NONUNIQUENESS OF IMMEDIATE MAXIMAL EXTENSIONS OF A VALUATION*

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An extension \( w \) of a valuation \( v \) is immediate if the residue field and value group of \( w \) are the same as those of \( v \), that is, if \( e(w/v) = f(w/v) = 1 \) where \( e(w/v) \) is the ramification index, \( f(w/v) \) the residue class degree of \( w \) over \( v \). A valuation is maximal if it admits no proper immediate extensions. A maximal valuation of \( K \) is henselian [5, p. 231], that is, \( v \) admits only one extension to any algebraic extension of \( K \) (we adopt the convention that an extension of a valuation \( v \) of \( K \) to an algebraic extension takes its values in the divisible group generated by the value group of \( v \)).

Krull [3, § 13] established that every valuation has an immediate maximal extension, and in [2] Kaplansky obtained conditions insuring the uniqueness of an immediate maximal extension. (An immediate maximal extension \( w \) to \( M \) of a valuation \( v \) of \( K \) is unique if for every immediate maximal extension \( w' \) to \( M' \) of \( v \), there is a \( K \)-isomorphism \( \sigma \) from \( M \) to \( M' \) satisfying \( w' \circ \sigma = w \).)

If \( w \) is a valuation of a field \( M \) whose residue field \( k \) has prime characteristic \( p \), we shall say that \( w \) is a Kaplansky valuation if \( w \) is maximal, the value group \( G \) of \( w \) satisfies \( p \cdot G = G \), and \( k \) satisfies the following condition:

\[(K) \quad \text{For any } \beta, \beta_0, \beta_1, \ldots, \beta_{n-1} \in k, \text{ the polynomial } X^p + \beta_{n-1}X^{p-1} + \ldots + \beta_1X^p + \beta_0X + \beta \text{ has a root in } k.\]

If \( w \) is a maximal valuation of \( M \) that is an immediate extension of valuation \( v \) of \( K \), we shall say that \( w \) satisfies the Uniqueness Condition relative to \( v \) if for every subfield \( L \) of \( M \) containing \( K \), \( w \) is a unique immediate maximal extension of its restriction to \( L \).

Kaplansky [2, Theorem 5] proved that an immediate maximal extension \( w \) of a valuation \( v \) is unique if either the residue field \( k \) of \( w \) has characteristic zero or \( k \) has prime characteristic and \( w \) is a Kaplansky valuation. Since these conditions pertain only to \( w \), we conclude:

**Theorem A (Kaplansky).** Let \( w \) be an immediate maximal extension of a

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valuation \( v \). If the residue field \( k \) of \( w \) has characteristic zero, or if \( k \) has prime characteristic and \( w \) is a Kaplansky valuation, then \( w \) satisfies the Uniqueness Condition relative to \( v \).

Kaplansky showed by an example [2, pp. 318–9] that an immediate maximal extension did not always satisfy the Uniqueness Condition. Our principal purpose here is to show that if the residue field of a valuation \( v \) of \( K \) has prime characteristic, and if the value group of \( v \) is archimedean in case \( K \) has zero characteristic, then an immediate maximal extension \( w \) of \( v \) to \( M \) satisfying the Uniqueness Condition is necessarily a Kaplansky valuation unless \( M \) is a rather small extension of \( K \), more precisely, unless the algebraic closure of \( K \) in \( M \) is dense. For this, we need to recall some concepts and theorems concerning maximal valuation:

An Ostrowski net \( (a_\beta)_{\beta \in B} \) for a valuation \( v \) of \( K \) is a family of elements of \( K \) indexed by a totally ordered set \( B \) having no largest element such that \( \nu(a_\lambda - a_\mu) < \nu(a_\mu - a_\nu) \) whenever \( \lambda < \mu < \nu \). The gauge of an Ostrowski net \( (a_\beta)_{\beta \in B} \) is the family \( \gamma_{\beta} = \nu(a_\beta - a_\lambda) \) for all \( \lambda > \beta \); it is a strictly increasing family. Obviously, an Ostrowski net for a valuation \( v \) is also one for any extension of \( v \). Committing an abuse of language, we shall say that an element \( c \in K \) is adherent to an Ostrowski net \( (a_\beta)_{\beta \in B} \) in \( K \) (or for \( v \)) if \( c \) is adherent to the filter base \( (a_\beta + M_{\gamma_\beta})_{\beta \in B} \) on \( K \), where \( (\gamma_\beta)_{\beta \in B} \) is the gauge of \( (a_\beta)_{\beta \in B} \) and

\[
M_{\gamma_\beta} = \{ x \in K : \nu(x) \geq \gamma_\beta \}.
\]

As each \( M_{\gamma_\beta} \) is closed, \( c \) is adherent to \( (a_\beta)_{\beta \in B} \) if and only if

\[
c \in \bigcap_{\beta \in B} (a_\beta + M_{\gamma_\beta}),
\]

or equivalently, if and only if \( \nu(c - a_\beta) = \gamma_\beta \) for all \( \beta \in B \). We note that the associated filter base \( (a_\beta + M_{\gamma_\beta})_{\beta \in B} \) is a Cauchy filter base if and only if the gauge \( (\gamma_\beta)_{\beta \in B} \) is cofinal in \( G \), or equivalently, if and only if \( (\gamma_\beta)_{\beta \in B} \) is unbounded above in \( G \). Kaplansky [2, Theorem 4] proved that a valuation \( v \) of \( K \) is maximal if and only if each Ostrowski net for \( v \) has an adherent point in \( K \).

