \varphi(w)|\varphi(rw)| = E_1(rw), w)(1-r)^2 - E_2(rw, w)(1-r)^2\varphi(w)$$

whenever $w \in I$ and $rw \in N$. Since $E_1$ and $E_2$ are bounded in $N \times N$ we have established (2.3).

If $n(s) = 0$ for some $s \in (0,1)$, then $e \to |\varphi(sw)|$ would be analytic in a neighborhood of $I$ and hence *constant* in $\Omega$. Therefore, by (2.3) the function

$$r \to \min_{r \leqq s \leqq 1} n(s)^{-1/2}, \quad 0 \leqq r < 1$$

is strictly positive, non-decreasing and unbounded. Thus, it *dominates* a growth rate $M_4$ which therefore satisfies

$$\lim_{r \uparrow 1} M_4(r) n(r) = 0.$$

Now set

$$\alpha(r) = \max_{w \in T} |\varphi(rw)|, \quad 0 \leqq r \leqq 1.$$

The functions $r \to \alpha(r)$ and $r \to (\alpha(r)+1)/2$ are strictly increasing on $[0,1]$ and $\alpha(1) = 1$. Hence, there are growth rates $M_2, M_3$ satisfying

$$M_2(\alpha(r)) = M(r) \text{ and } M_3((\alpha(r)+1)/2) = M_4(r), \quad 0 \leqq r < 1.$$

Set

$$M_1(s) = \min[M_2(s), M_3(s)] \quad 0 \leqq s < 1.$$

Then $M_1$ is a growth rate which satisfies

$$\lim_{r \uparrow 1} M_1((\alpha(r)+1)/2) n(r) = 0, \text{ and} \tag{2.4}$$

$$M_1(\alpha(r)) \leqq M(r), \text{ all } r \in [0,1). \tag{2.5}$$

We may now define $f$. Let $g$ be of growth $M_1$ and satisfy

$$\lim_{r \uparrow 1} \operatorname{Re} g(rw) = +\infty,$$

almost all $w \in T$ (Lemma 1). Set $f = g \circ \varphi$.

By (2.5), $|f(z)| \leqq M_1(\alpha(|z|))$ whenever $z \in D$. So $f$ is of growth $M$. Since $\varphi$ is analytic in a neighborhood of $B$, $\varphi(B)$ is a compact subset of $D$. Therefore statement (2.2) is also valid.

To verify (2.1) we first prove statement (2.6) (below). Fix $w \in 1$. Then

(a)

$$|g(\varphi(rw)) - g(|\varphi(rw)|\varphi(w))| \leqq \left(\max_{|z| \leqq |\varphi(rw)|} |g'(z)|\right) \cdot |\varphi(rw) - |\varphi(rw)|\varphi(w)|$$

$$\leqq (1-r)n(r)\left(\max_{|z| \leqq \alpha(r)} |g'(z)|\right), \quad 0 \leqq r < 1.$$

We estimate $|g'(z)|$ by use of the Cauchy integral formula on the circle $|\xi| = (|z|+1)/2$ and the estimate $|g(\xi)| \leqq M_1(|\xi|)$. The result may be expressed as

(b) $$\max_{|z| \leqq \alpha(r)} |g'(z)| \leqq M_1\left(\frac{\alpha(r)+1}{2}\right)\left(\frac{1-\alpha(r)}{2}\right)^{-1}, \quad 0 \leqq r < 1.$$

Finally, let $D \xrightarrow{s} D$ be an *analytic bijection* satisfying $\varphi(s(0)) = 0$ (note 0 lies in the range of $\varphi$). Then $|\varphi(s(z))| \leqq |z|$, (by Schwarz's lemma) and

$$\sup_{0 \leqq r < 1} \frac{1-r}{1-\alpha(r)} = \sup_{z \in D} \frac{1-|z|}{1-|\varphi(z)|} = \sup_{z \in D} \frac{1-|s(z)|}{1-|\varphi(s(z))|} \leqq \sup_{z \in D} \frac{1-|s(z)|}{1-|z|}$$

