

A REVISED AUGMENTED CUNTZ SEMIGROUP

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Abstract

We revise the construction of the augmented Cuntz semigroup functor used by the first author to classify inductive limits of 1-dimensional noncommutative CW complexes. The original construction has good functorial properties when restricted to the class of C^* -algebras of stable rank one. The construction proposed here has good properties for all C^* -algebras: we show that the augmented Cuntz semigroup is a stable, continuous, split exact functor, from the category of C^* -algebras to the category of Cu-semigroups.

1. Introduction

The Elliott classification program purports to distinguish “well behaved” simple nuclear separable C^* -algebras up to isomorphism relying on essentially two kinds of data: K -groups and the cone of traces. This classification program has recently come almost to completion, culminating decades of research. In these recent developments the Cuntz semigroup plays a key role as a tool to define and exploit the regularity properties of the classifiable C^* -algebras. When looking into classifying non-simple C^* -algebras, the Cuntz semigroup itself becomes a good candidate for a classification invariant. One limitation of the Cuntz semigroup, however, is that it may fail to capture the K_0 -group for non-unital C^* -algebras. To remedy this, a variation on the Cuntz semigroup was introduced in [10]. This ordered semigroup, which we call here the *augmented Cuntz semigroup*, is built out of the Cuntz semigroup of the unitization of the C^* -algebra, resembling the way in which the K_0 -group is built out of the monoid of Murray-von Neumann classes of projections of the unitization of the C^* -algebra. The augmented Cuntz semigroup was used in [10] to classify certain inductive limits of 1-dimensional noncommutative CW complexes.

Let $\widetilde{\text{Cu}}$ denote the augmented Cuntz semigroup functor. A shortcoming of the construction of $\widetilde{\text{Cu}}$ given in [10] is that it yields a well behaved functor only on the class of stable rank one C^* -algebras. In the present paper we address this issue by revising the definition of $\widetilde{\text{Cu}}$. More specifically, we change the way in which formal differences of Cuntz semigroup elements are built. The new

construction agrees with the old one for C^* -algebras of finite stable rank. But we are now able to establish the basic properties of the functor Cu^\sim in greater generality. We show that Cu^\sim is a functor from the category of C^* -algebra to the category of Cu-semigroups (Theorem 4.4), that it preserves inductive limits (Theorem 5.1), and that Cu^\sim is stable and split exact (Theorems 5.4 and 5.3). Further, we calculate $\text{Cu}^\sim(A)$ for A simple, separable, and pure (Theorem 6.11). Again, we do this without assuming that A has stable rank one. Under more restrictive hypotheses, although also circumventing the stable rank one hypothesis, this calculation is obtained in [7, Appendix A].

This paper is organized as follows: §2 is devoted to preliminaries on Cu-semigroups. In §3 we describe a “formal differences” construction for Cu-semigroups, which we use to define $\text{Cu}^\sim(A)$. In §4 we define $\text{Cu}^\sim(A)$ and show, among other results, that it is a Cu-semigroup. In §5 we prove the functorial properties of Cu^\sim mentioned above. In §6 we show that if A is stably finite then $K_0(A)$ and the cone of densely finite 2-quasitraces on A can be read off of $\text{Cu}^\sim(A)$. Further, if A is simple and pure, $\text{Cu}^\sim(A)$ can be calculated in terms of $K_0(A)$, the cone of densely finite 2-quasitraces on A , and the pairing between these two invariants.

2. Preliminaries

Here we briefly recall some facts on abstract Cu-semigroups. The reader is referred to [1] and references therein for more on this topic.

Let S be an ordered set such that every increasing sequence has a supremum. Given $x, y \in S$ we write $x \ll y$ if whenever $y \leq \sup_n y_n$ for some increasing sequence $(y_n)_n$, then there exists $n_0 \in \mathbb{N}$ such that $x \leq y_{n_0}$. We say then that x is compactly contained in y , or way below y . An element $e \in S$ is called compact if $e \ll e$.

Suppose now that S is an ordered semigroup with neutral element 0. Consider the following properties on S :

- O0 0 is a compact element
- O1 Every increasing sequence in S has a supremum.
- O2 For each $x \in S$ there exists an \ll -increasing sequence $(x_n)_n$ such that $x = \sup x_n$.
- O3 If $x_1 \ll y_1$ and $x_2 \ll y_2$ then $x_1 + x_2 \ll y_1 + y_2$.
- O4 If $(x_n)_n$ and $(y_n)_n$ are increasing sequences then $\sup(x_n + y_n) = \sup x_n + \sup y_n$.

In the literature ([1], [6]) a Cu-semigroup is typically understood to be a positively ordered semigroup (i.e., one for which $0 \leq x$ for all $x \in S$) satisfying O1–O4. Notice that such semigroups automatically satisfy O0. We will

consider here ordered semigroups that are not necessarily positively ordered. Instead, by a Cu-semigroup we shall understand an ordered semigroup that satisfies axioms O0–O4.

By a Cu-morphism we understand a map $\alpha: S \rightarrow T$ between Cu-semigroups that is an ordered semigroup morphism (preserves order and addition), preserves neutral elements, the suprema of increasing sequences, and the compact containment relation. These last two properties mean that if $(x_n)_{n=1}^\infty$ is an increasing sequence in S then $\alpha(\sup x_n) = \sup \alpha(x_n)$ and if $x, y \in S$ are such that $x \ll y$ then $\alpha(x) \ll \alpha(y)$.

The category Cu is defined as the category of Cu-semigroups with Cu-morphisms. If A is a C^* -algebra then its Cuntz semigroup $\text{Cu}(A)$ is a Cu-semigroup, [6, Theorem 1]. By [6, Theorem 2], the Cuntz semigroup functor $\text{Cu}(\cdot)$ is a functor from the category of C^* -algebras to the category Cu. Moreover, this functor preserves inductive limits, [1, Theorem 2.1.10].

We will make use below of an additional axiom satisfied by the Cuntz semigroup of a C^* -algebra:

O5 For all $x' \ll x \leq y$ and $w' \ll w$ such that $x + w \leq y$ there exists z such that $x' + z \leq y \leq x + z$ and $w' \ll z$.

We some times refer to O5 as the “almost algebraic order” axiom. That $\text{Cu}(A)$ satisfies O5 as stated above is proven in [1]. A slightly weaker version, which does not seem to suffice for our purposes, is proven in [14].

We call a positive element $z \in S$ full if $\infty \cdot z := \sup_n nz$ is the largest element of S .

3. The S_{cc} construction

Throughout this section S is a positively ordered Cu-semigroup that satisfies the almost algebraic order axiom O5.

Recall that an element $e \in S$ is called compact if $e \ll e$. We denote the subsemigroup of compact elements of S by S_c , i.e., $S_c = \{e \in S : e \ll e\}$. Let us define on $S \times S_c$ a relation as follows: $(x, e) \lesssim_{cc} (y, f)$ if for all $x' \ll x$ there exists $g \in S_c$ such that

$$x' + f + g \ll y + e + g.$$

Since y is the supremum of a \ll -increasing sequence, once this inequality holds we can then choose $y' \ll y$ such that $x' + f + g \leq y' + e + g$. Thus, $(x, e) \lesssim_{cc} (y, f)$ if and only if for all $x' \ll x$ there exist $y' \ll y$ and $g \in S_c$ such that $x' + f + g \leq y' + e + g$. Equipped with this observation it is straightforward to check that the relation \lesssim_{cc} is transitive. Another observation

easily checked, and which we will use frequently, is that if $x' \lesssim_{\text{cc}} y$ for all $x' \ll x$ then $x \lesssim_{\text{cc}} y$.

By anti-symmetrizing \lesssim_{cc} we obtain an equivalence relation: we write $(x, e) \sim_{\text{cc}} (y, f)$ if $(x, e) \lesssim_{\text{cc}} (y, f)$ and $(y, f) \lesssim_{\text{cc}} (x, e)$. We denote the equivalence class of $(x, e) \in S \times S_c$ by $\overline{(x, e)}$.

Let $S_{\text{cc}} = (S \times S_c) / \sim_{\text{cc}}$. Let us endow S_{cc} with an order and an addition operation. The order on S_{cc} is the one induced by the pre-order \lesssim_{cc} :

$$\overline{(x, e)} \leq \overline{(y, f)} \quad \text{if } (x, e) \lesssim_{\text{cc}} (y, f).$$

Addition in S_{cc} is defined in the obvious way:

$$\overline{(x, e)} + \overline{(y, f)} := \overline{(x + y, e + f)}.$$

It is straightforward to check that addition is well defined and compatible with the order. Thus, S_{cc} is an ordered monoid with neutral element $\overline{(0, 0)}$.

Given $x \in S$ let us denote $\overline{(x, 0)}$, the equivalence class of $(x, 0)$, simply by \bar{x} . Let us denote the neutral element $\overline{(0, 0)}$ by 0 . If e is a compact element then $\bar{e} + \overline{(0, e)} = 0$, i.e., $\overline{(0, e)}$ is the additive inverse of \bar{e} . We can thus write $\overline{(x, e)}$ as $\bar{x} - \bar{e}$. This is the form in which we typically write the elements of S_{cc} :

$$S_{\text{cc}} = \{ \bar{x} - \bar{e} : x \in S, e \in S_c \}.$$

Suppose that $e \in S_c$ is an order unit of the semigroup S_c . That is, for any $f \in S_c$ we have $f \leq ne$ for some $n \in \mathbb{N}$. If $f \leq ne$, with f, e compact, then $f + f' = ne$ for some $f' \in S_c$ (a consequence of the almost algebraic order axiom O5). It follows that every element of S_{cc} can be expressed in the form $\bar{x} - n\bar{e}$ for some $x \in S$ and $n \in \mathbb{N}$. That is,

$$S_{\text{cc}} = \{ \bar{x} - n\bar{e} : x \in S, n \in \mathbb{N} \}.$$

EXAMPLE 3.1. If $S = \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ then $S_{\text{cc}} \cong \overline{\mathbb{Z}} := \mathbb{Z} \cup \{\infty\}$.

LEMMA 3.2. *The map $x \mapsto \bar{x}$ from S to S_{cc} is a surjection onto the set of positive elements of S_{cc} .*

PROOF. If $0 \leq \bar{x} - \bar{e}$ then $e + f \leq x + f$ for some $f \in S_c$. By O5, $e + f$ is complemented in $x + f$. That is, there exists $x' \in S$ such that $x + f = x' + e + f$. Then $\bar{x}' = \bar{x} - \bar{e}$.

LEMMA 3.3. *If $\bar{x} \leq \bar{y}$ and $x' \ll x$ then there exists $z \in S$ such that $x' \ll z$ and $\bar{z} = \bar{y}$.*

PROOF. Choose x'' such that $x' \ll x'' \ll x$. Since $\bar{x} \leq \bar{y}$, there exists $e \in S_c$ such that $x'' + e \leq y + e$. By O5, e is complemented in $y + e$ and

this complement may be chosen way above x' . That is, there exists z such that $z + e = y + e$ and $x' \ll z$. Thus, z is as desired.

THEOREM 3.4. *Every increasing sequence in S_{cc} has a supremum. Moreover, if $(x_n)_{n=1}^\infty$ is an increasing sequence of positive elements in S_{cc} then we can choose an increasing sequence $(z_n)_{n=1}^\infty$ in S such that $\bar{z}_n \leq x_n$ for all n and $\sup_n x_n = \overline{\sup_n z_n}$.*

PROOF. Let $(x_n)_{n=1}^\infty$ be an increasing sequence in S_{cc} . Given $e \in S_c$, the translation map $x \mapsto x + \bar{e}$ is an ordered set isomorphism of S_{cc} . Thus, applying one such translation we may assume that $x_1 \geq 0$. Then, by Lemma 3.2, for each n we can choose $y_n \in S$ such that $x_n = \bar{y}_n$.

We construct recursively sequences $(z_n)_n$ and $(z'_n)_n$ in S satisfying that

- (1) $\bar{z}'_n = \bar{y}_n$ for all n ,
- (2) $z_n \ll z_{n+1} \ll z'_n$ for all n ,
- (3) if $y \ll y_m$ for some m then $\bar{y} \leq \bar{z}_n$ for some n .

Let us first choose for each n a \ll -increasing sequence $(y_{n,k})_k \subseteq S$ such that $y_n = \sup_k y_{n,k}$. Set $z_1 = 0$ and $z'_1 = y_1$. Assume that $z_1, z'_1, \dots, z_n, z'_n$ have already been chosen. Since $\overline{y_{n-1}} \leq \bar{y}_n = \bar{z}'_n$, there exists $e \in S_c$ such that $y_{n-1,n+1} + e \leq z'_n + e$. We can thus choose $z_{n+1} \ll z'_n$ such that $y_{n-1,n} + e \leq z_{n+1} + e$. Hence, $\overline{y_{n-1,n}} \leq \overline{z_{n+1}}$. Moreover, increasing z_{n+1} along an \ll -increasing sequence with supremum z'_n , we can arrange that $\overline{y_{k,n}} \leq \overline{z_{n+1}}$ for $k = 1, \dots, n-1$ and also that $z_n \ll z_{n+1}$ (since $z_n \ll z'_n$). Now, using that $\bar{z}'_n = \bar{y}_n \leq \bar{y}_{n+1}$ and Lemma 3.3, we choose $z'_{n+1} \in S$ such that $z_{n+1} \ll z'_{n+1}$ and $\overline{z'_{n+1}} = \bar{y}_{n+1}$. We continue this process ad infinitum. The sequences $(z_n)_n$ and $(z'_n)_n$ obtained in this way satisfy (1), (2), and (3), above.