Among the theorems we shall use are the following:

**Theorem B** [2, Theorem 1]. If a valuation \( w \) of a field \( L \) is an immediate extension of a valuation \( v \) of \( K \), then for any \( c \in L \setminus K \) there is an Ostrowski net \( (a_\beta)_{\beta \in B} \) for \( v \) such that the points adherent to \((a_\beta)_{\beta \in B}\) for \( w \) include \( c \) but no point of \( K \).
Theorem C [2, p. 306; 5, pp. 94–5]. If \((a_\beta)_{\beta \in B}\) is an Ostrowski net for a valuation \(v\) of \(K\) and if \(f\) is a nonzero polynomial over \(K\), then \(\langle v(f(a_\beta)) \rangle_{\beta \in B}\) is either eventually strictly increasing or eventually stationary, and the former case occurs if and only if some root of \(f\) in the algebraic closure \(\Omega\) of \(K\) is adherent to \((a_\beta)_{\beta \in B}\) for some extension of \(v\) to \(\Omega\).

If there is a nonzero polynomial \(f \in K[X]\) such that \(\langle v(f(a_\beta)) \rangle_{\beta \in B}\) is eventually strictly increasing, \((a_\beta)_{\beta \in B}\) is called algebraic, and any monic polynomial \(q\) of least degree such that \(\langle v(q(a_\beta)) \rangle_{\beta \in B}\) is eventually strictly increasing is called a minimal polynomial of \((a_\beta)_{\beta \in B}\). If \((a_\beta)_{\beta \in B}\) is not algebraic, it is called transcendental. The following theorem was proved in a different way by Kaplansky [2, pp. 307–8]:

Theorem D. Let \((a_\beta)_{\beta \in B}\) be an Ostrowski net for a valuation \(v\) of \(K\), and let \(w\) be an extension of \(v\) to \(L\). If \(c \in L\) is adherent to \((a_\beta)_{\beta \in B}\) for \(w\), then for any nonzero polynomial \(f \in K[X]\) such that \(\langle v(f(a_\beta)) \rangle_{\beta \in B}\) is eventually stationary, \(w(f(c)) = w(f(a_\mu))\) for all sufficiently large \(\mu \in B\).

Proof. There is a polynomial \(h \in L[X]\) such that

\[
f(c) - f(X) = (c - X)h(X) .
\]

As \(\langle w(c - a_\beta) \rangle_{\beta \in B}\) is strictly increasing and as \(\langle w(h(a_\beta)) \rangle_{\beta \in B}\) is either eventually stationary or eventually strictly increasing, \(\langle w(f(c) - f(a_\beta)) \rangle_{\beta \in B}\) is eventually strictly increasing. Therefore for sufficiently large \(\lambda\), if \(\mu > \lambda\), then

\[
w(f(c) - f(a_\mu)) > w(f(c) - f(a_\lambda)) \geq \min \{w(f(c)), w(f(a_\lambda))\}
\]

\[= \min \{w(f(c)), w(f(a_\mu))\} .
\]

Consequently, either \(w(f(c)) < w(f(c) - f(a_\mu))\) or \(w(f(a_\mu)) < w(f(c) - f(a_\mu))\), and in either case

\[
w(f(c)) = \min \{w(f(c) - f(a_\mu)), w(f(a_\mu))\} = w(f(a_\mu)) .
\]

This theorem is used to establish, in particular, the following two theorems:

Theorem E [2, Theorem 2]. If \((a_\beta)_{\beta \in B}\) is a transcendental Ostrowski net for a valuation \(v\) of \(K\) and if \(c\) is adherent to \((a_\beta)_{\beta \in B}\) for an extension \(w\) of \(v\) to \(L\), then \(c\) is transcendental over \(K\), for any polynomial \(f \in K[X]\),

\[
w(f(c)) = w(f(a_\mu))\quad \text{for all sufficiently large } \mu \in B ,
\]

and the restriction of \(w\) to \(K(c)\) is an immediate extension of \(v\).
THEOREM F [2, Theorem 3]. If \( q \) is a minimal polynomial of an algebraic Ostrowski net \((a_\beta)_{\beta \in B}\) for a valuation \( v \) of \( K \), then \( q \) is a prime polynomial, and there is an immediate extension of \( v \) to a stem field of \( q \).