We conclude

(c) $$\sup_{0 \leqq r < 1} \frac{1-r}{1-\alpha(r)} < \infty.$$

Combining (a), (b), (c), (2.4), and recalling that $f(rw) = g(\varphi(rw))$, gives

(2.6) $$\lim_{r \uparrow 1} |f(rw) - g(|\varphi(rw)|\varphi(w))| = 0, \text{ all } w \in I.$$

Next, we must verify statement (2.7) (below). Set

$$L = \{w \in I : \varliminf_{r \uparrow 1} \operatorname{Re} g(r\varphi(w)) < +\infty\}.$$

Since $\varphi^{-1}$ is analytic in a neighborhood of $\varphi(L)$, we have

$$m(L) = m[\varphi^{-1}(\varphi(L))] \leqq (\text{constant})m[\varphi(L)].$$

But

$$m[\varphi(L)] \leqq m\{u \in T : \varliminf_{r \uparrow 1} \operatorname{Re} g(ru) < +\infty\} = 0.$$

So

$$\varliminf_{r \uparrow 1} \operatorname{Re} g(r\varphi(w)) = +\infty, \text{ almost all } w \in I.$$

Since

$$\lim_{r\uparrow 1}|\varphi(rw)| = 1, \text{ all } w\in I,$$

we conclude

$$\lim_{r\uparrow 1}\operatorname{Re} g(|\varphi(rw)|\varphi(w)) = +\infty, \text{ almost all } w\in I. \tag{2.7}$$

Statements (2.6) and (2.7) together imply (2.1).

## 3. Approximation of indicator functions in measure.

LEMMA 3. *Let $I$ be a closed arc on $T$, let $M$ be a growth rate, and let $\varepsilon\in(0,1)$. Define $T\xrightarrow{s}\mathrm{R}$ by $s(w)=1$, all $w\in I$, and $s(w)=0$, all $w\in T-I$. Then there exists a set $\hat{T}$ and a function $f$ meeting these conditions:*

(3.1) *$\hat{T}\subset T$, $\hat{T}$ is a Borel set, $m(T-\hat{T})<\varepsilon$, and $f$ is analytic in a neighborhood of $\bar{D}$;*

(3.2) *$|f(z)|\leqq M(|z|)$, all $z\in D$;*

(3.3) *$|f(w)-s(w)|\leqq\varepsilon$, all $w\in\hat{T}$;*

(3.4) *$|f(rw)|\leqq 2|s(w)|+\varepsilon$, all $w\in\hat{T}$, all $r\in[0,1]$.*

PROOF. First, define a function $[0,1)\xrightarrow{M_1}\mathrm{R}$ as follows:

$$M_1(r)=\log(M(r)-1), \text{ all } r \text{ for which } M(r)>e^2+1;$$
$$M_1(r)=\min[e^{-1}M(r),2], \text{ all } r \text{ for which } M(r)\leqq e^2+1.$$

Then $M_1$ is a growth rate satisfying

$$\begin{aligned} &M_1(r)\leqq e^{-1}M(r), \text{ all } r \text{ for which } M_1(r)\leqq 2,\\ &1+\exp M_1(r)=M(r), \text{ all } r \text{ for which } M_1(r)>2. \end{aligned} \tag{3.5}$$

Secondly, let $B$ be a closed arc on $T$ such that

$$I\cap B=\varphi \quad\text{and}\quad m[T-(I\cup B)]<\varepsilon/4 \tag{3.6}$$

(closed arcs are proper subsets of $T$). Now apply Lemma 2. There is a function $g$ with these properties:

$$\begin{aligned} &g \text{ is of growth } M_1; \lim_{r\uparrow 1}\operatorname{Re} g(rw) = -\infty, \text{ almost all } w\in I;\\ &\sup\{|g(rw)|: w\in B,\ r\in[0,1)\}<\infty. \end{aligned} \tag{3.7}$$