Observe that $(z_n)_n$ is increasing. Let $z = \sup z_n$. Let us show that $\bar{z} = \sup \bar{y}_n$. Fix y_m . Let $y \ll y_m$. Then $\bar{y} \leq \bar{z}_n$ for some n and so $\bar{y} \leq \bar{z}$. Since this holds for all $y \ll y_m$, we conclude that $\overline{y_m} \leq \bar{z}$. Thus, \bar{z} is an upper bound for the sequence $(\bar{y}_n)_n$. Suppose that $w \in S$ is such that $\bar{y}_n \leq \bar{w}$ for all n . Choose $z' \ll z$. Then $z' \ll z_n \leq z'_n \sim_{cc} y_n$ for some n . Hence, $z' \lesssim_{cc} w$. Since $z' \ll z$ is arbitrary, it follows that $z \lesssim_{cc} w$, as desired.

LEMMA 3.5. *The map $x \mapsto \bar{x}$ preserves suprema of increasing sequences and the compact containment relation.*

PROOF. Let $(x_n)_n$ be an increasing sequence in S with $x = \sup x_n$. Choose $z \in S$ such that $\bar{z} = \sup \bar{x}_n$. Since $\bar{x}_n \leq \bar{x}$ for all n , we have that $\bar{z} \leq \bar{x}$. Let us prove the opposite inequality. Let $x' \ll x$. Then $x' \ll x_n$ for some n . Since $\overline{x_n} \leq \bar{z}$, there exists $e \in S_c$ such that $x' + e \leq z + e$. This shows that $\bar{x} \leq \bar{z}$, as desired.

Let us now prove preservation of compact containment. Let $x', x \in S$ with $x' \ll x$. Suppose that $\bar{x} \leq \sup \bar{y}_n$, where $(\bar{y}_n)_n$ is increasing. By Theorem 3.4, there exists an increasing sequence $(z_n)_n$ in S such that $\overline{\sup z_n} = \sup \bar{y}_n$ and $\bar{z}_n \leq \bar{y}_n$ for all n . Set $z = \sup z_n$. Since $\bar{x} \leq \bar{z}$, there exist $z' \ll z$ and $e \in S_c$ such that $x' + e \leq z' + e$. Then $z' \leq z_{n_0}$ for some n_0 . Hence, $\bar{x}' \leq \bar{z}_{n_0} \leq \bar{y}_{n_0}$. This shows that $\bar{x}' \ll \bar{x}$.

THEOREM 3.6. *Let S be a positively ordered Cu-semigroup satisfying O5. Then the ordered semigroup S_{cc} defined above is a Cu-semigroup also satisfying O5.*

PROOF. We have already proven the existence of sequential suprema in S_{cc} .

Let us prove O2. Given $\bar{x} - \bar{e} \in S_{cc}$, we can choose an \ll -increasing sequence $(x_n)_n$ in S with supremum x . Since, by the previous lemma, the map $z \mapsto \bar{z}$ is supremum and \ll preserving, the sequence (\bar{x}_n) is \ll -increasing and has supremum \bar{x} . Finally, the map $\bar{z} \mapsto \bar{z} - \bar{e}$ is an order isomorphism. Thus, $(\bar{x}_n - \bar{e})$ is \ll -increasing and has supremum $\bar{x} - \bar{e}$.

Let us prove O3. Translating the two sequences in this axiom by a suitable \bar{e} , with $e \in S_c$, we may assume that their terms are positive. Let (\bar{x}_n) and (\bar{y}_n) be increasing sequences of positive elements in S_{cc} . One inequality is clear: since $\bar{x}_n + \bar{y}_n \leq \sup \bar{x}_n + \sup \bar{y}_n$ for all n ,

$$\sup(\bar{x}_n + \bar{y}_n) \leq \sup \bar{x}_n + \sup \bar{y}_n.$$

Using Theorem 3.4, choose an increasing sequences $(z_n)_n$ in S such that $\bar{z}_n \leq \bar{x}_n$ and $\overline{\sup z_n} = \sup \bar{x}_n$. Similarly, choose (w_n) increasing and such that $\bar{w}_n \leq \bar{y}_n$ and $\overline{\sup w_n} = \sup \bar{y}_n$. By O3 in S we have that

$$\sup z_n + \sup w_n = \sup(z_n + w_n).$$

Passing to S_{cc} , and applying that $z \mapsto \bar{z}$ is supremum preserving on the left side, we get

$$\sup_n \bar{x}_n + \sup_n \bar{y}_n = \sup_n (\bar{z}_n + \bar{w}_n) \leq \sup_n (\bar{x}_n + \bar{y}_n).$$

Let us prove O4. Translating by a suitable $\bar{e} \in S_c$, we may again assume that all the elements involved are positive. Suppose that $\bar{x}_1 \ll \bar{x}_2$ and $\bar{y}_1 \ll \bar{y}_2$. Then $\bar{x}_1 \leq \bar{x}'_2$ for some $x'_2 \ll x_2$ and $\bar{y}_1 \leq \bar{y}'_2$ for some $y'_2 \ll y_2$. By O4 in S , $x'_2 + y'_2 \ll x_2 + y_2$. Thus, using that $z \mapsto \bar{z}$ preserves compact containment,

$$\bar{x}_1 + \bar{y}_1 \leq \bar{x}'_2 + \bar{y}'_2 \ll \bar{x}_2 + \bar{y}_2.$$

Finally, let us prove O5. Translating by a suitable $\bar{e} \in S_c$, we may again assume that all the elements involved are positive. Suppose that we have elements $\bar{x}, \bar{y}, \bar{w} \in S_{cc}$ such that $\bar{x} + \bar{w} \leq \bar{y}$. Let $x' \ll x'' \ll x, w' \ll w'' \ll w$. Then

$$x'' + w'' + e \leq y + e$$

for some $e \in S_c$. By O5 in S , there exists z such that $w' \ll z$ and

$$x' + z + e \leq y + e \leq x'' + e + z.$$

Passing to S_{cc} we find that $\bar{x}' + \bar{z} \leq \bar{y} \leq \bar{x} + \bar{z}$ and $\bar{w}' \ll \bar{z}$. This proves O5 in S_{cc} .

Let $\alpha: S \rightarrow T$ be a Cu-morphism. Since α maps compact elements to compact elements, we have a map $(x, e) \mapsto (\alpha(x), \alpha(e))$ from $S \times S_c$ to $T \times T_c$.

LEMMA 3.7. *The map $(x, e) \mapsto (\alpha(x), \alpha(e))$ preserves the \lesssim_{cc} relation.*

PROOF. Let (x, e) and (y, f) be pairs in $S \times S_c$ such that $(x, e) \lesssim_{cc} (y, f)$. Let $z \ll \alpha(x)$. Choose $x' \ll x$ such that $z \leq \alpha(x') \ll \alpha(x)$. Then $x' + f + g \leq y + e + g$ for some $g \in S_c$. Then,

$$z + \alpha(f) + \alpha(g) \leq \alpha(x') + \alpha(f) + \alpha(g) \leq \alpha(y) + \alpha(e) + \alpha(g).$$

It follows that $(\alpha(x), \alpha(e)) \lesssim_{cc} (\alpha(y), \alpha(f))$.

In view of the previous lemma, we can define $\alpha_{cc}: S_{cc} \rightarrow T_{cc}$ by

$$\alpha_{cc}(\bar{x} - \bar{e}) = \overline{\alpha(x)} - \overline{\alpha(e)}.$$

THEOREM 3.8. *The map $\alpha_{cc}: S_{cc} \rightarrow T_{cc}$ is a Cu-morphism.*

PROOF. Additivity, preservation of order, and preservation of 0 are straightforward to check.

Let us prove the preservation of sequential suprema. Translating by compact elements, it suffices to consider sequences of positive elements. Let $(\bar{x}_n)_n$ be an increasing sequence in S_{cc} . The inequality $\sup \alpha_{cc}(\bar{x}_n) \leq \alpha_{cc}(\sup \bar{x}_n)$ is clear. Choose an increasing sequence $(z_n)_n \subseteq S$ such that $\overline{\sup z_i} = \sup \bar{x}_n$ and $\bar{z}_n \leq \bar{x}_n$ for all n . Set $z = \sup z_n$. Since α is a Cu-morphism, $\sup_n \alpha(z_n) = \alpha(z)$. Passing to S_{cc} we get

$$\alpha_{cc}(\bar{z}) = \sup \alpha_{cc}(\bar{z}_n) \leq \sup \alpha_{cc}(\bar{x}_n),$$

where we have used that the map $x \mapsto \bar{x}$ is supremum preserving. Thus, $\alpha_{cc}(z) = \sup \alpha_{cc}(\bar{x}_n)$.

Finally, let us prove preservation of compact containment: Again translating all the elements involved it suffices to deal with positive elements only. Say $\bar{x} \ll \bar{y}$ in S_{cc} . Since $\bar{y} = \sup_{y' \ll y} \bar{y}'$, there exists $y' \ll y$ such that $\bar{x} \leq \bar{y}'$. Now, $\alpha(y') \ll \alpha(y)$. Hence,

$$\alpha_{\text{cc}}(\bar{x}) \leq \alpha_{\text{cc}}(\bar{y}') = \overline{\alpha(y')} \ll \overline{\alpha(y)} = \alpha(\bar{y}),$$

where we have used that the map $z \mapsto \bar{z}$ preserves \ll .

It is straightforward to check that $S \mapsto S_{\text{cc}}, \alpha \mapsto \alpha_{\text{cc}}$ is a functor from the category Cu to itself.

In [6, Theorem 2] it is shown that sequential inductive limits exist in the category Cu . This is extended to arbitrary inductive limits in [1] (see [1, Theorem 2.1.10]). It follows from the proofs of these theorems that inductive limits in the category Cu can be characterized as follows: Let $(S_i, \alpha_{i,j})_{i,j \in I}$ be an inductive system in Cu . A semigroup S with maps $\alpha_{i,\infty}: S_i \rightarrow S$ is the inductive limit of $(S_i, \alpha_{i,j})_{i,j}$ in the category Cu if and only if the following two conditions are satisfied:

- L1 Every element in S is supremum of an increasing sequence of elements in $\bigcup_i \alpha_{i,\infty}(S_i)$.
- L2 If $s', s, t \in S_i$, for some i , are such that $\alpha_{i,\infty}(s) \leq \alpha_{i,\infty}(t)$ and $s' \ll s$, then there exists $j \geq i$ such that $\alpha_{i,j}(s') \leq \alpha_{i,j}(t)$.

THEOREM 3.9. *The functor $S \mapsto S_{\text{cc}}, \alpha \mapsto \alpha_{\text{cc}}$ preserves inductive limits.*

PROOF. Let $(S_i, \alpha_{i,j})_{i,j}$ be an inductive system with inductive limit $(S, \alpha_{i,\infty})_i$. We must show that L1 and L2 above are satisfied after applying the cc -construction.

Let us prove L1, i.e., that every element in S_{cc} is the supremum of an increasing sequence of elements in $\bigcup_i \text{im}((\alpha_{i,\infty})_{\text{cc}})$. Let $\bar{x} - \bar{e} \in S_{\text{cc}}$. Every compact element in S is the image of some compact element in some S_i . So there exists $e' \in S_i$ such that $\alpha_{i,\infty}(e') = e$. Passing to the inductive system $(S_j, \phi_{j,k})_{j,k \geq i}$, we can find an increasing sequence $(x_n)_n \subset \bigcup_{j \geq i} \text{im}(\alpha_{j,\infty})$ such that $x = \sup x_n$. Then $\bar{x} - \bar{e} = \sup \bar{x}_n - \bar{e}$, and the sequence $(\bar{x}_n - \bar{e})$ is contained in $\bigcup_i \text{im}((\alpha_{i,\infty})_{\text{cc}})$.

Let us prove L2. Let $z', z, w \in (S_i)_{\text{cc}}$ be such that $(\alpha_{i,\infty})_{\text{cc}}(z) \leq (\alpha_{i,\infty})_{\text{cc}}(w)$ and $z' \ll z$. Say $z = \bar{x} - \bar{e}$ and $w = \bar{y} - \bar{e}$, where $x, y, e \in S_i$ and e is a compact element. Then there exists $x' \ll x$ such that $z' \leq \bar{x}' - \bar{e}$. So it suffices to assume that $z' = \bar{x}' - \bar{e}$. Choose x'' such that $x' \ll x'' \ll x$. Then $\alpha_{i,\infty}(x'') + f \leq \alpha_{i,\infty}(y) + f$, for some compact element $f \in S$. There exists $j_1 \geq i$ and a compact $f_1 \in S_{j_1}$ mapped into f by $\alpha_{j_1,\infty}$. By moving the elements x'' and x' to S_{j_1} along the inductive limit, and relabeling them as

x'' and x' , we may assume that $j_1 = i$. Now $\alpha_{i,\infty}(x'' + f_1) \leq \alpha_{i,\infty}(y + f_1)$. Since S is the inductive limit of the system $(S_j, \alpha_{j,k})$ in the category Cu , there exists $j \geq i$ such that $\alpha_{i,j}(x' + f_1) \leq \alpha_{i,j}(y + f_1)$. This readily implies that $(\alpha_{i,j})_{\text{cc}}(\bar{x}' - \bar{e}) \leq (\alpha_{i,j})_{\text{cc}}(\bar{y} - \bar{e})$, as desired.

4. The augmented Cuntz semigroup

4.1. Definition and basic properties

Let A be a C^* -algebra. We now turn to the definition of the ordered semigroup $\text{Cu}^\sim(A)$, which we call the augmented Cuntz semigroup of the C^* -algebra A .