Theorems E and F are used to establish the following theorem, which is implicit in [2]:

THEOREM G. Let \( v \) be a valuation of \( K \). An immediate maximal extension of \( v \) satisfies the Uniqueness Condition relative to \( v \) if and only if for every immediate maximal extension \( w \) of \( v \) to a field \( M \) and for each subfield \( L \) of \( M \) containing \( K \), every minimal polynomial of every algebraic Ostrowski net on \( L \) has a root in \( M \).

We begin with some results concerning henselian valuations:

THEOREM 1. The restriction of a henselian valuation \( w \) of \( L \) to a subfield \( K \) of \( L \) that is algebraically closed in \( L \) is henselian.

PROOF. A valuation \( u \) of a field \( F \) is henselian if and only if every polynomial over \( F \) of the form

\[ X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 , \]

where \( u(a_i) \geq 0 \) for all \( i \in [0,n-1] \), \( u(a_1) = 0 \), \( u(a_0) > 0 \), has a root \( c \) in \( F \) satisfying \( u(c) > 0 \) [4, pp. 94-5]. Let \( f \) be such a polynomial over \( K \) for \( v \). Then \( f \) has a root \( c \in L \) satisfying \( w(c) > 0 \). But as \( K \) is algebraically closed in \( L \), \( c \in K \). Therefore \( v \) is henselian.

Theorem 1 provides an easy proof of the fact that a henselization of a valuation [5, pp. 175-8] is an immediate extension:

THEOREM 2 [5, p. 184]. A henselization of a valuation \( v \) of \( K \) is an immediate extension of \( v \).

PROOF. By Krull's theorem, \( v \) has an immediate maximal extension \( w \) to a field \( M \). As \( w \) is henselian, its restriction to the algebraic closure \( H \) of \( K \) in \( M \) is henselian by Theorem 1. As \( H \) is an algebraic extension of \( K \), \( H \) contains a henselization of \( K \), which therefore is an immediate extension of \( K \). Since any two henselizations of \( v \) are equivalent, the assertion follows.

The same proof establishes the following:

THEOREM 3. If \( w \) is a henselian valuation of a field \( L \), every subfield of \( L \) has a henselization contained in \( L \).
We may now identify those immediate maximal extensions of a valuation that are so small their uniqueness is assured without any further hypothesis:

**Theorem 4.** Let $w$ be an immediate maximal extension to $M$ of a valuation $v$ of $K$. If $M = \hat{K}$, the completion of a henselization $K_h$ of $K$, then $w$ satisfies the Uniqueness Condition relative to $v$.

**Proof.** We first observe that if $K_r$ is another henselization of $K$ contained in $M$, then $M = \hat{K_r}$. Indeed, there is a $K$-isomorphism from $K_h$ to $K_r$ satisfying $(w \circ \sigma)(x) = w(x)$ for all $x \in K_h$ and $\sigma$ extends by continuity to a $K$-isomorphism $\hat{\sigma}$ from $M = \hat{K_h}$ to $\hat{K_r}$ satisfying $w \circ \hat{\sigma} = w$. Therefore $K_r$ is a maximal field. As $M$ is a immediate extension of $\hat{K_r}$, $M = \hat{K_r}$.

Now let $L$ be any subfield of $M$ containing $K$. By Theorem 3, $M$ contains a henselization $L_h$ of $L$. Again by Theorem 3, $L_h$ contains a henselization $K_r$ of $K$. As $M$ is complete, $M \supseteq L_h \supseteq K_r = M$. Let $w'$ be an immediate maximal extension to $M'$ of the restriction of $w$ to $L$. By Theorem 3, $M'$ contains a henselization $L_h'$ of $L$. Thus there is an $L$-isomorphism $\tau$ from $L_h'$ to $L_h$ satisfying $(w' \circ \tau)(x) = w(x)$ for all $x \in L_h$, and $\tau$ extends by continuity to an $L$-monomorphism $\hat{\tau}$ from $M = L_h$ into $M'$ satisfying $w' \circ \hat{\tau} = w$. Again, $M'$ is an immediate maximal extension of the maximal field $\hat{\tau}(M)$, so $M' = \hat{\tau}(M)$.

**Corollary.** If $v$ is a valuation of a field $K$ and if its completion $\hat{v}$ on $\hat{K}$ is a maximal valuation, then $\hat{v}$ satisfies the Uniqueness Condition relative to $v$.

Our principal theorem is the following:

**Theorem 5.** Let $v$ be a valuation of a field $K$ whose residue field $k$ has prime characteristic $p$, let $w$ be an immediate maximal extension of $v$ to $M$, let $L$ be the algebraic closure of $K$ in $M$, and let $G$ be the value group of $v$ and $w$. If $w$ satisfies the Uniqueness Condition relative to $v$ and if $M + L\hat{\gamma}$, then $k$ satisfies (K); in addition, if either $K$ has characteristic $p$ or $G$ is archimedean, $p \cdot G = G$.