We now construct $\hat{T}$ by reference to Egoroff's theorem. There is a set $\hat{T}$ with these properties:

$$\text{(3.8)} \qquad \begin{aligned}&\hat{T} \subset I \cup B;\ \hat{T} \text{ is a Borel set; } m[(I \cup B) - \hat{T}] < \varepsilon/4;\\ &\operatorname{Re} g(rw) \to -\infty \text{ uniformly on } I \cap \hat{T} \text{ as } r \uparrow 1.\end{aligned}$$

We now construct $f$. First, set

$$\begin{aligned}K(I) &= \sup\{\operatorname{Re} g(rw) : w \in I \cap \hat{T},\ r \in [0,1)\}\\ K(B) &= \sup\{|g(rw)| : w \in B \cap \hat{T},\ r \in [0,1)\}.\end{aligned}$$

From (3.7) and (3.8) it follows that

$$\text{(3.9)} \qquad -\infty < K(I) < \infty \quad \text{and} \quad 0 < K(B) < \infty.$$

Second, choose $\rho$ so that

$$\text{(3.10)} \quad \rho \in (0, \tfrac{1}{2}),\ 1 + \exp(\rho K(I)) < 2 + \varepsilon, \ \text{ and } \ (e^2 + 1)\rho K(B) < \varepsilon.$$

This is possible by (3.9). Third, choose $\delta$ with these properties:

$$\text{(3.11)} \qquad \delta \in (0,1);\ \exp[\rho \operatorname{Re}(g(\delta w))] < \varepsilon, \ \text{all } w \in I \cap \hat{T}.$$

This is possible by (3.8). Now, set

$$\text{(3.12)} \qquad f(z) = 1 - \exp[\rho g(\delta z)],\ z \in \bar{D}.$$

We proceed to verify statements (3.1) through (3.4). In the course of verification we shall occasionally make use of the fact that

$$\text{(3.13)} \qquad |1 - e^x| \leqq e|x|, \ \text{if } |x| \leqq 1.$$

Statement (3.1) follows from (3.6), (3.7), (3.8), and (3.12).

We verify (3.2). Let $z \in D$ with $M_1(|z|) \leqq 2$. By (3.7), (3.11), (3.10)

$$\rho|g(\delta z)| \leqq \rho M_1(\delta|z|) \leqq \rho M_1(|z|) \leqq 2\rho \leqq 1.$$

So, by (3.12), (3.13), (3.10), and (3.5)

$$|f(z)| \leqq e\rho|g(\delta z)| \leqq eM_1(|z|) \leqq M(|z|).$$

Now suppose $z \in D$ and $M_1(|z|) > 2$. Then, by (3.12) and (3.5),

$$|f(z)| \leqq 1 + \exp(\rho M_1(\delta|z|)) \leqq 1 + \exp M_1(|z|) = M(|z|).$$

This establishes (3.2).

We verify (3.3). If $w \in B \cap \hat{T}$, then

$$\rho|g(\delta w)| \leqq \rho K(B) \leqq \varepsilon(e^2 + 1)^{-1} < 1$$

by (3.10) (recall $\varepsilon \in (0,1)$). So, by (3.12). (3.13), and (3.10)

$$|f(w) - s(w)| = |f(w)| \leqq e\rho|g(\delta w)| \leqq e\rho K(B) < \varepsilon, \ \text{all } w \in B \cap \hat{T}.$$

If $w \in I \cap \hat{T}$, then

$$|f(w)-s(w)| = |f(w)-1| = \exp \mathrm{Re}(\rho g(\delta w)) < \varepsilon$$

by (3.11). Since $\hat{T} \subset I \cup B$, we have established (3.3).