We let $\text{Cu}(A)$ denote the Cuntz semigroup of A , i.e., the set of Cuntz equivalence classes of positive elements of the stabilization $A \otimes \mathcal{K}$ endowed with a suitable order and addition operation (see [1] and references therein). Since $\text{Cu}(A)$ is a Cu -semigroup satisfying O5, we can apply the cc -functor from the previous section to it thereby obtaining a Cu -semigroup $(\text{Cu}(A))_{\text{cc}}$. We shall denote by Cu_{cc} the functor obtained by first applying the Cuntz semigroup functor followed by the cc -functor. We thus write $\text{Cu}_{\text{cc}}(A)$ and $\text{Cu}_{\text{cc}}(\phi)$, rather than $(\text{Cu}(A))_{\text{cc}}$ and $(\text{Cu}(\phi))_{\text{cc}}$, although these mean the same thing.

Let \tilde{A} denote the forced unitization of A . Observe that $[1] \in \text{Cu}(\tilde{A})$ is a full compact element: for any compact $e \in \text{Cu}(\tilde{A})$ there exists $n \in \mathbb{N}$ such that $e \leq n[1]$. It follows that

$$\text{Cu}_{\text{cc}}(\tilde{A}) = \{ \overline{[a]} - n\overline{[1]} : [a] \in \text{Cu}(\tilde{A}), n \in \mathbb{N} \}.$$

Let $\pi: \tilde{A} \rightarrow \mathbb{C}$ denote the canonical quotient map. The Cu -morphism $\text{Cu}(\pi)$ from $\text{Cu}(\tilde{A})$ to $\text{Cu}(\mathbb{C}) \cong \overline{\mathbb{N}}$ gives rise to a Cu -morphism $\text{Cu}_{\text{cc}}(\pi)$ from $\text{Cu}(\tilde{A})$ to $\text{Cu}_{\text{cc}}(\mathbb{C}) \cong \overline{\mathbb{Z}}$. We call $\text{Cu}_{\text{cc}}(\pi)$ the rank map on $\text{Cu}_{\text{cc}}(\tilde{A})$ and we alternatively denote it by $\text{rank}(\cdot)$. Notice that under the identification of $\text{Cu}(\mathbb{C})$ with $\overline{\mathbb{N}}$, $\text{Cu}(\pi)$ assigns to a Cuntz class $[a] \in \text{Cu}(\tilde{A})$, where $a \in (\tilde{A} \otimes \mathcal{K})_+$, the rank of $\pi(a) \in \mathcal{K}$. We also denote this number by $\text{rank}(a)$, $\text{rank}([a])$, or $\text{rank}(\overline{[a]})$.

We define $\text{Cu}^\sim(A)$ as the kernel of the rank map on $\text{Cu}_{\text{cc}}(\tilde{A})$, i.e.,

$$\text{Cu}^\sim(A) := \{ \overline{[a]} - n\overline{[1]} : [a] \in \text{Cu}(\tilde{A}), n \in \mathbb{N} \text{ such that } \text{rank}(\pi(a)) = n \}.$$

We endow $\text{Cu}^\sim(A)$ with the addition and the order from $\text{Cu}_{\text{cc}}(\tilde{A})$.

Let $\iota: A \rightarrow \tilde{A}$ denote the inclusion of A in \tilde{A} . This map induces an embedding of $\text{Cu}(A)$ as a subsemigroup (indeed, as an ideal) of $\text{Cu}(\tilde{A})$. Let us regard $\text{Cu}(A)$ as a subsemigroup of $\text{Cu}(\tilde{A})$ via this embedding. Observe then that the rank map on $\text{Cu}(\tilde{A})$ is zero on $\text{Cu}(A)$. It follows that the map $\text{Cu}_{\text{cc}}(A) \xrightarrow{\text{Cu}_{\text{cc}}(\iota)} \text{Cu}_{\text{cc}}(\tilde{A})$, induced by the inclusion of $\text{Cu}(A)$ in $\text{Cu}(\tilde{A})$ ranges in $\text{Cu}^\sim(A)$. We thus get a map $\text{Cu}_{\text{cc}}(\iota): \text{Cu}_{\text{cc}}(A) \rightarrow \text{Cu}^\sim(A)$,

$$\text{Cu}_{\text{cc}}(A) \ni \bar{x} - \bar{e} \mapsto \bar{x} - \bar{e} \in \text{Cu}^\sim(A).$$

PROPOSITION 4.1. *Let A be a unital C^* -algebra. Then the map $\text{Cu}_{\text{cc}}(\iota): \text{Cu}_{\text{cc}}(A) \rightarrow \text{Cu}^{\sim}(A)$ described above is an isomorphism of ordered semi-groups.*

PROOF. Since A is unital, $\tilde{A} \cong A \oplus \mathbb{C}$ via the homomorphism $a + \lambda \tilde{1} \mapsto (a + \lambda 1, \lambda)$ (where $\tilde{1}$ is the unit in \tilde{A} and 1 the unit in A). Let us identify \tilde{A} with $A \oplus \mathbb{C}$ via this isomorphism. Observe then that A , as a subalgebra of \tilde{A} , is $A \times \{0\}$, and that the quotient homomorphism $\tilde{A} \rightarrow \mathbb{C}$ is $(a, \lambda) \mapsto \lambda$.

We have $\text{Cu}(\tilde{A}) = \text{Cu}(A) \times \overline{\mathbb{N}}$ (on the right side order and addition are taken coordinatewise). The set of compact elements of $\text{Cu}(A) \times \overline{\mathbb{N}}$ is $\text{Cu}_{\text{c}}(A) \times \mathbb{N}$. It is straightforward to calculate that $\text{Cu}_{\text{cc}}(\tilde{A}) \cong \text{Cu}_{\text{cc}}(A) \times \overline{\mathbb{Z}}$. Further, the rank map $\text{rank}: \text{Cu}_{\text{cc}}(A) \rightarrow \overline{\mathbb{Z}}$ is simply $([\bar{a}] - n[\bar{1}], m) \mapsto m$. In particular, the subsemigroup of rank zero elements $\text{Cu}^{\sim}(A)$ is $(\text{Cu}_{\text{cc}}(A), 0)$. This is precisely the range of the map $\text{Cu}_{\text{cc}}(\iota): \text{Cu}_{\text{cc}}(A) \rightarrow \text{Cu}^{\sim}(A)$ described above. Indeed, this map is $\bar{x} - \bar{e} \mapsto (\bar{x} - \bar{e}, 0)$.

EXAMPLE 4.2. Recall that $\text{Cu}(\mathbb{C}) \cong \overline{\mathbb{N}}$. Thus, the preceding proposition and Example 3.1 imply that $\text{Cu}^{\sim}(\mathbb{C}) \cong \overline{\mathbb{Z}}$.

As remarked above, if $x \in \text{Cu}(A)$ then \bar{x} in $\text{Cu}_{\text{cc}}(\tilde{A})$ has rank 0, and thus belongs to $\text{Cu}^{\sim}(A)$. Let us define $q: \text{Cu}(A) \rightarrow \text{Cu}^{\sim}(A)$ as $q(x) = \bar{x}$.

LEMMA 4.3. *The map q is a surjection onto the positive elements of $\text{Cu}^{\sim}(A)$.*

PROOF. Let $z \in \text{Cu}^{\sim}(A) \subseteq \text{Cu}_{\text{cc}}(\tilde{A})$ be a positive element. By Lemma 3.2, $z = \bar{x}$ for some $x \in \text{Cu}(\tilde{A})$. But $\text{rank}(x) = 0$, so $x \in \text{Cu}(A)$. Thus, $q(x) = z$.

THEOREM 4.4. *$\text{Cu}^{\sim}(A)$ is a Cu -semigroup. Moreover, it satisfies the following additional axioms.*

- (i) *For all $x \in \text{Cu}^{\sim}(A)$ there exists $z \geq 0$ such that $x + z \geq 0$.*
- (ii) *For all $x' \ll x$ there exists z such that $x' + z \leq 0 \leq x + z$. Moreover, if $x + y \leq 0$ for some y and $y' \ll y$ then z can be chosen such that $y' \leq z$.*

PROOF. In Theorem 3.6 we have proved these properties for $\text{Cu}_{\text{cc}}(\tilde{A})$. We show here that they pass on to $\text{Cu}^{\sim}(A)$. This is essentially a consequence of the rank map being a Cu -morphism.

Let $(z_n)_n$ be an increasing sequence in $\text{Cu}^{\sim}(A)$. Since $\text{rank}(\cdot)$ is supremum preserving, the supremum of this sequence in $\text{Cu}_{\text{cc}}(\tilde{A})$ has rank zero, i.e., it belongs to $\text{Cu}^{\sim}(A)$. Hence $\text{Cu}^{\sim}(A)$ is closed under sequential suprema and further the supremum of a sequence in $\text{Cu}^{\sim}(A)$ agrees with the supremum of the same sequence taken in $\text{Cu}_{\text{cc}}(\tilde{A})$. It is now clear that O3 holds in $\text{Cu}^{\sim}(A)$, given that it holds $\text{Cu}_{\text{cc}}(\tilde{A})$.

Let $z \in \text{Cu}^{\sim}(A)$. Choose an \ll -increasing sequence $(z_n)_n$ in $\text{Cu}_{\text{cc}}(\tilde{A})$ with supremum z . Then $\text{sup rank } z_n = 0$. Since 0 is a compact element of

$\overline{\mathbb{Z}}$, $\text{rank } z_n = 0$ for large enough n , i.e., $z_n \in \text{Cu}^\sim(A)$ for large enough n . Since the relation \ll taken in $\text{Cu}_{\text{cc}}(\tilde{A})$ is stronger than the same relation in $\text{Cu}^\sim(A)$, the sequence is also \ll -increasing in $\text{Cu}^\sim(A)$. This proves O2.

Let us show that the restriction of the relation \ll in $\text{Cu}_{\text{cc}}(\tilde{A})$ to $\text{Cu}^\sim(A)$ agrees with the same relation in $\text{Cu}^\sim(A)$. As already pointed out, one direction is clear. Suppose that $x, y \in \text{Cu}^\sim(A)$ are such that $x \ll y$ in $\text{Cu}^\sim(A)$. Choose $(y_n)_n \subset \text{Cu}_{\text{cc}}(\tilde{A})$ that is \ll -increasing and with supremum y . As observed in the previous paragraph, $y_n \in \text{Cu}^\sim(A)$ for large enough n . Thus, there exists n_0 such that $x \leq y_{n_0} \ll y$, where \ll is taken in $\text{Cu}_{\text{cc}}(\tilde{A})$. Thus, $x \ll y$ in $\text{Cu}_{\text{cc}}(\tilde{A})$. Axiom O4 in $\text{Cu}^\sim(A)$ is now immediate from its holding in $\text{Cu}_{\text{cc}}(\tilde{A})$.

Let us prove property (i). Let $x = \overline{[a]} - n\overline{[1]} \in \text{Cu}^\sim(A)$, where $\text{rank}(a) = n$. By the properties of the Cuntz semigroup functor under C^* -algebra quotients ([5]), there exist $[b], [c] \in \text{Cu}(A)$ such that

$$n\overline{[1]} + [c] = \overline{[a]} + [b], \quad \text{in } \text{Cu}(\tilde{A}).$$

Let $z = \overline{[b]} \in \text{Cu}^\sim(A)$, i.e., z is the image of $[b]$ under the map q defined above. Then, working in $\text{Cu}_{\text{cc}}(\tilde{A})$, we have

$$x + z = \overline{[a]} - n\overline{[1]} + \overline{[b]} = \overline{[a] + [b]} - n\overline{[1]} = \overline{[c]} \geq 0.$$

Finally, let us prove (ii). Let $e = \overline{[1]} \in \text{Cu}_{\text{cc}}(\tilde{A})$. Let $x', x \in \text{Cu}^\sim(A)$ be such that $x' \ll x$. Choose x'' such that $x' \ll x'' \ll x$. Since $\overline{[1]}$ is a full element in $\text{Cu}_{\text{cc}}(\tilde{A})$, there exists $N \in \mathbb{N}$ such that $x'' \leq N\overline{[1]}$. By O5 in $\text{Cu}_{\text{cc}}(\tilde{A})$, there exists $w \in \text{Cu}_{\text{cc}}(\tilde{A})$ such that $x' + w \leq N\overline{[1]} \leq x'' + w$. Set $z = w - N\overline{[1]}$. Then

$$x' + z \leq 0 \leq x'' + z \leq x + z.$$

From the fact that x' and x both have rank 0 we deduce that z has rank 0 as well, i.e., it belongs to $\text{Cu}^\sim(A)$. Further, if y', y are such that $x + y \leq 0$ and $y' \ll y$, then

$$x'' + (y + N\overline{[1]}) \leq N\overline{[1]}.$$

By O5 in $\text{Cu}_{\text{cc}}(\tilde{A})$, w may be chosen such that $y' + N\overline{[1]} \leq w$, i.e., $y' \leq z$. So z is as desired.

REMARK 4.5. If there is a greatest element in $\text{Cu}^\sim(A)$ (e.g., A is separable), axiom (i) of the previous theorem boils down to saying that $x + \omega = \omega$ for all $x \in \text{Cu}^\sim(A)$.