**Proof.** Let $c \in M \setminus L\hat{\gamma}$, and let $v'$ be the restriction of $w$ to $L$. By Theorem B there is an Ostrowski net $(a_p)_{p \in B}$ for $v'$ whose adherent points in $M$ include $c$ but no point of $L$. As $c \notin L\hat{\gamma}$, the filter base corresponding to $(a_p)_{p \in B}$ is not a Cauchy filter base, so the gauge $(\gamma_p)_{p \in B}$ of $(a_p)_{p \in B}$ is bounded above by some $\gamma \in G$. Let $a \in L$ satisfy $v'(a) = \gamma$. Then $(a^{-1}a_p)_{p \in B}$ is an Ostrowski net for $v'$ whose gauge is bounded above by zero and whose adherent points in $M$ include $a^{-1}c$ but no points of $a^{-1}L = L$. Therefore we may assume that $(\gamma_p)_{p \in B}$ is bounded above by zero.

Suppose $(a_p)_{p \in B}$ were an algebraic Ostrowski net, and let $q \in L[X]$ be a
minimal polynomial. By Theorem G, \( M \) contains a root \( d \) of \( q \), and \( d \in L \) as \( L \) is algebraically closed in \( M \). By Theorem C, some conjugate \( d' \) of \( d \) in an algebraic closure \( \Omega \) of \( L \) is adherent to \( (a_\beta)_{\beta \in B} \) for an extension \( v'' \) of \( v' \) to \( \Omega \). As \( w \) is henselian, so is \( v' \) by Theorem 1. If \( \sigma \) is an \( L \)-automorphism of \( \Omega \) such that \( \sigma(d) = d' \), then \( v'' \circ \sigma = v'' \) as \( v' \) is henselian, so

\[
\gamma_\beta = v''(d' - a_\beta) = v''(\sigma(d) - \sigma(a_\beta)) = (v'' \circ \sigma)(d - a_\beta) = v'(d - a_\beta),
\]

and hence \( d \in L \) is adherent to \( (a_\beta)_{\beta \in B} \), a contradiction. Thus \( (a_\beta)_{\beta \in B} \) is a transcendental Ostrowski net. (If \( K \) has characteristic \( p \), we are now in the situation of Kaplansky's counter-example [2, pp. 318-9].)

We shall show that if \( b, b_0, b_1, \ldots, b_{n-1} \) belong to the valuation ring \( A_v \) of \( v' \) and if

\[
g(X) = X^p + b_{n-1}X^{p-1} + \ldots + b_1X + b_0X,
\]

then there exists \( c_1 \in M \) such that \( g(c) - g(c_1) = -b \). To do so, we shall first establish that for some subset \( C \) of \( B \) of the form \( B \setminus I \), where \( I = \emptyset \) or \( I \) is an initial segment of \( B \) (a set of the form \( \{ \beta \in B : \beta \leq \delta \} \) for some \( \delta \in B \), \( (g(a_\beta))_{\beta \in C} \) is an Ostrowski net for \( v' \) whose gauge is bounded above by zero, and \( g(c) \) is adherent to \( (g(a_\beta))_{\beta \in C} \) for \( w \).

Case 1. \( K \) has characteristic \( p \). We take \( C = B \). For any \( x \in M \) such that \( w(x) < 0 \), clearly \( w(g(x)) = p^nw(x) \). Since \( g(y) - g(z) = g(y - z) \) for all \( y, z \in M \), it follows readily that \( (g(a_\beta))_{\beta \in B} \) is an Ostrowski net whose gauge is \( (p^n \gamma_\beta)_{\beta \in B} \) and that \( g(c) \) is adherent to \( (g(a_\beta))_{\beta \in B} \) for \( w \).

Case 2. \( K \) has zero characteristic. We first observe that if \( x, b \in M \) satisfy \( w(x) < w(x - b) < 0 \) and if \( w(p) \neq (p - 1)(w(x - b) - w(b)) \), then

\[
(1) \quad w(x^p - b^p) = \min\{pw(x - b), w(p) + (p - 1)w(b) - w(x - b)\}.
\]

Indeed, let \( f(X) = X^p - b^p \). Expanding \( f \) in Taylor series about \( b \), we obtain

\[
f(X) = \sum_{j=1}^{p} \binom{p}{j} b^{p-j} (x-b)^j.
\]

If \( 1 \leq j \leq p \), \( p \) divides \( \binom{p}{j} \) in the valuation \( A_w \) of \( w \) but \( p^2 \) does not, so \( w((\binom{p}{j} b^{p-j}(x-b)^j)) = w(p) \). Consequently, if \( 1 < j < p \), then as \( w(x-b) > w(b) \),

\[
w\left(\binom{p}{j} b^{p-j}(x-b)^j\right) = w(p) + (p-j)w(b) + jw(x-b)
\]

\[
> w(p) + (p-1)w(b) + w(x-b) = w(pb^{p-1}(x-b)).
\]

Thus
\[
   w\left(\sum_{j=0}^{p-1} \binom{p}{j} b^{p-j} (x-b)^j\right) = w(p) + (p-1)w(b) + w(x-b).
\]

Also
\[
   w((x-b)^p) = pw(x-b) = w(p) + (p-1)w(b) + w(x-b)
\]
by hypothesis. Hence (1) holds.