We verify (3.4). If $w \in B \cap \hat{T}$ and $r \in [0,1]$ we have $\rho|g(r\delta w)| \leqq \rho K(B) < 1$ by (3.10). Hence, by (3.12), (3.13), and (3.10)

$$|f(rw)| \leqq e\rho K(B) < \varepsilon = 2|s(w)| + \varepsilon, \text{ all } w \in B \cap \hat{T}, \text{ all } r \in [0,1].$$

If $w \in I \cap \hat{T}$ and $r \in [0,1]$, then

$$|f(rw)| \leqq 1 + \exp \mathrm{Re}(\rho g(r\delta w)) \leqq 1 + \exp \rho K(I) < 2 + \varepsilon = 2|s(w)| + \varepsilon,$$

by (3.12) and (3.10). This establishes (3.4).

## 4. Approximation in measure of real functions on $T$.

LEMMA 4. *Let $T \xrightarrow{s} \mathrm{R}$ be measurable, let $\varepsilon < 0$, and let $M$ be a growth rate. Then there exists a set $\hat{T}$ and a function $f$ with these properties:*

(4.1) $\hat{T} \subset T$, *$\hat{T}$ is a Borel set,* $m(T - \hat{T}) < \varepsilon$,
*$f$ is analytic in a neighborhood of* $\bar{D}$;

(4.2) $|f(z)| \leqq M(|z|)$, $z \in D$;

(4.3) $|f(w) - s(w)| < \varepsilon$, $w \in \hat{T}$;

(4.4) $|f(rw)| \leqq 2|s(w)| + \varepsilon$, *all* $w \in \hat{T}$, *all* $r \in [0,1]$.

PROOF. By reference to general results on real functions, we may select mutually disjoint closed arcs $I(1), I(2), \ldots, I(N)$ and corresponding real numbers $c_1, c_2, \ldots, c_n$, and we may form a function $T \xrightarrow{\varphi} \mathrm{R}$ with the following properties:

$$\varphi(w) = \sum_{n=1}^{N} c_n \varphi_n(w), \text{ all } w \in T;$$

(4.5) $\varphi_n(w) = 1$, all $w \in I(n)$, and $\varphi_n(w) = 0$, all $w \in T - I(n)$, all $n \in \{1, 2, \ldots, N\}$;

$$m\{w \in T : |\varphi(w) - s(w)| > \varepsilon/4\} < \varepsilon/2; \quad \sum_{n=1}^{N} |c_n| \neq 0.$$

By reference to Lemma 3, there is a sequence of sets $(\hat{T})_{n=1}^{N}$ and a sequence of functions $(f_n)_{n=1}^{N}$ such that if $n \in \{1, 2, \ldots, N\}$ the following statements are valid:

(4.6) $\hat{T}_n \subset T$, $\hat{T}_n$ is a Borel set, $m(T - \hat{T}_n) \leqq \varepsilon(2N)^{-1}$,
and $f_n$ is analytic in a neighborhood of $\bar{D}$;

(4.7) $|f_n(z)| \leqq \left(\sum_{k=1}^{N} |c_k|\right)^{-1} M(|z|)$, all $z \in D$;

$$(4.8)\quad |f_n(w)-\varphi_n(w)| \leqq \left(2\sum_{k=1}^{N}|c_k|\right)^{-1}\varepsilon, \text{ all } w\in \hat{T}_n;$$

$$(4.9)\quad |f_n(rw)| \leqq 2|\varphi_n(w)| + \left(2\sum_{k=1}^{N}|c_k|\right)^{-1}\varepsilon, \text{ all } w\in \hat{T}_n, \text{ all } r\in[0,1].$$

Set

$$(4.10)\ f=\sum_{n=1}^{N} c_n f_n \text{ and } \hat{T} = \left(\bigcap_{n=1}^{N}\hat{T}_n\right) - \{w\in T: |\varphi(w)-s(w)|>\varepsilon/4\}$$

By (4.10), (4.5), (4.6)

$$m(T-\hat{T}) \leqq \sum_{n=1}^{N} m(T-\hat{T}) \leqq \sum_{n=1}^{N} m(T-\hat{T}_n)+\varepsilon/2 \leqq \varepsilon.$$

This verifies (4.1).

Statement (4.2) follows from (4.10) and (4.7).