Weak cancellation in a Cu -semigroup is defined as follows: $x + z \ll y + z$ implies $x \ll y$ for all x, y, z . The Cuntz semigroup of a C^* -algebra of stable rank one has weak cancellation, [14, Theorem 4.3]. This property, however,

is not true in general for the Cuntz semigroup of a C^* -algebra. On the other hand, $\text{Cu}^\sim(A)$ always has weak cancellation:

COROLLARY 4.6. *If $x + z \ll y + z$ in $\text{Cu}^\sim(A)$ then $x \ll y$.*

PROOF. Let $z' \ll z$ and $y' \ll y$ be such that $x + z \leq y' + z'$. Choose $w \in \text{Cu}^\sim(A)$ such that $z' + w \leq 0 \leq z + w$. Then

$$x \leq x + z + w \leq y' + z' + w \leq y' \ll y.$$

Let $\phi: A \rightarrow B$ be a homomorphism of C^* -algebras. Let $\tilde{\phi}: \tilde{A} \rightarrow \tilde{B}$ denote the homomorphism obtained by applying the forced unitization functor. This homomorphism gives rise to a Cu-morphism $\text{Cu}(\tilde{\phi}): \text{Cu}(\tilde{A}) \rightarrow \text{Cu}(\tilde{B})$. Then, as shown in the previous section, we obtain a Cu-semigroup morphism $\text{Cu}(\tilde{\phi})_{\text{cc}}: \text{Cu}_{\text{cc}}(\tilde{A}) \rightarrow \text{Cu}_{\text{cc}}(\tilde{B})$. It is not hard to deduce that $\text{Cu}_{\text{cc}}(\tilde{\phi})$ preserves the rank. In particular, it maps $\text{Cu}^\sim(A)$ to $\text{Cu}^\sim(B)$. We define $\text{Cu}^\sim(\phi)$ as the restriction of $\text{Cu}(\tilde{\phi})_{\text{cc}}$ to $\text{Cu}^\sim(A)$. More concretely,

$$\text{Cu}^\sim(\phi)(\overline{[a]} - n\overline{[1]}) = \overline{[\tilde{\phi} \otimes \text{id}(a)]} - n\overline{[1]}.$$

(Here id is the identity on \mathcal{K} , so that $\tilde{\phi} \otimes \text{id}: \tilde{A} \otimes \mathcal{K} \rightarrow \tilde{B} \otimes \mathcal{K}$.)

THEOREM 4.7. *$\text{Cu}^\sim(\phi)$ is a Cu-morphism.*

PROOF. In the course of the prove of Theorem 4.4 we have shown that both the suprema of increasing sequences and the compact containment relation in $\text{Cu}^\sim(A)$ agree with their counterparts taken in the larger ordered semigroup $\text{Cu}_{\text{cc}}(\tilde{A})$. It is then clear that the preservation of sequential suprema and of compact containment by $\text{Cu}^\sim(\phi)$ follow from the same properties for the map $\text{Cu}_{\text{cc}}(\tilde{\phi})$. But, indeed, $\text{Cu}_{\text{cc}}(\tilde{\phi})$ is a Cu-morphism by Theorem 3.8.

4.2. Comparison with the original definition

Let us compare the definition of $\text{Cu}^\sim(A)$ given above with the original definition in [10]. The main difference lies in the definition of the relation \preceq_{cc} , which in [10] is taken as $(x, e) \preceq_{\text{cc}} (y, f)$ if

$$x + f + g \leq y + e + g$$

for some compact element g . The rest of the construction is the same. Observe that the relation used in [10] is a stronger relation, i.e., if $(x, e) \preceq_{\text{cc}} (y, f)$ as in [10] then $(x, e) \preceq_{\text{cc}} (y, f)$ as defined above. If these two relations are the same in $\text{Cu}(\tilde{A}) \times \text{Cu}(\tilde{A})_{\text{c}}$, then the two constructions of $\text{Cu}^\sim(A)$ agree. In Corollary 4.10 below we show that this is the case for C^* -algebras of finite stable rank. Thus, in this case the two constructions agree.

Let us recall the definition of the stable rank of a C*-algebra. Let $m \in \mathbb{N}$ and let A be a unital C*-algebra. An m -tuple $(a_1, \dots, a_m) \in A^m$ is called left invertible if $\sum_{i=1}^m b_i a_i = 1$ for some $(b_1, \dots, b_m) \in A^m$. The C*-algebra A is said to have stable rank at most m if the set of left invertible m -tuples is dense in A^m . This can be rephrased in the language of Hilbert C*-modules as follows: A has stable rank at most m if the left invertible operators from A to A^m are dense in the set of bounded operators from A to A^m . The stable rank of A is the least m with this property. If no such m exists then the stable rank is infinite. For non-unital C*-algebras the stable rank is defined as the stable rank of the unitization.

THEOREM 4.8. *Let A be a unital C*-algebra of stable rank at most $m < \infty$. Let H and G be Hilbert C*-modules over A such that $A^{m+1} \oplus H \cong A \oplus G$. Then $A^m \oplus H \cong G$.*

PROOF. Let $H' = A^m \oplus H$, so that $A \oplus H' \cong A \oplus G$. Let $V: A \oplus G \rightarrow A \oplus H'$ be a Hilbert C*-module isomorphism. Let us write V in matrix form:

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \in \begin{pmatrix} B(A) & B(G, A) \\ B(A, H') & B(G, H') \end{pmatrix}.$$

Recall that $H' = A^m \oplus H$. Let us write $V_{21} = (U_1, U_2)$, where $U_1 \in B(A, A^m)$ and $U_2 \in B(A, H)$. Let $U'_1 \in B(A, A^m)$ be left invertible and $\|U_1 - U'_1\| < 1$. Let $V'_{21} = (U'_1, U_2)$. Now V'_{21} is left invertible and $\|V'_{21} - V_{21}\| < 1$. If we replace V_{21} by V'_{21} in the matrix of V we get V' such that $\|V - V'\| < 1$. Thus V' is an invertible adjointable operator. Thus, without loss of generality, we may assume that V is an invertible adjointable operator such that $V_{21} \in B(A, H')$ is left invertible. Let $W \in B(H', A)$ be a left inverse of V_{21} . Set $(1 - V_{11})W = X$.

Then

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} 1 & V'_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 0 \\ -V_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & V'_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} 1 & -V'_{12} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & V'_{22} \end{pmatrix}.$$

The map on the right side is still an invertible adjointable operator. It follows that $V'_{22} \in B(G, H)$ is adjointable and invertible. Hence $G \cong H$ as Hilbert C*-modules, via the polar decomposition of V'_{22} .

THEOREM 4.9. *Let A be a unital C*-algebra of stable rank at most $m < \infty$. If $x, y \in \text{Cu}(A)$ are such that $x + (m + 1)[1] \leq y + [1]$ then $x + m[1] \leq y$.*

PROOF. We rely on the Hilbert C*-modules picture of $\text{Cu}(A)$ (see [6]). Suppose that $x = [H]$ and $y = [G]$, where H and G are countably generated

Hilbert C^* -modules over A . Let $E \subseteq H$ be a compactly contained submodule. The assumption $x + (m + 1)[1] \leq y + [1]$ implies that $E \oplus A^{m+1}$ embeds as a Hilbert C^* -submodule in $G \oplus A$. Let $\phi: E \oplus A^{m+1} \rightarrow G \oplus A$ be an embedding. Set $E' = \phi(0 \oplus A^{m+1})^\perp$. That is, E' is the orthogonal complement of the image of A^{m+1} in $G \oplus A$. Since A^{m+1} is a projective module, we have that $E' \oplus A^{m+1} \cong G \oplus A$. By the previous theorem, $E' \oplus A^m \cong G$. Since E embeds in E' via ϕ , $E \oplus A^m$ embeds in G . Hence $[E] + m[A] \leq [G]$. Passing to the supremum over all E compactly contained in H we get $[H] + m[A] \leq [G]$, as desired.

COROLLARY 4.10. *Let A be a C^* -algebra of stable rank at most $m < \infty$. Let $[a], [b] \in \text{Cu}(\tilde{A})$. Then $\overline{[a]} \leq \overline{[b]}$ in $\text{Cu}_{\text{cc}}(\tilde{A})$ if and only if $[a] + m[1] \leq [b] + m[1]$ in $\text{Cu}(\tilde{A})$.*

PROOF. Let $[a'] \ll [a]$. Then, by the definition of \lesssim_{cc} , $[a'] + n[1] \leq [b] + n[1]$ for some $n \in \mathbb{N}$. Since the stable rank of \tilde{A} is at most m , by the previous theorem we have $[a'] + m[1] \leq [b] + m[1]$. Passing to the supremum over all $[a'] \ll [a]$, we get $[a] + m[1] \leq [b] + m[1]$.

4.3. An example

Here we calculate $\text{Cu}^\sim(C_0(\mathbb{R}^2))$. Let S^2 denote the 2-dimensional sphere. Let us recall the computation of the Cuntz semigroup of $C(S^2)$ obtained in [11]. Let $V(C(S^2))$ denote the Murray-von Neumann monoid of projections. Then

$$\text{Cu}(C(S^2)) \cong V(C(S^2)) \sqcup \text{Lsc}(S^2, \overline{\mathbb{N}}) / \sim,$$

where \sim is defined such that $n[1] \in V(S^2)$ and the constant function $n \in \text{Lsc}(S^2, \overline{\mathbb{N}})$ are equivalent for all $n = 0, 1, \dots$. Addition on the right side is as follows: within the sets $V(C(S^2))$ and $\text{Lsc}(S^2, \overline{\mathbb{N}})$ we simply use the addition operation with which these sets are endowed. If $[p] \in V(C(S^2))$ and $f \in \text{Lsc}(S^2, \overline{\mathbb{N}})$ is non-constant then $[p] + f = \text{rank}(p) + f \in \text{Lsc}(S^2, \overline{\mathbb{N}})$. The order again need only be defined for $[p] \in V(C(S^2))$ and $f \in \text{Lsc}(S^2, \overline{\mathbb{N}})$: $f \leq [p]$ if $f \leq \text{rank}(p)$ and $[p] \leq f$ if $\text{rank}(p) \leq f$ and f is non-constant.

It is straightforward to calculate that

$$\text{Cu}_{\text{cc}}(C(S^2)) \cong K_0(C(S^2)) \sqcup \text{Lsc}(S^2, \overline{\mathbb{Z}}) / \sim,$$

where \sim identifies $n[1] \in K_0(C(S^2))$ and the constant function $n \in \text{Lsc}(S^2, \overline{\mathbb{Z}})$ for all $n \in \mathbb{Z}$.

Let us regard $C(S^2)$ as unitization of $C_0(\mathbb{R}^2)$ and $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$. The rank function on $\text{Cu}_{\text{cc}}(C(S^2))$ coming from this unitization is then $\text{rank}([p] - [q]) =$

$\text{rank}(p) - \text{rank}(q)$ for $[p] - [q] \in K_0(C(S^2))$ and $\text{rank}(f) = f(\infty)$ for $f \in \text{Lsc}(S^2, \overline{\mathbb{Z}})$. We thus get

$$\begin{aligned} \text{Cu}^\sim(C_0(\mathbb{R}^2)) &\cong K_0(C_0(\mathbb{R}^2)) \sqcup \text{Lsc}_0(\mathbb{R}^2, \overline{\mathbb{Z}}) / \sim \\ &\cong \mathbb{Z} \sqcup \text{Lsc}_0(\mathbb{R}^2, \overline{\mathbb{Z}}) / \sim, \end{aligned}$$

where \sim identifies $0 \in \mathbb{Z}$ with $0 \in \text{Lsc}_0(\mathbb{R}^2, \overline{\mathbb{Z}})$. Let us describe the order and addition on the set on the right. Order: The elements of \mathbb{Z} are pairwise not comparable. They are greater than any non-zero function in $\text{Lsc}_0(\mathbb{R}^2, \overline{\mathbb{Z}})$ that is non-positive and smaller than any non-zero function that is non-negative. The order on $\text{Lsc}_0(\mathbb{R}^2, \overline{\mathbb{Z}})$ is pointwise. Addition: if $n \in \mathbb{Z}$ and $f \in \text{Lsc}_0(\mathbb{R}^2, \overline{\mathbb{Z}})$ then $n + f = f$. Observe that the subsemigroup spanned by the positive and negative elements is $\text{Lsc}_0(\mathbb{R}^2, \overline{\mathbb{Z}})$.

5. Properties of the functor Cu^\sim

By Theorems 4.4 and 4.7 of the previous section, $A \mapsto \text{Cu}^\sim(A), \phi \mapsto \text{Cu}^\sim(\phi)$, defines a functor from the category of C^* -algebras to the category Cu . Here we investigate its properties.

5.1. Continuity with respect to inductive limits

THEOREM 5.1. *The functor Cu^\sim is continuous, i.e., it preserves inductive limits.*

PROOF. Let $(A_i, \phi_{i,j})_{i,j \in I}$ be an inductive system of C^* -algebras and $(A, \phi_{i,\infty})_{i \in I}$ its inductive limit. Both the forced unitization functor and the Cuntz semigroup functor are continuous, [1, Corollary 3.2.9]. Thus, we have $\text{Cu}(\tilde{A}) = \varinjlim \text{Cu}(\tilde{A}_i)$. Moreover, by Theorem 3.9, applying the cc-construction still yields an inductive limit of Cu -semigroups: $\text{Cu}_{\text{cc}}(\tilde{A}) = \varinjlim \text{Cu}_{\text{cc}}(\tilde{A}_i)$.