As \((a_\beta)_{\beta \in B}\) is transcendental, \((v'(a_\beta))_{\beta \in B}\) is eventually stationary, so there exists \(\beta_0 \in B\) and \(\gamma \in G\) such that \(v'(a_\beta) = \gamma\) for all \(\beta \geq \beta_0\).

Let
\[
   B_0 = \{\beta \in B : \beta > \beta_0\}.
\]

For any \(\beta \in B_0\),
\[
   \gamma_\beta > \gamma_{\beta_0} = v'(a_\beta - a_{\beta_0}) \geq \min \{v'(a_\beta), v'(a_{\beta_0})\} = \gamma.
\]

Thus \((a_\beta)_{\beta \in B_0}\) is an Ostrowski net for \(v'\) whose gauge \((\gamma_{\beta,0})_{\beta \in B_0}\) satisfies
\[
   v'(a_\beta) = \gamma < \gamma_{\beta,0} \quad \text{for all} \ \beta \in B_0.
\]

As \((\gamma_{\beta,0})_{\beta \in B_0}\) is strictly increasing, there is at most one \(\tau_0 \in B\) such that \((p-1)(\gamma_{\tau_0,0} - \gamma) = v'(p)\); let
\[
   B_1 = \{\beta \in B_0 : \beta > \tau_0\}
\]
if there is such a \(\tau_0\), \(B_1 = B_0\) otherwise. By (1), for any \(\lambda, \mu, \nu \in B_1\) such that \(\lambda < \mu < \nu\),
\[
   v'(a_\lambda - a_\mu) = \min \{p\gamma_{\lambda,0}, v'(p) + (p-1)\gamma + \gamma_{\lambda,0}\}
   < \min \{p\gamma_{\mu,0}, v'(p) + (p-1)\gamma + \gamma_{\mu,0}\} = v'(a_\mu - a_\nu).
\]

Thus \((a_\beta)_{\beta \in B_1}\) is an Ostrowski net whose gauge \((\gamma_{1,\beta})_{\beta \in B_1}\) satisfies
\[
   \gamma_{1,\beta} = \min \{p\gamma_{0,\beta}, v'(p) + (p-1)\gamma + \gamma_{0,\beta}\}.
\]

In particular,
\[
   \gamma_{1,\beta} \leq p\gamma_{0,\beta} < \gamma_{0,\beta} < 0 \quad \text{for all} \ \beta \in B_1.
\]

Moreover, \(v'(a_\beta) = p\gamma < \gamma_{1,\beta}\) for all \(\beta \in B_1\). Furthermore, by (1)
\[
   w(c^p - a_\beta) = \min \{pw(c - a_\beta), w(p) + (p-1)w(a_\beta) + w(c - a_\beta)\}
   = \min \{p\gamma_{0,\beta}, v'(p) + (p-1)\gamma + \gamma_{0,\beta}\} = \gamma_{1,\beta}
\]
for all \(\beta \in B_1\), so \(c^p\) is adherent to \((a_\beta)_{\beta \in B_1}\). Applying this result successively to
\[
   (a_0^p), (a_1^p), \ldots, (a_{p-1}^p),
\]
we obtain a decreasing sequence $B_0, B_1, \ldots, B_n$ of subsets of $B$, each either $B$ or
the complement of an initial segment of $B$, such that

$$(a_\beta)_{\beta \in B_0} (a_\beta^p)_{\beta \in B_1}, \ldots, (a_\beta^p)_{\beta \in B_n}$$

are Ostrowski nets whose respective gauges

$$\nu_{0, \beta} = (a_\beta^p)_{\beta \in B_0}, \nu_{1, \beta} = (a_\beta^p)_{\beta \in B_1}, \ldots, (a_\beta^p)_{\beta \in B_n}$$

satisfy $\nu_{n, \beta} < \ldots < \nu_{1, \beta} < \nu_{0, \beta} < 0$ for all $\beta \in B_n$ and $\nu' (a_\beta^p) = p^j \nu < \nu_{j, \beta}$ for all
$j \in [0, n]$ and all $\beta \in B_j$ and whose adherent points for $w$ respectively include $c, c^p, \ldots, c^{p^n}$. Let $C = B_n$. If $j \in [0, n-1]$, then $\nu' (b_j) \geq 0$, so if $\lambda, \mu \in C$ and $\lambda < \mu$,

$$\nu' (b_j a_\lambda^p - b_j a_\mu^p) \geq \nu' (a_\lambda^p - a_\mu^p) = \nu_{j, \lambda} > \nu_{n, \lambda} = \nu (a_\lambda^p - a_\mu^p),$$

whence

$$\nu' (g (a_\lambda) - g (a_\mu)) = \nu' \left( (a_\lambda^p - a_\mu^p) + \sum_{j=0}^{n-1} b_j (a_\lambda^p - a_\mu^p) \right) = \nu_{n, \lambda}.$$

As $(\nu_{n, \beta})_{\beta \in C}$ is the gauge of the Ostrowski net $(a_\beta^p)_{\beta \in C}$, therefore, $(g (a_\beta))_{\beta \in C}$ is an
Ostrowski net whose gauge $(\nu_{n, \beta})_{\beta \in C}$ is bounded above by zero. By the same reasoning,

$$w (g(c) - g(a_\beta)) = \nu_{n, \beta} \quad \text{for all } \beta \in C,$$

so $g(c)$ is adherent in $M$ to $(g(a_\beta))_{\beta \in C}$.