Let $w\in\hat{T}$. Then, by (4.10) and (4.8)

$$|f(w)-s(w)| \leqq \sum_{n=1}^{N}|f_n(w)-\varphi_n(w)||c_n| + |\varphi(w)-s(w)| \leqq \varepsilon/2+\varepsilon/4.$$

This verifies (4.3).

Finally, if $r\in[0,1]$ and $w\in\hat{T}$, we have $w\in\bigcap_{n=1}^{N}\hat{T}_n$ and

$$|f(rw)| \leqq \sum_{n=1}^{N}|f_n(rw)||c_n| \leqq \sum_{n=1}^{N}\left(2|\varphi_n(w)|+\left(2\sum_{k=1}^{N}|c_k|\right)^{-1}\varepsilon\right)|c_n|$$
$$\leqq 2\sum_{n=1}^{N}|c_n\varphi_n(w)|+\varepsilon/2 = 2|\varphi(w)|+\varepsilon/2$$

because of (4.10) and (4.9) and because the arcs $I_n$, $n=1,2,\dots N$, are disjoint. If $w\in\hat{T}$, then $w\in\hat{T}_m$ for all $m\in\{1,2,\dots,N\}$ and $|\varphi(w)-s(w)| < \varepsilon/4$. Hence, if $r\in[0,1]$ and $w\in\hat{T}$, we have

$$|f(rw)| \leqq 2|\varphi(w)|+\varepsilon/2 \leqq 2(|\varphi(w)-s(w)|+|s(w)|)+\varepsilon/2$$
$$\leqq 2|s(w)|+2(\varepsilon/4)+\varepsilon/2.$$

This verifies (4.4).

## 5. Real measurable radial limits.

LEMMA 5. *Let $T\xrightarrow{s}\mathsf{R}$ be measurable and let $M$ be a growth rate. Then there exists a function $f$, of growth $M$, such that*

(5.1) $$\lim_{r\uparrow 1} f(rw) = s(w), \text{ almost all } w \in T.$$

PROOF. By Lemma 4 we may define inductively a sequence of functions $(f_n)_{n=0}^{\infty}$ and a sequence of sets $(T_n)_{n=0}^{\infty}$ with these properties:

(5.2) $f_0 \equiv 0,\ T_0 = \emptyset$;

(5.3) $T_n \subset T$, $T_n$ is a Borel set, $m(T - T_n) \leqq 2^{-n}$, and $f_n$ is analytic on $\bar{D}$, all $n \geqq 1$;

(5.4) $|f_n(z)| \leqq 2^{-n-1}M(|z|)$, all $z \in D$, all $n \geqq 0$;

(5.5) $|f_n(w) - (s(w) - f_0(w) - \ldots - f_{n-1}(w))| \leqq 2^{-n}$, all $w \in T_n$, all $n \geqq 1$;

(5.6) $|f_n(rw)| \leqq 2|s(w) - f_0(w) - \ldots - f_n^{-1}(w)| + 2^{-n}$, all $r \in [0,1]$, all $w \in T_n$, all $n \geqq 1$.

By (5.4), $\sum_{n=0}^{\infty}$ converges uniformly on compact subsets of $D$. Set

$$f = \sum_{n=0}^{\infty} f_n,\ E = \bigcap_{n=0}^{\infty}\left[\bigcup_{k=n}^{\infty}(T - T_k)\right] \text{ and } \hat{T} = T - E.$$

By (5.4), $|f(z)| \leqq M(|z|)$, all $z \in D$. Also, by (5.3)

$$m(E) \leqq \sum_{k=n}^{\infty} m(T - T_k) \leqq 2^{-n+1}, \text{ all } n \geqq 1.$$

So $m(E) = 0$ and $m(\hat{T}) = 2\pi$.