For brevity of notation, let us set $S = \text{Cu}_{\text{cc}}(\tilde{A}), S_i = \text{Cu}_{\text{cc}}(\tilde{A}_i)$, and $\alpha_{i,j} = \text{Cu}_{\text{cc}}(\tilde{\phi}_{i,j})$. Each of the Cu -semigroups S_i and the Cu -semigroup S carry a rank map into $\overline{\mathbb{Z}}$. As argued in the definition of the functor $\text{Cu}^\sim(\cdot)$ in the previous section, the Cu -morphisms $\alpha_{i,j}$ and $\alpha_{i,\infty}$ map rank zero elements to rank zero elements. So their restrictions to the rank zero elements form an inductive system as well. In order to prove that the Cu -subsemigroup of rank zero elements in S is the inductive limit of the Cu -subsemigroups of rank zero elements in the S_i s, we must show that conditions L1 and L2 of an inductive limit in the category Cu are valid (see Section 3). This is quite straightforward: Say $\bar{x} - \bar{e} \in S$ has rank zero. By L1 applied to the inductive limit $S = \varinjlim S_i$, we can choose an increasing sequence $(\bar{x}_n - \bar{e})_{n=1}^\infty \subset \bigcup \text{im}(\alpha_{i,\infty})$ with supremum $\bar{x} - \bar{e}$. The terms of this sequence eventually have rank zero (since $0 \ll 0$ in $\overline{\mathbb{Z}}$). This proves L1. Condition L2 follows also from the same condition in the inductive limit $S = \varinjlim S_i$, bearing in mind that the compact

containment relation on the the set of rank zero elements agrees with the compact containment relation on the larger set S restricted to the rank zero elements (as argued in the proof of Theorem 4.4).

5.2. *Split exactness*

A projection p in a C^* -algebra is called finite if it is not Murray-von Neumann equivalent to a proper subprojection of itself. In terms of the monoid of Murray-von Neumann classes, this is expressible as $[p]_{\text{MvN}} + [q]_{\text{MvN}} = [p]_{\text{MvN}}$ implies $[q]_{\text{MvN}} = 0$ for all $[q]_{\text{MvN}} \in V(A)$. A unital C^* -algebra is called stably finite if any $[p]_{\text{MvN}} \in V(A)$ is finite; equivalently, if $n[1]_{\text{MvN}}$ is finite for all $n \in \mathbb{N}$.

LEMMA 5.2. *Let A be a stably finite unital C^* -algebra and let $x \in \text{Cu}(A)$. If $\bar{x} \in \text{Cu}_{\text{cc}}(A)$ is compact then $x = [p]$ for some projection $p \in A \otimes \mathcal{K}$.*

PROOF. We will make use of the characterization given in [3] of the compact elements in the Cuntz semigroup of a stably finite C^* -algebra. By [3, Proposition 5.7], if A is unital and stably finite and $[a] \in \text{Cu}(A)$ is compact, then 0 is an isolated point of the spectrum of a . It follows, by functional calculus, that we can choose a projection p such that $[a] = [p]$. It also follows that if $[a] + [b]$ is compact then both $[a]$ and $[b]$ are compact. Indeed, once 0 is an isolated point of the spectrum of $a \oplus b$, it is also an isolated point of the spectrum of a and b .

Let $(x_n)_n$ be a \ll -increasing in $\text{Cu}(A)$ with supremum x . Since the map $z \mapsto \bar{z}$ preserves sequential suprema, $\bar{x}_n = \bar{x}$ for all sufficiently large n . Assume without loss of generality that this is the case for all n . Fix n . From $\overline{x_{n+1}} = \bar{x}_1$ we get that $x_n + k_n[1] \leq x_1 + k_n[1]$ for some $k_n \in \mathbb{N}$. It follows that $x_n + k_n[1] = x_1 + k_n[1]$ is compact, and so, as remarked above, $x_n = [p_n]$ for some projection p_n . Moreover, by stable finiteness, we must have that $[p_1] = [p_n]$. Indeed, if we write $[p_1] + [q] = [p_n]$, with q a projection, then $[p_1] + k_n[1] + [q] = [p_1] + k_n[1]$, which implies that $[q] = 0$. Thus, $[p_1] = [p_n]$ for all n , and so $x = [p_1]$.

THEOREM 5.3. *Let*

$$0 \longrightarrow I^\iota \longrightarrow A^\pi \longrightarrow A/I \longrightarrow 0$$

be a short exact sequence of C^ -algebras. Apply the functor Cu^\sim to it to get*

$$\text{Cu}^\sim(I) \xrightarrow{\text{Cu}^\sim(\iota)} \text{Cu}^\sim(A) \xrightarrow{\text{Cu}^\sim(\pi)} \text{Cu}^\sim(A/I).$$

The following are true:

- (i) *If \tilde{A}/I is a stably finite C^* -algebra then $\text{im}(\text{Cu}^\sim(\iota)) = \text{Ker}(\text{Cu}^\sim(\pi))$.*
- (ii) *If π splits then $\text{Cu}^\sim(\pi)$ is surjective and $\text{Cu}^\sim(\iota)$ is an order embedding.*

PROOF. (i) Let us first go over some notation: We denote by $\tilde{\pi}: \tilde{A} \rightarrow \tilde{A}/I$ the unital extension of π . Further, we also denote by $\tilde{\pi}$ the homomorphism $\tilde{\pi} \otimes \text{id}$ from $\tilde{A} \otimes \mathcal{K}$ to $\tilde{A}/I \otimes \mathcal{K}$. The same conventions apply to $\tilde{\iota}$. Finally, we identify \tilde{A}/I with \tilde{A}/I and regard \tilde{I} as a subalgebra of \tilde{A} .

From $\pi \circ \iota = 0$ we get that $\text{Cu}^\sim(\pi) \circ \text{Cu}^\sim(\iota) = 0$. Therefore, $\text{im}(\text{Cu}^\sim(\iota)) \subseteq \text{Ker}(\text{Cu}^\sim(\pi))$. Next we show the opposite inclusion.

Let $\overline{[a]} - n\overline{[1]}$ be an element of $\text{Cu}^\sim(A)$ mapped to 0 by $\text{Cu}^\sim(\pi)$. That is, $\overline{[\tilde{\pi}(a)]} = n\overline{[1]}$ in $\text{Cu}_{\text{cc}}(\tilde{A}/I)$. Since $n\overline{[1]}$ is compact, we have by Lemma 5.2 that $[\tilde{\pi}(a)]$ is a compact element of $\text{Cu}(\tilde{A}/I)$. From the definition of \preceq_{cc} , this means that $[\tilde{\pi}(a)] + k[1] = n[1] + k[1]$ for some $k \in \mathbb{N}$. Let us replace $[a]$ by $[a] + k[1]$ and n by $n + k$, which does not change $\overline{[a]} - n\overline{[1]}$, so as to now have $[\tilde{\pi}(a)] = n[1]$ in $\text{Cu}(\tilde{A}/I)$. Since $\tilde{\pi}(a)$ is Cuntz equivalent to a projection and we have assumed that \tilde{A}/I is stably finite, 0 is an isolated point of the spectrum of $\tilde{\pi}(a)$, [3, Proposition 5.7]. Hence, for a suitable choice of a strictly positive $f \in C_0(\mathbb{R}^+)$, $\tilde{\pi}(f(a))$ is a projection. Let $a' = f(a)$. Then $[a'] = [a]$ (since we have chosen f strictly positive) and $\tilde{\pi}(a')$ is a projection. Let us simply rename a' as a and assume that $\tilde{\pi}(a)$ is a projection. Now $\tilde{\pi}(a)$ and 1_n are Cuntz equivalent projections in $(\tilde{A}/I) \otimes \mathcal{K}$. Since \tilde{A}/I is stably finite, $\tilde{\pi}(a)$ and 1_n are Murray-von Neumann equivalent. In a stable C^* -algebra, if p, q are Murray-von Neumann equivalent projections then there exists a unitary u in the unitization of the algebra such that $upu^* = q$; moreover u may be chosen in the connected component of the identity. (Proof: If $\|p - q\| < 1$ this is well known. From this we deduce the case when p and q are homotopic. But in a stable C^* -algebra Murray-von Neumann equivalent projections are homotopic.) Applying this fact to $\tilde{\pi}(a)$ and 1_n , we obtain a unitary u in the unitization of $(\tilde{A}/I) \otimes \mathcal{K}$ such that $u\tilde{\pi}(a)u^* = 1_n$ and u is in the connected component of the identity. Since u is connected to the identity, it has a lift to a unitary v in the unitization of $\tilde{A} \otimes \mathcal{K}$. Set $v^*av = a_1 \in \tilde{A} \otimes \mathcal{K}$. Then $[a] = [a_1]$ and $\tilde{\pi}(a_1) = 1_n$. The latter implies that $a_1 \in \tilde{I} \otimes \mathcal{K}$. Thus,

$$\overline{[a]} - n\overline{[1]} = \overline{[a_1]} - n\overline{[1]} \in \text{im}(\text{Cu}^\sim(\iota)),$$

as desired.

(ii) Let $\lambda: A/I \rightarrow A$ be a homomorphism such that $\pi \circ \lambda = \text{id}_{A/I}$. Then $\text{Cu}^\sim(\psi) \circ \text{Cu}^\sim(\lambda) = \text{id}_{\text{Cu}^\sim(A/I)}$, which implies that $\text{Cu}^\sim(\psi)$ is surjective.

Let us prove that $\text{Cu}^\sim(\phi)$ is an order embedding. We maintain the conventions introduced in the first paragraph of the proof of (i). Let $a, b \in \tilde{I} \otimes \mathcal{K}$ be positive elements such that $\overline{[a]} - n\overline{[1]}$ and $\overline{[b]} - n\overline{[1]}$ belong to $\text{Cu}^\sim(A)$ and $\overline{[a]} - n\overline{[1]} \leq \overline{[b]} - n\overline{[1]}$. Then $\text{rank}(a) = \text{rank}(b) = n$ (these are the ranks of their images in \mathcal{K} by the canonical map $\tilde{I} \otimes \mathcal{K} \rightarrow \mathcal{K}$) and $\overline{[a]} \leq \overline{[b]}$ in $\text{Cu}_{\text{cc}}(\tilde{A})$. We will show that $\overline{[a]} \leq \overline{[b]}$ in $\text{Cu}_{\text{cc}}(\tilde{I})$. Choose $[a'] \ll [a]$. Then

there exists $k \in \mathbb{N}$ such that $[a'] + k[1] \leq [b] + k[1]$ in $\text{Cu}(\tilde{A})$. Suppose that we have shown that $[a'] + k[1] \leq [b] + k[1]$ in $\text{Cu}(\tilde{I})$. Since $[a'] \ll [a]$ has been chosen arbitrarily, we conclude that $[\bar{a}] \leq [\bar{b}]$ in $\text{Cu}_{\text{cc}}(\tilde{I})$. It thus suffices to show that if $[a] \leq [b]$ in $\text{Cu}(\tilde{A})$ and $\text{rank}(a) = \text{rank } b = n$, then $[a] \leq [b]$ in $\text{Cu}(\tilde{I})$. We prove this next.

We may assume without loss of generality that $a = 1_n + a'$ and $b = 1_n + b'$, for some selfadjoint elements $a', b' \in I \otimes \mathcal{K}$. Let $\varepsilon > 0$. Since $[a] \leq [b]$ in $\text{Cu}(\tilde{A})$, there exists $x \in \tilde{A} \otimes \mathcal{K}$ such that $(a - \varepsilon/2)_+ = x^*x$ and $xx^* \leq Mb$ for some $M > 0$. By the almost stable rank one property of $A \otimes \mathcal{K}$ ([2, Lemma 4.3.2]) there exists a unitary $u \in (A \otimes \mathcal{K})^\sim$ such that $u^*(a - \varepsilon)_+ u \leq Mb$.

Consider the elements

$$u_1 = (\tilde{\lambda} \circ \tilde{\pi})(u) \in (\tilde{A} \otimes \mathcal{K})^\sim \quad \text{and} \quad a_1 = (uu_1^*)^* a (uu_1^*) \in \tilde{I} \otimes \mathcal{K}.$$

We have $\tilde{\pi}(uu_1^*) = 1$, which implies that uu_1^* is a unitary in the unitization of $\tilde{I} \otimes \mathcal{K}$. Thus, a and a_1 are unitarily equivalent in $\tilde{I} \otimes \mathcal{K}$, which implies that $[(a - \varepsilon)_+] = [(a_1 - \varepsilon)_+]$ in $\text{Cu}(\tilde{I})$. From $u^*(a - \varepsilon)_+ u \leq Mb$ and the definition of u_1 we get that $u_1^*(a_1 - \varepsilon)_+ u_1 \leq Mb$. Thus, on one hand we have that $[(a - \varepsilon)_+] = [(a_1 - \varepsilon)_+]$ in $\text{Cu}(\tilde{I})$, and on the other hand we have that $u_1^*(a_1 - \varepsilon)_+ u_1 \leq Mb$. We show next that $[(a_1 - \varepsilon)_+] = [u_1^*(a_1 - \varepsilon)_+ u_1]$ in $\text{Cu}(\tilde{I})$. In fact, we will show that a and a_1 are approximately unitarily equivalent in $\tilde{I} \otimes \mathcal{K}$ (with unitaries chosen in the unitization of this C^* -algebra).

Applying $\tilde{\lambda} \circ \tilde{\pi}$ on both sides of $u_1^*(a_1 - \varepsilon)_+ u_1 \leq Mb$, and using that $(\tilde{\lambda} \circ \tilde{\pi})(u_1) = u_1$, we get that $u_1^* 1_n u_1 = 1_n$, i.e., u_1 commutes with 1_n . We also have that

$$(\tilde{\lambda} \circ \tilde{\pi})(a_1) = (\tilde{\lambda} \circ \tilde{\pi})(a) = 1_n.$$

This implies that $a_1 = 1_n + a'_1$ for some selfadjoint $a'_1 \in I \otimes \mathcal{K}$. Let us choose an approximate unit $(e_\lambda) \in I_+$ of I . For each λ and $k \in \mathbb{N}$ define

$$v_{\lambda,k} = 1_n + (u_1 - 1_n)(e_\lambda \otimes 1_k).$$

Then $v_{\lambda,k}$ belongs to $\tilde{I} \otimes \mathcal{K}$ for all λ and k . We have

$$v_{\lambda,k} a'_1 = 1_n a'_1 + (u_1 - 1_n)(e_\lambda \otimes 1_k) a'_1 \rightarrow u_1 a'_1,$$

as $\lambda, k \rightarrow \infty$. Here we have used that $(e_\lambda \otimes 1_k) a'_1 \rightarrow a'_1$ as $\lambda, k \rightarrow \infty$. Similarly, we deduce that $v_{\lambda,k}^* a'_1 \rightarrow u_1^* a'_1$. Since $u_1 - 1_n$ and $e_\lambda \otimes 1_k$ commute with 1_n , it is clear from its definition that $v_{\lambda,k}$ commutes with 1_n for all λ, k .