Thus in both cases $(g(a_\beta))_{\beta \in C}$ is an Ostrowski net for $\nu'$ whose gauge is
bounded above by zero, and $g(c)$ is adherent in $M$ to $(g(a_\beta))_{\beta \in C}$. As $\nu' (b) \geq 0$, therefore, $g(c) + b$ is also adherent in $M$ to $(g(a_\beta))_{\beta \in C}$. Moreover, $(g(a_\beta))_{\beta \in C}$ is
clearly transcendental as $(a_\beta^p)_{\beta \in C}$ is. Therefore by Theorem E, $g(c)$ and $g(c) + b$
are transcendental over $L$, and for any $f \in L[X]$

$$w (f (g(c) + b)) = w (f (g(a_\beta))) = w (f (g(c)))$$

for all sufficiently large $\mu \in C$. Let $N = L (g(c)) = L (g(c) + b)$, and let $\tau$ be the $L$-
automorphism of $N$ satisfying $\tau (g(c)) = g(c) + b$. Then $(w \circ \tau)(z) = w(z)$ for all
$z \in N$.

Let $\bar{\tau}$ be the automorphism of $N[X]$ induced by $\tau$. In $N(c)$, $c$ is a root of $g(X) - g(c)$. Let $r$ be the minimal polynomial of $c$ over $N$ and let $c'$ be a root of $\bar{\tau}(r)$
in a stem field $N(c')$ of $\bar{\tau}(r)$. Then $c'$ is also a root of

$$\bar{\tau}(g(X) - g(c)) = g(X) - \tau (g(c)) = g(X) - (g(c) + b),$$

that is, $g(c') = g(c) + b$. Let $\tau'$ be the unique isomorphism from $N(c)$ to $N(c')$
 extending $\tau$ such that $\tau' (c) = c'$, and let $w'$ be an immediate maximal extension to $M'$ of the valuation $w \circ \tau'^{-1}$ of $N(c')$. Both $w$ and $w'$ are then immediate maximal extensions of the restriction of $w$ to $N$, as $(w \circ \tau)(z) = w(z)$ for all $z \in N$.  


By hypothesis, there is an $N$-isomorphism $\sigma$ from $M$ to $M'$. As $c' \in M'$ is a root of $g(X) - g(c) - b \in N[X]$, $c_1 = \sigma^{-1}(c')$ is a root of $g(X) - g(c) - b$ in $M$. Thus $g(c) - g(c_1) = -b$.

**Case 1.** $K$ has characteristic $p$. Then $g(c - c_1) = g(c) - g(c_1) = -b$, so

$$X^{p^n} + b_{n-1}X^{p^{n-1}} + \ldots + b_1X + b$$

has the root $c - c_1$ in $M$, and clearly $w(c - c_1) \geq 0$. Consequently, $k$ satisfies (K). Moreover, $p \cdot G = G$, for if $\gamma \geq 0$ and if $b \in L$ satisfies $v(b) = \gamma$, then $X^{p} - b$ has a root $x$ in $M$, so $w(x) \in G$ and $p \cdot w(x) = \gamma$.

**Case 2.** $K$ has characteristic zero. As $p$ divides $g(c - c_1) - (g(c) - g(c_1))$ in the valuation ring $A_w$ of $w$,

$$g(c - c_1) \equiv g(c) - g(c_1) \pmod{pA_w},$$

that is $g(c - c_1) \equiv -b \pmod{pA_w}$. In particular, as $p$ belongs to the maximal ideal $M_w$ of $A_w$, $g(c - c_1) \equiv -b \pmod{M_w}$. Thus the polynomial in $k[X]$ corresponding to

$$X^{p^n} + b_{n-1}X^{p^{n-1}} + \ldots + b_1X + b$$

has a root in $k$. Hence $k$ satisfies (K). We shall next show that the isolated subgroup $H$ of $G$ generated by $v(p)$ satisfies $p \cdot H = H$. First, suppose $\gamma \in G$ satisfies $0 \leq \gamma < v(p)$. Applying the preceding to $X^{p} - b$, where $b \in L$ satisfies $v(b) = \gamma$, we conclude there exists $x \in A_w$ such that $x^{p} \equiv b \pmod{pA_w}$, that is, $w(x^{p} - b) \geq w(p)$. Hence as $w(b) < w(p)$, $w(x^{p}) = w(b)$, so $w(x) \in G$ and $p \cdot w(x) = \gamma$; in particular, $0 \leq w(x) \leq \gamma$, so $w(x) \in H$. Suppose next that $p^{-1} \cdot \sigma \in G$ whenever $0 \leq \sigma < m \cdot v(p)$, where $m \geq 1$, and let $\gamma \in G$ satisfy $m \cdot v(p) \leq \gamma < (m + 1) \cdot v(p)$. Then $0 \leq \gamma - m \cdot v(p) < v(p)$. By hypothesis, $M = K^c$, so $v$ is not discrete. Consequently, there exists $\tau \in G$ such that $\gamma - m \cdot v(p) < \tau < v(p) \leq \gamma$, whence $0 < \gamma - \tau < m \cdot v(p)$, so both $p^{-1} \cdot \tau$, $p^{-1} \cdot (\gamma - \tau) \in G$, and therefore