We shall prove that

$$\lim_{r\uparrow 1} f(rw) = s(w), \text{ all } w \in \hat{T}.$$

Fix $w \in \hat{T}$ and fix $\varepsilon > 0$. Then $w \in T_n$ if $n$ is sufficiently large. Hence, we may choose $N(w)$ so that $w \in T_n$, all $n \geqq N(w)$, and so that

$$\sum_{n=N(w)+1}^{\infty} 2^{-n} \leqq \varepsilon/16.$$

Each $f_n$ is analytic on $\bar{D}$; so we may choose $r(\varepsilon, w) \in (0,1)$ so that

(5.7) $$\left|\sum_{n=0}^{N(w)} f_n(rw) - \sum_{n=0}^{N(w)} f_n(w)\right| \leqq \varepsilon/4, \text{ if } r(\varepsilon,w) \leqq r \leqq 1.$$

Now suppose $n \leqq N(w) + 1$ and $0 \leqq r < 1$. Then $w \in T_n$ and reference to (5.6), and then (5.5), shows

$|f_n(rw)| \leqq 2|s(w) - f_0(w) - - f_{n-1}(w)| + 2^{-n} \leqq 2 \cdot 2^{-(n-1)} + 2^{-n} = 5 \cdot 2^{-n}$;

and

$$\sum_{n=N(w)+1}^{\infty} |f_n(rw)| \leqq 5 \sum_{n=N(w)+1}^{\infty} 2^{-n} \leqq 5\varepsilon/16.$$

Hence

$$\left| f(rw) - \sum_{n=0}^{N(w)} f_n(rw) \right| \leqq 5\varepsilon/16, \text{ all } r \in [0,1). \tag{5.8}$$

Finally, by (5.5) we have (since $w \in T_{N(w)}$ and $\sum_{n \geqq N(w)+1} 2^{-n} \leqq \varepsilon/16$

$$|s(w) - \sum_{n=0}^{N(w)} f_n(w)| \leqq 2^{-N(w)} \leqq \varepsilon/8. \tag{5.9}$$

Combining (5.7), (5.8), and (5.9) gives

$$|f(rw) - s(w)| \leqq \varepsilon, \text{ if } r(\varepsilon,w) \leqq r < 1.$$

This establishes (5.1).

## 6. Extended real values on measurable sets.

LEMMA 6. *Let $A \subset T$ be a Borel set and let $M$ be a growth rate. Then there exists a function $f$, of growth $M$, with these properties:*

(6.1) $\lim_{r \uparrow 1} \operatorname{Re} f(rw) = +\infty$, *almost all* $w \in A$;

(6.2) $\lim_{r \uparrow 1} \operatorname{Re} f(rw) = 0$, *almost all* $w \in T - A$;

(6.3) $\lim_{r \uparrow 1} \operatorname{Im} f(rw) = 0$, *almost all* $w \in T$.

PROOF. Let $G$ denote an entire function with the properties specified in (2) of Lemma 0 and fix $K > 0$ so that $|G(0)| < KM(0)$. Then for each $r \in [0,1)$ there is a unique positive real number $M_1(r)$ such that

$$\max_{|z| \leqq M_1(r)} |G(z)| = KM(r) \tag{6.4}$$

(since $G$ is non-constant and entire). The elementary properties of entire functions imply that the function $[0,1) \xrightarrow{M_1} (0,\infty)$ is a growth rate.

By Lemma 1 and Lemma 5, there is a function $g$, of growth $M_1$, satisfying these properties:

$$\lim_{r\uparrow 1} \operatorname{Re} g(rw) = +\infty, \quad \textit{almost all } w \in T,$$

$$(6.5) \quad \lim_{r\uparrow 1} \operatorname{Im} g(rw) = 1, \quad \text{almost all } w \in A,$$

$$\lim_{r\uparrow 1} \operatorname{Im} g(rw) = -1, \quad \text{almost all } w \in T - A.$$

Set $f = K^{-1} G \circ g$. Then

$$|f(z)| \leqq K^{-1} \sup_{|\xi| \leqq |g(z)|} |G(\xi)| \leqq K^{-1} \sup_{|\xi| \leqq M_1(|z|)} |G(\xi)| = M(|z|), \text{ all } z \in D$$

(by (6.5)).