Since $\tilde{I} \otimes \mathcal{K}$ has almost stable rank 1 ([2, Lemma 4.3.2]), for each λ, k there exists a unitary $w_{\lambda,k}$ in the unitization of $\tilde{I} \otimes \mathcal{K}$ such that $\|v_{\lambda,k} - w_{\lambda,k} |v_{\lambda,k}|\| <$

$1/k$. Then

$$w_{\lambda,k}^* a'_1 w_{\lambda,k} \rightarrow u_1^* a'_1 u_1 \quad \text{and} \quad w_{\lambda,k}^* 1_n w_{\lambda,k} \rightarrow 1_n, \quad \text{as } \lambda, k \rightarrow \infty.$$

It follows that $w_{\lambda,k}^* a_1 w_{\lambda,k} \rightarrow u_1^* a_1 u_1$. So a and a_1 are approximately unitarily equivalent in $\tilde{I} \otimes \mathcal{K}$. We now have

$$(a - \varepsilon)_+ \sim_{\text{Cu}} (a_1 - \varepsilon)_+ \sim_{\text{Cu}} u_1^* (a_1 - \varepsilon)_+ u_1 \leq Mb,$$

where the Cuntz comparisons are all taken in $\tilde{I} \otimes \mathcal{K}$. Since $\varepsilon > 0$ is arbitrary, we conclude that $[a] \leq [b]$ in $\text{Cu}(\tilde{I})$.

5.3. Stability

THEOREM 5.4. *Let A be a C^* -algebra. The inclusion $A \hookrightarrow A \otimes \mathcal{K}$ (in the top corner) induces an isomorphism of ordered semigroups $\text{Cu}^\sim(A) \rightarrow \text{Cu}^\sim(A \otimes \mathcal{K})$.*

PROOF. Consider the inductive limit

$$A \hookrightarrow M_2(A) \hookrightarrow M_4(A) \hookrightarrow \dots \hookrightarrow A \otimes \mathcal{K}.$$

By the continuity of Cu^\sim with respect to inductive limits, it suffices to show that the inclusion in the top corner $A \hookrightarrow M_2(A)$ induces an isomorphism at the level of Cu^\sim . We prove this next.

Let us first assume that A is unital. By Proposition 4.1, the map $\text{Cu}_{\text{cc}}(\iota): \text{Cu}_{\text{cc}}(A) \rightarrow \text{Cu}^\sim(A)$ is an isomorphism. Similarly, the inclusion of $M_2(A)$ in $M_2(A)^\sim$ induces an ordered semigroup isomorphism $\text{Cu}_{\text{cc}}(\iota): \text{Cu}_{\text{cc}}(M_2(A)) \rightarrow \text{Cu}^\sim(M_2(A))$. We thus have the diagram

$$\begin{array}{ccc} \text{Cu}_{\text{cc}}(A) & \longrightarrow & \text{Cu}_{\text{cc}}(M_2(A)) \\ \downarrow & & \downarrow \\ \text{Cu}^\sim(A) & \longrightarrow & \text{Cu}^\sim(M_2(A)) \end{array}$$

where the horizontal arrows are induced by the inclusion $A \hookrightarrow M_2(A)$ applying the functors $\text{Cu}_{\text{cc}}(\cdot)$ and Cu^\sim . It is known that $A \hookrightarrow M_2(A)$ induces an isomorphism in Cu (see [6, Appendix 6]). This in turn implies that the top horizontal arrow is an isomorphism from $\text{Cu}_{\text{cc}}(A)$ to $\text{Cu}_{\text{cc}}(M_2(A))$. It follows that the bottom horizontal arrow is an isomorphism as well.

The non-unital case is reduced to the unital case as follows. Let A be a non-unital C^* -algebra. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & A^\sim & \longrightarrow & \mathbb{C} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_2(A) & \longrightarrow & M_2(A^\sim) & \longrightarrow & M_2(\mathbb{C}) \longrightarrow 0, \end{array}$$

where the rows form short exact sequences that split and the vertical arrows are the natural inclusions. Applying the functor Cu^\sim we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Cu}^\sim(A) & \longrightarrow & \text{Cu}^\sim(A^\sim) & \longrightarrow & \text{Cu}^\sim(\mathbb{C}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Cu}^\sim(M_2(A)) & \longrightarrow & \text{Cu}^\sim(M_2(A^\sim)) & \longrightarrow & \text{Cu}^\sim(M_2(\mathbb{C})) \longrightarrow 0. \end{array}$$

A diagram chase, as in the proof of the five lemma, using the exactness of the rows of this diagram (in the sense of Proposition 5.3 (i) and (ii)), and that the two rightmost vertical arrows are isomorphisms, shows that $\text{Cu}^\sim(A) \rightarrow \text{Cu}^\sim(M_2(A))$ is an isomorphism.

From the stability of Cu^\sim we derive the following proposition, which will be needed later on:

PROPOSITION 5.5. *Suppose that A is σ -unital. Let $x, y \in \text{Cu}(A)$ be such that $q(x) \leq q(y)$ in $\text{Cu}^\sim(A)$. Let $z \in \text{Cu}(A)$ be a full element (i.e., such that $\infty \cdot z$ is the largest element in $\text{Cu}(A)$). Then for every $x' \ll x$ there exist $n \in \mathbb{N}$ and $y' \ll y$ such that $x' + nz \leq y' + nz$.*

PROOF. Let $z = [c]$, where $c \in A \otimes \mathcal{H}$ is full. Let $C = \overline{c(A \otimes \mathcal{H})c}$. Then $C \otimes \mathcal{H} \cong A \otimes \mathcal{H}$ by Brown's theorem. This isomorphism induces isomorphisms $\text{Cu}(A) \cong \text{Cu}(C)$ and $\text{Cu}^\sim(A) \cong \text{Cu}^\sim(C)$. Let $\tilde{x}, \tilde{y}, \tilde{z} \in \text{Cu}(C)$ denote the images of x, y, z via the isomorphism of $\text{Cu}(A)$ and $\text{Cu}(C)$. We claim that $q(\tilde{x}) \leq q(\tilde{y})$ in $\text{Cu}^\sim(C)$. Indeed, consider the following diagram:

$$\begin{array}{ccccccc} \text{Cu}(A) & \longrightarrow & \text{Cu}(A \otimes \mathcal{H}) & \longleftarrow & \text{Cu}(C \otimes \mathcal{H}) & \longleftarrow & \text{Cu}(C) \\ \downarrow q & & \downarrow q & & \downarrow q & & \downarrow q \\ \text{Cu}^\sim(A) & \longrightarrow & \text{Cu}^\sim(A \otimes \mathcal{H}) & \longleftarrow & \text{Cu}^\sim(C \otimes \mathcal{H}) & \longleftarrow & \text{Cu}^\sim(C). \end{array}$$

All squares in this diagram commute. This is straightforward to check for the first and third squares. The middle square commutes since the horizontal maps are induced by an isomorphism at the C^* -algebra level. It thus suffices to prove the proposition for $\tilde{x}, \tilde{y}, \tilde{z}$, in $\text{Cu}(C)$. Let $x' \ll \tilde{x}$. From $q(\tilde{x}) \leq q(\tilde{y})$ we get that $x' + n[1] \leq y' + n[1]$ in $\text{Cu}(\tilde{C})$, for some $y' \ll \tilde{y}$ and some $n \in \mathbb{N}$. We

can now “project” all the terms of this inequality to the ideal $\text{Cu}(C)$ in $\text{Cu}(\tilde{C})$. This operation is more explicit in the Hilbert C^* -modules picture of the Cuntz semigroup: given a Hilbert C^* -module H over a C^* -algebra B and a σ -unital closed two-sided ideal I , the map on the Cuntz semigroup $[H] \mapsto [HI]$ is well defined, additive, order preserving, and supremum preserving; see [5, §2 and Theorem 5]. Applying the map $[H] \mapsto [HC]$ in the inequality $x' + n[1] \leq y' + n[1]$ we get $x' + nz \leq y' + nz$. This proves the proposition.

6. Compact elements and functionals

Here we investigate the compact elements and functionals on the augmented Cuntz semigroup.

6.1. Compact elements

Let A be a C^* -algebra. Let $V(A)$ denote the semigroup of Murray-von Neumann classes of projections in $A \otimes \mathcal{K}$. Given a projection $p \in A \otimes \mathcal{K}$, we denote by $[p]_{MvN}$ the element in $V(A)$ with representative p . Since the Murray-von Neumann equivalence relation is stronger than the relation of Cuntz equivalence, the map $V(A) \ni [p]_{MvN} \mapsto [p] \in \text{Cu}(A)$ is well defined. Further, this map ranges in the subsemigroup of compact elements of $\text{Cu}(A)$.

Recall that

$$K_0(A) = \{ [p]_{MvN} - n[1]_{MvN} : p \in \tilde{A} \otimes \mathcal{K}, \text{rank}(p) = n \}.$$

We thus have a map

$$K_0(A) \ni [p]_{MvN} - n[1]_{MvN} \mapsto \overline{[p]} - n\overline{[1]} \in \text{Cu}^\sim(A). \tag{6.1}$$

It is easy to show that this is a well defined additive map.

A unital C^* -algebra is called stably finite if $n[1]_{MvN} + [q]_{MvN} = n[1]_{MvN}$ implies $[q]_{MvN} = 0$ in $V(\tilde{A})$. If A is stably finite, then $K_0(A)$ is an ordered group under the order induced by the image of $V(A)$ in $K_0(A)$.

Let $\text{Cu}^\sim(A)_c$ denote the set of compact elements of $\text{Cu}^\sim(A)$, i.e., the set of $x \in \text{Cu}^\sim(A)$ such that $x \ll x$.

THEOREM 6.1. *Let A be a C^* -algebra. Then $\text{Cu}^\sim(A)_c$ is a group such that if $x + y \in \text{Cu}^\sim(A)_c$ then $x, y \in \text{Cu}^\sim(A)_c$. Moreover, if A is stably finite, then $\text{Cu}^\sim(A)_c \cong K_0(A)$ as ordered groups via the map (6.1).*

PROOF. Let us show that $\text{Cu}^\sim(A)_c$ is a group. Since compact elements form a subsemigroup containing 0, it suffices to show that every compact element has a compact additive inverse. Let $x \in \text{Cu}^\sim(A)_c$. By Theorem 4.4 (ii), there exists $y \in \text{Cu}^\sim(A)$ such that $x + y = 0$. Since $0 \ll 0$, there exists $y' \ll y$

such that $x + y' = 0$. Then $x + y = 0 \ll 0 = x + y'$. We conclude by weak cancellation (Corollary 4.6) that $y = y'$. Hence, y is compact, as desired.

Suppose that $x + y$ is compact. Choose $x' \ll x$ such that $x' + y = x + y$. By weak cancellation, we get that $x' = x$. Hence, x is compact, and similarly, y is compact.

Suppose that A is stably finite. Let $x \in \text{Cu}^\sim(A)$ be compact. Since the relation \ll in $\text{Cu}^\sim(A)$ is the restriction of the same relation taken in $\text{Cu}_{\text{cc}}(\tilde{A})$, x is compact in $\text{Cu}_{\text{cc}}(\tilde{A})$. Translation by $n[1]$ in $\text{Cu}_{\text{cc}}(\tilde{A})$ is an order isomorphism. Hence, $x = \bar{e} - n[1]$, where $\bar{e} \in \text{Cu}_{\text{cc}}(\tilde{A})$ is compact and lifts to $e \in \text{Cu}(\tilde{A})$. By Lemma 5.2, there exists $p \in \tilde{A} \otimes \mathcal{K}$ such that $e = [p]$. It follows that $x = \overline{[p]} - n[1]$, i.e., x is in the range of the map (6.1).

Finally, let us show that the map (6.1) is an order embedding. Suppose that $\overline{[p]} - n[1] \leq \overline{[q]} - n[1]$, where $p, q \in \tilde{A} \otimes \mathcal{K}$ are projections of rank n . Then $\overline{[p]} \leq \overline{[q]}$. Since $[p] \ll [p]$, this implies that $[p] + m[1] \leq [q] + m[1]$ for some m . Cuntz subequivalence of projections agrees with Murray-von Neumann subequivalence. We thus get that $[p]_{MvN} + m[1]_{MvN} \leq [q]_{MvN} + m[1]_{MvN}$, which in turn implies that $[p]_{MvN} - n[1]_{MvN} \leq [q]_{MvN} - n[1]_{MvN}$, as desired.

6.2. Functionals

Let S be a positively ordered Cu-semigroup. A map $\lambda: S \rightarrow [0, \infty]$ is called a functional if it maps 0 to 0, it is additive, order preserving, and preserves the suprema of increasing sequences. It is called densely finite if $\lambda(x) < \infty$ whenever $x \ll y$ for some $y \in S$. If $S = \text{Cu}(A)$, then the densely finite functionals are in bijective correspondence with the densely finite 2-quasitraces on A_+ as well as with the densely finite rank functions, [8].