$$p^{-1} \cdot \gamma = p^{-1} \cdot \tau + p^{-1} \cdot (\gamma - \tau) \in G.$$

As $0 \leq p^{-1} \cdot \gamma \leq \gamma$, $p^{-1} \cdot \gamma \in H$. Thus $p \cdot H = H$. In particular, if $G$ is archimedean, then $H = G$, so $p \cdot G = G$.

**Corollary.** Let $v$ be a valuation of a field $K$ whose residue field has prime characteristic $p$, and let $w$ be an immediate maximal extension of $v$ to $M$. If $w$ satisfies the Uniqueness Condition relative to $v$, if the algebraic closure of $K$ in $M$ is not dense in $M$, and if either $K$ has characteristic $p$ or the value group of $w$ is archimedean, then $w$ is a Kaplansky valuation.

With the notation of Theorem 5, assume $K$ has prime characteristic or that
the value group is archimedean, and let $K_h$ be a henselization of $K$ in $M$, whence $K_h \subseteq L$. If $M = K_h^\wedge$, $w$ satisfies the Uniqueness Condition relative to $v$ without any further restrictions by Theorem 4, but if $M \supseteq L^\wedge$, $w$ satisfies the Uniqueness Condition relative to $v$ if and only if $w$ is a Kaplansky valuation by Theorems A and 5. Whether $w$ must be a Kaplansky valuation for it to satisfy the Uniqueness Condition remains an open question for the case $K_h^\wedge \subseteq L^\wedge = M$.

To demonstrate the ubiquity of nonunique immediate maximal extensions, we shall apply Theorem 5 to the classical example of an immediate maximal extension introduced by Krull. Let $k$ be a field, $G$ a totally ordered abelian group, and let $S(k, G)$ be the set of all functions from $G$ to $k$ whose support is a well-ordered subset of $G$ (the support, $\text{Supp} f$, of $f \in k^G$ is defined to be $\{x \in G : f(x) \neq 0\}$). Under the usual addition and convolution (defined by $(fg)(z) = \sum f(x)g(y)$, the sum over all $(x, y) \in G \times G$ such that $x + y = z$), $S(k, G)$ is a field; we equip $S(k, G)$ with the valuation $w$ satisfying $w(f)$ = the smallest element in $\text{Supp} f$ for every nonzero $f \in S(k, G)$. (See [1, Exercise 2, § 3, Ch. 6, pp. 173–4].) We denote by $F[k, G]$ the subring of all functions with finite support, and by $F(k, G)$ its field of quotients in $S(k, G)$. Krull proved that $w$ is a maximal valuation, and hence that $S(k, G)$ is an immediate maximal extension of $F(k, G)$ [3, Satz 26]. (We shall apply the terminology for valuations to the fields on which they are defined, as we are henceforth interested only in the restrictions of $w$ to subfields of $S(k, G)$.) For each $x \in G$, we denote by $\delta_x$ the member of $F[k, G]$ defined by $\delta_x(x) = 1$, $\delta_x(y) = 0$ if $y \neq x$. Clearly $(\delta_x f)(y) = f(y - x)$ for all $y \in G$.

Let $G$ be a cyclic group. There is a topological isomorphism from $S(k, G)$ to $k((X))$, equipped with the $X$-adic valuation, that takes $F(k, G)$ to $k(X)$. Consequently $S(k, G)$ is the completion of $F(k, G)$, so by the Corollary of Theorem 4, $S(k, G)$ satisfies the Uniqueness Condition relative to $F(k, G)$. This establishes the sufficiency of the condition in the following theorem:

**Theorem 6.** Let $k$ be a field of prime characteristic $p$, and let $G$ be a subgroup of the additive group $Q$ of rational numbers such that $p \cdot G \neq G$. Then $S(k, G)$ satisfies the Uniqueness Condition relative to $F(k, G)$ if and only if $G$ is cyclic.