Statements (6.1), (6.2), and (6.3) follow immediately from (2) of Lemma 0 and (6.5).

**7. Specified oscillation.**

LEMMA 7. *Let $\underline{m}$ and $\bar{m}$ be elements of $\overline{\mathrm{R}}$ for which $\underline{m} \leqq \bar{m}$. Let $A$ denote a Borel subset of $T$ and let $M$ be a growth rate. Then there exists a function $f$, of growth $M$, with these properties:*

$$(7.1) \quad \varliminf_{r\uparrow 1} \operatorname{Re} f(rw) = \underline{m}, \ \lim_{r\uparrow 1} \operatorname{Re} f(rw) = \bar{m}, \ \textit{and}$$

$$\lim_{r\uparrow 1} \operatorname{Im} f(rw) = 0, \ \textit{almost all } w \in A;$$

$$(7.2) \quad \lim_{r\uparrow 1} f(rw) = 0, \ \textit{almost all } w \in T - A.$$

PROOF. Let $G$ denote an entire function with the properties specified in (3) of Lemma 0 and choose $K > 0$ so that $|G(0)| < KM(0)^{1/2}$. For each $r \in [0,1)$, there is a unique real number $M_1(r)$ such that

$$(7.4) \qquad \max_{|z| \leqq M_1(r)} |G(z)| = KM(r)^{1/2}.$$

The function $[0,1) \xrightarrow{M_1} (0,\infty)$ is a growth rate, and Lemma 6 implies the existence of a function $g$, of growth $M_1$, with these properties:

$$\lim_{r\uparrow 1} \operatorname{Re} g(rw) = +\infty, \quad \text{almost all } w \in A;$$

$$(7.5) \quad \lim_{r\uparrow 1} \operatorname{Re} g(rw) = -\infty, \quad \text{almost all } w \in T - A;$$

$$\lim_{r\uparrow 1} \operatorname{Im} g(rw) = 0, \quad \text{almost all } w \in T.$$

For almost all $w \in A$ there is a corresponding number $s \in [0,1)$ such that the corresponding curve $\gamma(t) \equiv g(w(1-t)s + wt)$, $t \in (0,1)$ satisfies

$$|\operatorname{Im} \gamma(t)| \leq 1 \text{ all } t \in (0,1) \text{ and } \lim_{t \uparrow 1} \operatorname{Re} \gamma(t) = +\infty.$$

Thus, if we set $F = -ik^{-1}G \circ g$ and refer to part 3 of Lemma 0, (7.4), and (7.5), we see that $F$ has these properties:

$$\begin{aligned}
&F \text{ is of growth } M^{1/2}; \ \varliminf_{r \uparrow 1} \operatorname{Re} F(rw) = K^{-1}\underline{m}, \text{ almost all } w \in A\\
&\varlimsup_{r \uparrow 1} \operatorname{Re} F(rw) = K^{-1}\bar{m}, \text{ almost all } w \in A;\\
&\lim_{r \uparrow 1} \operatorname{Im} F(rw) = -K^{-1}, \text{ almost all } w \in A; \qquad (7.6)\\
&\lim_{r \uparrow 1} F(rw) = 0, \text{ almost all } w \in T - A.
\end{aligned}$$

It remains only to modify $F$. We first multiply $F$ by a function of growth $\frac{1}{2}M^{1/2}$ with radial limits $K$ almost everywhere on $T$ (Lemma 5). We then add a function of growth $\frac{1}{2}M$ with radial limits $i$ almost everywhere on $A$ and radial limits 0 almost everywhere on $T - A$ (again apply Lemma 5). By (7.6), the resulting function satisfies (7.1) and (7.2).

## 8. Proof of Theorem 1.

We shall prove Theorem 1 under the additional assumption that ${}_*f_2(w) = {}^*f_2(w) = 0$, all $w \in T$. The general case then follows immediately. For ease of notation set $g_1 = {}_*f_1$ and $g_2 = {}^*f_1$.