Let $\lambda: \text{Cu}^\sim(A)_+ \rightarrow [0, \infty]$ be a densely finite functional. Our goal is to extend it to all of $\text{Cu}^\sim(A)$. Let $x \in \text{Cu}^\sim(A)$. We know, by Theorem 4.4 (i), that there exists $z \geq 0$ such that $x + z \geq 0$. Moreover, since $0 \ll 0$, we can choose z such that $z \ll z'$ for some z' . Then λ is finite on z . Let us define

$$\lambda(x) := \lambda(x + z) - \lambda(z) \in (-\infty, \infty]. \quad (6.2)$$

Clearly, this is the only possible way to extend λ additively to $\text{Cu}^\sim(A)$.

LEMMA 6.2. *Let $\lambda: \text{Cu}^\sim(A)_+ \rightarrow [0, \infty]$ be a densely finite functional. The extension of λ to $\text{Cu}^\sim(A)$ defined above is additive, order preserving, and preserves the suprema of increasing sequences.*

PROOF. Let

$$P = \{z \in \text{Cu}^\sim(A)_+ : z \ll z' \text{ for some } z'\}. \quad (6.3)$$

Let $x \in \text{Cu}^\sim(A)$. Let us show first that $\lambda(x)$, defined as in (6.2), is independent of the choice of z , as long as $x + z \geq 0$ and $z \in P$. Suppose that z' is another element with these properties. Then

$$\lambda(x + z) + \lambda(z') = \lambda(x + z + z') = \lambda(x + z') + \lambda(z).$$

Hence, $\lambda(x + z) - \lambda(z) = \lambda(x + z') - \lambda(z')$.

Suppose that $x \leq y$ in $\text{Cu}^\sim(A)$. Choose $z \in P$ such that $x + z \geq 0$. Then $x + z \leq y + z$ and so

$$\lambda(x) = \lambda(x + z) - \lambda(z) \leq \lambda(y + z) - \lambda(z) = \lambda(y).$$

Thus, λ is order preserving.

Let $(x_n)_n$ be an increasing sequence in $\text{Cu}^\sim(A)$. Choose z such that $x_1 + z \geq 0$ and $z \in P$. Then

$$\sup \lambda(x_i) = \sup \lambda(x_i + z) - \lambda(z) = \lambda(\sup x_i + z) - \lambda(z) = \lambda(\sup x_i).$$

Additivity of the extension of λ is handled similarly.

Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. Let $F_0(\text{Cu}^\sim(A))$ denote the set of $\lambda: \text{Cu}^\sim(A) \rightarrow \overline{\mathbb{R}}$ that are additive, preserve 0, sequential suprema, and are densely finite. We endow $F_0(\text{Cu}^\sim(A))$ with the topology such that a net $(\lambda_i)_i$ converges to λ if

$$\limsup \lambda_i(x) \leq \lambda(y) \leq \liminf \lambda_i(y) \quad \text{for all } 0 \leq x \ll y.$$

Given $x \in \text{Cu}^\sim(A)$ we get a function $\hat{x}: F_0(\text{Cu}^\sim(A)) \rightarrow \overline{\mathbb{R}}$ defined by $\hat{x}(\lambda) = \lambda(x)$ for all $\lambda \in F_0(\text{Cu}^\sim(A))$.

LEMMA 6.3. *The function \hat{x} is linear and lower semicontinuous.*

PROOF. Let $\lambda_i \rightarrow \lambda$. Choose $z \in P$ such that $x + z \geq 0$ and $z \geq 0$, where P is the set defined in (6.3). Choose $0 \ll z' \ll z$ such that $x + z' \geq 0$ (it exists by the compactness of 0). Then

$$\begin{aligned} \lambda(x + z') - \lambda(z) &\leq \liminf \lambda_i(x + z') - \limsup \lambda_i(z') \\ &= \liminf \lambda_i(x). \end{aligned}$$

Passing to the supremum over all $z' \ll z$ on the left side we get that $\lambda(x) \leq \liminf \lambda_i(x)$. That is, \hat{x} is l.s.c.

Recall that the map $q: \text{Cu}(A) \rightarrow \text{Cu}^\sim(A)_+$ defined as $q(x) = \bar{x}$ is an onto Cu-morphism (Lemma 4.3). It follows that $\lambda \mapsto \lambda \circ q$ is an embedding of $F_0(\text{Cu}^\sim(A))$ into $F_0(\text{Cu}(A))$. The topology on $F_0(\text{Cu}^\sim(A))$ that we have defined above is precisely the one induced by the topology on $F_0(\text{Cu}(A))$ via this embedding.

LEMMA 6.4. *Suppose that A is σ -unital. Let $\lambda: \text{Cu}(A) \rightarrow [0, \infty]$ be a functional that is finite on a full element of $\text{Cu}(A)$. Then λ factors through q , i.e., $\lambda = \tilde{\lambda} \circ q$, where $\tilde{\lambda} \in F_0(\text{Cu}^\sim(A))$.*

PROOF. Let $z \in \text{Cu}(A)$ be a full element such that $\lambda(z) < \infty$. Let us show that if $x, y \in \text{Cu}(A)$ are such that $\bar{x} \leq \bar{y}$ in $\text{Cu}^\sim(A)$ then $\lambda(x) \leq \lambda(y)$. Let $x' \ll x$. By Proposition 5.5, $x' + nz \leq y + nz$ for some n . It follows that $\lambda(x') \leq \lambda(y)$. Passing to the supremum over all $x' \ll x$, we get that $\lambda(x) \leq \lambda(y)$. We can thus define $\tilde{\lambda}(\bar{x}) = \lambda(x)$, for $x \in \text{Cu}(A)$. Additivity, preservation of order, and preservation of 0, are readily passed from λ to $\tilde{\lambda}$. Let us show that $\tilde{\lambda}$ preserves sequential suprema: Suppose that $(x_n)_n$ is an increasing sequence in $\text{Cu}^\sim(A)_+$. By Theorem 3.4, there exists an increasing sequence $z_n \in \text{Cu}(\tilde{A})$ such that $\bar{z}_n \leq x_n$ for all n and $\overline{\sup z_n} = \sup x_n$. Since $0 \leq \text{rank}(z_n) \leq \text{rank}(x_n) = 0$, we must have that $\text{rank}(z_n) = 0$ for all n , i.e., $z_n \in \text{Cu}(A)$. Then

$$\tilde{\lambda}(\sup x_n) = \lambda(\sup z_n) = \sup \lambda(z_n) \leq \sup \tilde{\lambda}(x_n).$$

Hence, $\tilde{\lambda}(\sup x_n) \leq \sup \tilde{\lambda}(x_n)$. The opposite inequality is straightforward. Thus, $\tilde{\lambda}$ preserves sequential suprema. We have thus defined a densely finite functional $\tilde{\lambda}$ on $\text{Cu}^\sim(A)_+$ such that $\lambda = \tilde{\lambda}q$. Extending it to $\text{Cu}^\sim(A)$ as in (6.2), we obtain a functional in $F_0(\text{Cu}^\sim(A))$.

Let λ be a functional on $\text{Cu}(A)$ meeting the assumptions of the previous theorem, so that it arises as $\lambda = \tilde{\lambda} \circ q$, for $\tilde{\lambda} \in F_0(\text{Cu}^\sim(A))$. Let $x \in \text{Cu}^\sim(A)$. Then we can define a pairing of λ and x : $\langle \lambda, x \rangle := \tilde{\lambda}(x)$. If every densely finite functional on $\text{Cu}(A)$ factors through q , then we can define this pairing on all $F_0(\text{Cu}(A)) \times \text{Cu}^\sim(A)$. This is the case when the primitive spectrum of A is compact. We summarize the situation in the following theorem:

THEOREM 6.5. *Let A be a σ -unital C^* -algebra with compact primitive spectrum. Then the densely finite functionals on $\text{Cu}^\sim(A)$ are in bijection with the densely finite functionals on $\text{Cu}(A)$ via the map $\lambda \mapsto \lambda \circ q$. Moreover, given $x \in \text{Cu}^\sim(A)$ the function $\hat{x}: F_0(\text{Cu}(A)) \rightarrow \overline{\mathbb{R}}$ defined as $\hat{x}(\lambda) = \langle \lambda, x \rangle$ for all $\lambda \in F_0(\text{Cu}(A))$ is additive and lower semicontinuous.*

PROOF. Let $z = [a] \in \text{Cu}(A)$ be the class of a strictly positive element of A . Then z is full, i.e., $\infty \cdot z$ is the largest element of $\text{Cu}(A)$. Choose a \ll -increasing sequence $(z_n)_n$ with supremum z . Since the primitive spectrum of A is compact, z_n is full for large enough n . We thus find z_n that is full and such that every densely finite functional on $\text{Cu}(A)$ is finite on z_n . It follows that every densely finite functional on $\text{Cu}(A)$ factors through q by the previous lemma. This proves the first assertion of the theorem. The second part

follows from Lemma 6.3 and the agreement of the topologies on $F_0(\text{Cu}(A))$ and $F_0(\text{Cu}^\sim(A))$.

6.3. Simple pure C^* -algebras

A C^* -algebra is said to be pure if its Cuntz semigroup is almost unperforated and almost divisible, [16, Definition 3.6(i)]. Let us recall the definition of these properties: A Cu -semigroup S is called almost unperforated if $(k + 1)x \leq ky$ implies $x \leq y$ for all $x, y \in S$ and $k \in \mathbb{N}$. Almost unperforation is equivalent to the property of strict comparison, defined as follows: for all $x, y \in \text{Cu}(A)$ and $\varepsilon > 0$ if $\lambda(x) \leq (1 - \varepsilon)\lambda(y)$ for all $\lambda \in F(S)$ then $x \leq y$. S is called almost divisible if for all $x', x \in S$ with $x' \ll x$, and $n \in \mathbb{N}$, there exists $y \in S$ such that $ny \leq x$ and $x' \leq (n + 1)y$. C^* -algebras that tensorially absorb the Jiang-Su C^* -algebra are pure, [13].

Let us denote the cone of densely finite functionals $F_0(\text{Cu}(A))$ simply by Q . Recall that Q may also be regarded as the cone of densely finite 2-quasitraces on A . For the remainder of this section, A denotes a simple, separable, pure C^* -algebra such that $Q \neq \{0\}$ (whence, A is stably finite). Our goal is to calculate $\text{Cu}^\sim(A)$. We accomplish this in Theorem 6.11 below. This calculation, under the additional assumption that A has stable rank one, is obtained in [10]. We follow closely the same arguments, while circumventing the stable rank one assumption.

LEMMA 6.6. *$\text{Cu}^\sim(A)$ is simple in the sense that for every nonzero $z \in \text{Cu}^\sim(A)_+$ we have that $\infty \cdot z$ is the largest element of $\text{Cu}^\sim(A)$.*

PROOF. The order ideals of $\text{Cu}(A)$ that are closed under addition and sequential suprema are in bijection with the closed two-sided ideals of A , [4]. Since A is simple, the lemma is true for $\text{Cu}(A)$: if $\tilde{z} \in \text{Cu}(A)$ is nonzero then $\infty \cdot \tilde{z}$ is the largest element of $\text{Cu}(A)$. But $\text{Cu}(A)$ is mapped onto $\text{Cu}^\sim(A)_+$ by the Cu -morphism q , Lemma 4.3. This readily implies the lemma. More explicitly, suppose that $y \in \text{Cu}^\sim(A)$ and $z \in \text{Cu}^\sim(A)_+$, with $z \neq 0$. Then $y \leq x$ for some $x \geq 0$. Let $\tilde{x}, \tilde{z} \in \text{Cu}(A)$ be lifts of x and z along q . Then $\tilde{x} \leq \infty \cdot \tilde{z}$, yielding that $y \leq x \leq \infty \cdot z$, as desired.

Let us call an element $x \in \text{Cu}^\sim(A)$ soft if for each $x' \ll x$ there exists a nonzero $z \in \text{Cu}^\sim(A)_+$ such that $x' + z \leq x$. (This definition is well suited for $\text{Cu}^\sim(A)$ with simple A , but should be amended in the nonsimple case.)

LEMMA 6.7. *The soft elements form an absorbing subsemigroup: if $x, y \in \text{Cu}^\sim(A)$ and y is soft then $x + y$ is soft.*

PROOF. If $w \ll x + y$ then $w \leq x + y'$ for some $y' \ll y$. Find $z \geq 0$ nonzero such that $y' + z \leq y$. Then $w + z \leq x + y' + z \leq x + y$.

LEMMA 6.8. *Each $x \in \text{Cu}^\sim(A)$ is either soft or compact and not both.*

PROOF. Let $x' \ll x$. Choose $x'' \in \text{Cu}^\sim(A)$ such that $x' \ll x'' \ll x$. Find $z \geq 0$ such that $x' + z \leq x \leq x'' + z$, Theorem 4.4(ii). If $z = 0$ then $x = x'' \ll x$, so x is compact. If it is the case that $z \neq 0$ for all $x' \ll x$ then x is soft.

If x is both soft and compact, then $x + z = x$, with z positive and nonzero. Since densely finite functionals are finite on compact elements, $\lambda(z) = 0$ for any $\lambda \in \mathcal{Q}$. By the simplicity of $\text{Cu}^\sim(A)$, z is full. Hence $\lambda = 0$, which contradicts our assumption that $\mathcal{Q} \neq \{0\}$.