Consider the following six sets:

$$\begin{aligned}
A_1 &= \{w \in T : g_1(w) = +\infty &&\text{and} \quad g_2(w) = +\infty\},\\
A_2 &= \{w \in T : g_1(w) \in \mathsf{R} &&\text{and} \quad g_2(w) = +\infty\},\\
A_3 &= \{w \in T : g_1(w) = -\infty &&\text{and} \quad g_2(w) = +\infty\},\\
A_4 &= \{w \in T : g_1(w) \in \mathsf{R} &&\text{and} \quad g_2(w) \in \mathsf{R}\},\\
A_5 &= \{w \in T : g_1(w) = -\infty &&\text{and} \quad g_2(w) \in \mathsf{R}\},\\
A_6 &= \{w \in T : g_1(w) = -\infty &&\text{and} \quad g_2(w) = -\infty\}.
\end{aligned}$$

We shall construct functions $f_j, j = 1,2,\ldots,6$, each of growth $\frac{1}{6}M$, with the following properties:

$$\varliminf_{r \uparrow 1} \operatorname{Re} f_j(rw) = g_1(w) \text{ and } \varlimsup_{r \uparrow 1} \operatorname{Re} f_j(rw) = g_2(w), \text{ almost all } w \in A_j;$$

$$\lim_{r \uparrow 1} \operatorname{Re} f_j(rw) = 0, \text{ almost all } w \in T - A_j;$$

$\lim_{r\uparrow 1} \operatorname{Im} f_j(rw) = 0$, almost all $w \in T$.

Once this is done, $f = \sum_{j=1}^{6} f_j$ has the required properties.

For $j = 1, 3, 6$, the existence of $f_j$ follows from Lemma 7.

Consider the case $j = 2$. By Lemma 7 there is a function $h_1$, of growth $(12)^{-1}M$, with these properties:

$$\varliminf_{r\uparrow 1} \operatorname{Re} h_1(rw) = 0, \ \varlimsup_{r\uparrow 1} \operatorname{Re} h_1(rw) = +\infty,$$

$$\lim_{r\uparrow 1} \operatorname{Im} h_1(rw) = 0, \text{ almost all } w \in A_2;$$

$$\lim_{r\uparrow 1} h_1(rw) = 0, \text{ almost all } w \in T - A_2.$$

By Lemma 5 there is a function $h_2$, of growth $(12)^{-1}M$, with these properties:

$$\lim_{r\uparrow 1} h_2(rw) = g_1(w), \text{ almost all } w \in A_2;$$

$$\lim_{r\uparrow 1} h_2(rw) = 0, \text{ almost all } w \in T - A_2.$$

Clearly $f_2 = h_1 + h_2$ has the required properties.

Existence in case $j = 5$ follows from that in case 2 (multiply all functions by $(-1)$).

Consider the case $j = 4$. By Lemmas 5 and 7 there exist functions $h_1$ and $h_2$ of growth $(12^{-1}M)^{1/2}$ with these properties:

$$\lim_{r\uparrow 1} h_1(rw) = \frac{g_2(w) - g_1(w)}{2}, \ \varliminf_{r\uparrow 1} \operatorname{Re} h_2(rw) = -1,$$

$$\varlimsup_{r\uparrow 1} \operatorname{Re} h_2(rw) = 1, \text{ and } \lim_{r\uparrow 1} \operatorname{Im} h_2(rw) = 0, \text{ almost all } w \in A_4;$$

$$\lim_{r\uparrow 1} h_1(rw) = \lim_{r\uparrow 1} h_2(rw) = 0, \text{ almost all } w \in T - A_4.$$

By Lemma 5 there is a function $h_3$, of growth $12^{-1}M$, with these properties:

$$\lim_{r\uparrow 1} h_3(rw) = \frac{g_1(w) + g_2(w)}{2}, \text{ almost all } w \in A_4;$$

$$\lim_{r\uparrow 1} h_3(rw) = 0, \text{ almost all } w \in T - A_4.$$

Clearly $f_4 = h_3 + h_1 \cdot h_2$ has the required properties.

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