Let $\text{Lsc}(\mathcal{Q}, \overline{\mathbb{R}})$ denote the set of functions $f: \mathcal{Q} \rightarrow \overline{\mathbb{R}}$ that are lower semi-continuous, linear, and map 0 to 0. We have shown in Theorem 6.5 that $\hat{x} \in \text{Lsc}(\mathcal{Q}, \overline{\mathbb{R}})$ for every $x \in \text{Cu}^\sim(A)$. Let $\text{Lsc}(\mathcal{Q})_{++}$ denote the functions in $\text{Lsc}(\mathcal{Q}, \overline{\mathbb{R}})$ that are strictly positive on the nonzero functionals. By the simplicity of $\text{Cu}^\sim(A)$, if $x \in \text{Cu}^\sim(A)_+$ is nonzero then $\hat{x} \in \text{Lsc}(\mathcal{Q})_{++}$.

We will make use of the calculation of $\text{Cu}(A)$ (see [15, Theorem 6.2]). Here are the main facts that we will need:

Fact 1: Every element of $\text{Cu}(A)$ is either compact or soft (same definition as above).

Fact 2: The map $x \mapsto \hat{x}$ is an isomorphism of the semigroup of soft elements in $\text{Cu}(A)$ with $\text{Lsc}(\mathcal{Q})_{++}$.

LEMMA 6.9. *The map $x \mapsto \hat{x}$ is an ordered semigroup isomorphism from the set of positive soft elements in $\text{Cu}^\sim(A)_+$ to $\text{Lsc}(\mathcal{Q})_{++}$.*

PROOF. Recall that $\hat{x} = \widehat{q(x)}$ for all $x \in \text{Cu}(A)$. Further, by Fact 2 above, $x \mapsto \hat{x}$ is an isomorphism from the soft elements of $\text{Cu}(A)$ to $\text{Lsc}(\mathcal{Q})_{++}$. It thus suffices to show that $q: \text{Cu}(A) \rightarrow \text{Cu}^\sim(A)_+$ is an ordered semigroup isomorphism from the subsemigroup of soft element of $\text{Cu}(A)$ to the subsemigroup of positive soft elements in $\text{Cu}^\sim(A)$. We prove this next.

Suppose that $y \in \text{Cu}(A)$ is soft. Let $x' \ll q(y)$. Choose $y' \ll y$ such that $x' \leq q(y')$ (recall that q is a Cu-morphism). Choose a nonzero $w \geq 0$ such that $y' + w \leq y$. Then $x' + q(w) \leq q(y)$. Further, we cannot have $q(w) = 0$, for otherwise $\hat{w} = 0$, which implies that $w = 0$ (by strict comparison of $\text{Cu}(A)$). Thus, q maps soft elements of $\text{Cu}(A)$ to soft elements of $\text{Cu}^\sim(A)$.

To see that q is an order embedding on the soft elements, recall that every functional on $\text{Cu}(A)$ factors through q . Hence, $\widehat{q(y)} = \hat{y}$ for all $y \in \text{Cu}(A)$. Further, $\text{Cu}(A) \ni y \mapsto \hat{y} \in \text{Lsc}(\mathcal{Q})_{++}$ is an order isomorphism on the soft elements (Fact 2 recalled above). Thus, q is an order embedding.

Finally, let us prove surjectivity: any $x \in \text{Cu}^\sim(A)_+$ has a lift in $\text{Cu}(A)$. If the lift is compact, then x is compact, since q preserves \ll . Thus, soft elements in $\text{Cu}^\sim(A)$ lift to soft elements in $\text{Cu}(A)$.

LEMMA 6.10. Let $\text{Cu}^\sim(A)_{\text{sft}}$ denote the set of soft elements in $\text{Cu}^\sim(A)$.

- (i) For all $x, y \in \text{Cu}^\sim(A)$, with x soft, $\hat{x} \leq \hat{y}$ implies $x \leq y$.
- (ii) The map $x \mapsto \hat{x}$ is an isomorphism from $\text{Cu}^\sim(A)_{\text{sft}}$ to $\text{Lsc}(Q, \overline{\mathbb{R}})$.

PROOF. (i) Let $x, y \in \text{Cu}^\sim(A)$ be such that $\hat{x} \leq \hat{y}$ and x is soft. Choose a sequence of functions $f_n \in \text{Lsc}(Q)_{++}$ that are finite-valued, continuous, and with supremum the function equal to infinity on all Q except 0. By the isomorphism of the set of positive soft elements in $\text{Cu}^\sim(A)$ with $\text{Lsc}(Q)_{++}$, there exists an increasing sequence of positive soft elements $(z_n)_n$ in $\text{Cu}^\sim(A)_+$ such that $\hat{z}_n = f_n$. Since

$$x + \infty = y + \infty = \infty \geq 0$$

and $0 \in \text{Cu}^\sim(A)$ is compact, there exists n such that $x + z_n \geq 0$ and $y + z_n \geq 0$. Set $z = z_n$. Then $\hat{x} + \hat{z} \leq \hat{y} + \hat{z}$. Since $x + z$ and $y + z$ are soft and positive, we have $x + z \leq y + z$ in $\text{Cu}^\sim(A)_+$. Let $x' \ll x$. Let us show that $x' + z \ll x + z$. Choose x'' such that $x' \ll x'' \ll x$. Since x is soft, there exists a nonzero $w \geq 0$ such that $x'' + w \leq x$. Then $x'' + w + z \leq x + z$. In $\text{Lsc}(Q)_{++}$, we have $\hat{z} \ll \hat{w} + \hat{z}$, since \hat{w} is strictly positive and \hat{z} is continuous and finite. It follows that $z \ll z + w$ in $\text{Cu}^\sim(A)$. Hence,

$$x' + z \ll x'' + (w + z) \leq x + z \leq y + z.$$

By weak cancellation (Corollary 4.6), $x' \leq y$. Passing to the supremum over all $x' \ll x$, we conclude that $x \leq y$.

(ii) It is clear from (i) that $x \mapsto \hat{x}$ is an order embedding. Let us prove surjectivity. We have already shown that the functions in $\text{Lsc}(Q)_{++}$ are in the range of this map. Any function in $\text{Lsc}(Q, \overline{\mathbb{R}})$ can be expressed in the form $g - f$, where $f, g \in \text{Lsc}(Q)_{++}$ and f is continuous and finite. (If $h \in \text{Lsc}(Q, \overline{\mathbb{R}})$, then h is bounded from below on any compact base of Q by lower semicontinuity. Choose $f_0 \in \text{Lsc}(Q)_{++}$ continuous. Then the desired decomposition is $h = (h + Cf_0) - Cf_0$, with $C > 0$ sufficiently large.) It thus suffices to show that if $f \in \text{Lsc}(Q)_{++}$ is continuous and finite then $-f$ is in the range of the map $x \mapsto \hat{x}$. Let $\varepsilon > 0$. Choose $x, y \in \text{Cu}^\sim(A)$ positive, soft, such that $\hat{x} = f$ and $\hat{y} = \varepsilon f$. We have $(1 + \varepsilon)f \ll (1 + 2\varepsilon)f$ in $\text{Lsc}(Q)_{++}$ (since f is continuous, finite, and strictly positive). Hence $x + y \ll x + 2y$. Using Theorem 4.4 (ii), choose $w \in \text{Cu}^\sim(A)$ such that

$$x + y + w \leq 0 \leq x + 2y + w.$$

Set $z = y + w$. Then z is soft, since y is, and

$$-(1 + 2\varepsilon)f \leq \hat{z} \leq -(1 + \varepsilon)f.$$

Applying the construction of z for $\varepsilon_n = 1/2^n$, with $n = 1, 2, \dots$, we obtain a sequence $(z_n)_n$ of soft elements such that \hat{z}_n is increasing and has supremum $-f$. Since the z_n are soft, the sequence is increasing in $\text{Cu}^\sim(A)$ (by (i)). Letting $z = \sup z_n$ we have $\hat{z} = -f$, as desired.

Let $\text{Cu}^\sim(A)_c$ and $\text{Cu}^\sim(A)_{\text{sft}}$ denote the subsemigroups of compact and soft elements respectively. By Lemma 6.8, $\text{Cu}^\sim(A)$ is the disjoint union of these two subsemigroups. By Theorem 6.1, $\text{Cu}^\sim(A)_c$ is isomorphic to $K_0(A)$ via the map (6.1) (recall that our assumptions on A imply that it is stably finite). By Lemma 6.10, $\text{Cu}^\sim(A)_{\text{sft}}$ is isomorphic to $\text{Lsc}(Q, \overline{\mathbb{R}})$ via the map $x \mapsto \hat{x}$, where $Q = F_0(\text{Cu}(A))$. Let us define a bijection

$$\text{Cu}^\sim(A) \rightarrow K_0(A) \sqcup \text{Lsc}(Q, \overline{\mathbb{R}})$$

by combining these two isomorphisms:

$$\begin{aligned} \overline{[p]} - \overline{[q]} &\mapsto [p]_{MvN} - [q]_{MvN} \in K_0(A), & \text{if } \overline{[p]} - \overline{[q]} \in \text{Cu}^\sim(A)_c, \\ x &\mapsto \hat{x} \in \text{Lsc}(Q, \overline{\mathbb{R}}), & \text{if } x \in \text{Cu}^\sim(A)_{\text{sft}}. \end{aligned} \tag{6.4}$$

We obtain a map from $K_0(A)$ to $\text{Lsc}(Q, \overline{\mathbb{R}})$ by first regarding $x \in K_0(A)$ as a compact element in $x \in \text{Cu}^\sim(A)$ and then finding $\hat{x} \in \text{Lsc}(Q, \overline{\mathbb{R}})$. Let us continue to denote this map with a hat: $K_0(A) \ni x \mapsto \hat{x} \in \text{Lsc}(Q, \overline{\mathbb{R}})$. We can now endow $K_0(A) \sqcup \text{Lsc}(Q, \overline{\mathbb{R}})$ with an order and an addition operation as follows: On the sets $K_0(A)$ and $\text{Lsc}(Q, \overline{\mathbb{R}})$, these are already defined. Let $x \in K_0(A)$ and $f \in \text{Lsc}(Q, \overline{\mathbb{R}})$. We define

$$x + f := \hat{x} + f \in \text{Lsc}(Q, \overline{\mathbb{R}}).$$

We define $f \leq x$ if $f \leq \hat{x}$ and $x \leq f$ if $\hat{x} + h = f$ for some $h \in \text{Lsc}(Q)_{++}$.

THEOREM 6.11. *Let A be a simple, separable, pure C^* -algebra such that $F_0(\text{Cu}(A)) \neq \{0\}$. Then the map defined in (6.4) is an isomorphism of ordered semigroups.*

PROOF. We have already shown that this map is bijective and an isomorphism when restricted both to the subsemigroups of compact elements and of soft elements. It remains to show that it is additive and an order embedding.

Let $x, y \in \text{Cu}^\sim(A)$ with x soft and y compact. Then $x + y$ is soft. So the map is additive by the definition of addition on $K_0(A) \sqcup \text{Lsc}(Q, \overline{\mathbb{R}})$.

Suppose that the image of x is less than or equal to the image of y , i.e., $\hat{x} \leq \hat{y}$. Then, by Lemma 6.10, $x \leq y$.

Suppose on the other hand that the image of y is less than the image of x . By the definition of the order in the codomain, this means that $\hat{y} + h = \hat{x}$, where $h \in \text{Lsc}(Q)_{++}$. Let $z \in \text{Cu}^\sim(A)_+$ be a soft element such that $\hat{z} = h$.

Then $y + z$ is soft and $\hat{y} + \hat{z} = \hat{x}$. Hence, $y + z = x$, by Lemma 6.10. Since $z \geq 0$, we have that $y \leq x$.

EXAMPLE 6.12. Let \mathcal{W} denote the Jacelon-Razak C^* -algebra, [9]. Then $\text{Cu}(\mathcal{W}) \cong [0, \infty]$ and $K_0(\mathcal{W}) = \{0\}$. Thus, $\text{Cu}^\sim(\mathcal{W}) \cong \{0\} \sqcup \overline{\mathbb{R}}$. Notice that the neutral element is $0 \in \{0\}$ and not $0 \in \overline{\mathbb{R}}$, which is soft.

The calculation of $\text{Cu}^\sim(A)$ in Theorem 6.11 applies to simple \mathcal{L} -stable, projectionless C^* -algebras. Moreover, in this case $\text{Cu}^\sim(A)$ agrees with the original construction in [10], since these C^* -algebras have finite stable rank (see §4.2), as we now show.

THEOREM 6.13. *Let A be a simple, projectionless, \mathcal{L} -stable C^* -algebra. Then the stable rank of A is at most two.*

PROOF. By [12, Corollary 3.2], A almost has stable rank one, in the sense that A is contained in the closure of the invertible elements of \tilde{A} . Let us show that if A almost has stable rank one then it has stable rank at most two. Let $(\alpha_1 \cdot 1 + a_1, \alpha_2 \cdot 1 + a_2) \in \tilde{A} \times \tilde{A}$, where $\alpha_1, \alpha_2 \in \mathbb{C}$ and $a_1, a_2 \in A$. Our goal is to show that they are in the closure of the left invertible pairs in $\tilde{A} \times \tilde{A}$. By a small perturbation, we may assume without loss of generality that $(\alpha_1, \alpha_2) \neq (0, 0)$. Multiplying the vector $(\alpha_1 \cdot 1 + a_1, \alpha_2 \cdot 1 + a_2)$ by a suitable invertible scalar matrix, we may further assume that $(\alpha_1, \alpha_2) = (1, 0)$. The pair now has the form $(1 + a_1, a_2)$, with $a_1, a_2 \in A$. But, by the almost stable rank one property, a_2 is in the closure of the invertible elements in \tilde{A} . Let $\tilde{a}_2 \in \tilde{A}$ be invertible. Then $(1 + a_2, \tilde{a}_2)$ is left invertible, with left inverse $(0, \tilde{a}_2^{-1})$. This proves the theorem.

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