We note that each element of $F[k, G]$ is contained in $F[k, aZ]$ for some rational $a > 0$, and consequently each element of $F(k, G)$ is contained in $F(k, bZ)$ for some rational $b > 0$. To establish the necessity of the condition, we need two lemmas:

**Lemma 1.** Let $L$ be a subfield of $S(k, Q)$ containing $F(k, e aZ)$, where $a$ is a positive rational, $e$ a positive integer. If the value group of $L$ is $aZ$ and if $p \nmid e$, then $L \subseteq S(k, aZ)$.
Proof. We shall first prove that any uniformizer $u$ of $L$ belongs to $S(k, a\mathbb{Z})$. As $w(u) = a$, $u(a) = 0$; multiplying $u$ by $u(a)^{-1} \in k$, we may assume $u(a) = 1$. Suppose $u \notin S(k, a\mathbb{Z})$. Then there is a smallest $t \in \text{Supp } u \setminus a\mathbb{Z}$. Let $n \geq 1$ be the largest integer such that $na < t$. We define recursively $u_1, \ldots, u_n \in L$ by $u_1 = u$, $u_{r+1} = u_r - u_r((r+1)a)u_r^{r+1}$ for all $r \in [1, n-1]$. Straightforward calculations establish that for each $r \in [1, n]$, $u_r(a) = 1$, $u_r(t) = u(t) \neq 0$, and the numbers $< t$ in Supp $u_r$ are among $a, (r+1)a, (r+2)a, \ldots, na$. Let $z = u_n \in L$. Then $z(a) = 1$, $z(t) = 0$, and $a, t$ are the smallest numbers in Supp $z$. Easy calculations establish that for each integer $q \geq 0$, $z^{q+1}((q+1)a) = 1$, $z^{q+1}(qa + t) = (q+1)z(t)$, and $z^{q+1}(x) = 0$ for all $x < qa + t$. In particular, let $q = e$; then $\delta_e \in F(k, ea\mathbb{Z}) \subseteq L$, so $z^{e+1} - \delta_e z \in L$,

$$(z^{e+1} - \delta_e z)(ea + t) = ez(t) \neq 0 \quad \text{as } p \nmid e,$$

and consequently $w(z^{e+1} - \delta_e z) = ea + t \notin a\mathbb{Z}$, a contradiction. Therefore, $u \in S(k, a\mathbb{Z})$.

Next, we shall show that a unit $y$ of the valuation ring of $L$ belongs to $S(k, a\mathbb{Z})$. As before, we may assume $y(0) = 1$. Then $w(y - \delta_0) > 0$, so $w(y - \delta_0) \geq a$. If $w(y - \delta_0) = a$, then by the preceding $y - \delta_0 \in S(k, a\mathbb{Z})$, whence $y \in S(k, a\mathbb{Z})$. If $w(y - \delta_0) > a$, let $u$ be a uniformizer of $L$; then $w(y - \delta_0 + u) = a$, so $y - \delta_0 + u \in S(k, a\mathbb{Z})$, whence again $y \in S(k, a\mathbb{Z})$. Therefore $L \subseteq S(k, a\mathbb{Z})$ as each nonzero element of $L$ is a product of a unit and a power of a uniformizer.

Lemma 2. Let $(G_n)_{n \geq 1}$ be an increasing sequence of cyclic groups such that $G = \bigcup_{n=1}^{\infty} G_n$. The algebraic closure $A(k, G)$ of $F(k, G)$ in $S(k, G)$ is contained in $\bigcup_{n=1}^{\infty} S(k, G_n)$.

Proof. Since $p \cdot G \subseteq G$, there is a largest integer $s$ such that $p^{-s} \in G$; we may assume $p^{-s} \in G_1$. Let $z \in A(k, G)$. There exists $m \geq 1$ such that each coefficient of the minimal polynomial of $z$ belongs to $F(k, G_m)$, so $z \in A(k, G_m)$. Let $G_m = b\mathbb{Z}$, where $b > 0$, and let $e$ be the ramification index of $F(k, b\mathbb{Z})[z]$ over $F(k, b\mathbb{Z})$. The value group of $F(k, b\mathbb{Z})[z]$ is then $a\mathbb{Z}$ where $b = ea$. As $z \in S(k, G)$ and $b\mathbb{Z} \subseteq G$, $F(k, b\mathbb{Z})[z] \subseteq S(k, G)$, and therefore $a\mathbb{Z} \subseteq G$. As $p^{-s} \in G_1 \subseteq b\mathbb{Z}$, $p^{-s} = bc$ for some integer $c$. If $e = pd$ for some integer $d$, then $p^{-(s+1)} = acd \in a\mathbb{Z} \subseteq G$, a contradiction. Thus $p \nmid e$, so by Lemma 1, $F(k, b\mathbb{Z})[z] \subseteq S(k, a\mathbb{Z})$. As $a\mathbb{Z} \subseteq G$, $a\mathbb{Z} \subseteq G_n$ for some $n \geq 1$, whence $z \in S(k, G_n)$.

To complete the proof of Theorem 6, it suffices by Theorem 5 and Lemma 2 to show that if $G$ is not cyclic, then $\bigcup_{n=1}^{\infty} S(k, G_n)$ is not dense in $S(k, G)$, where $(G_n)_{n \geq 1}$ is an increasing sequence of cyclic groups whose union is $G$. As $G$ is not
cyclic, \( G \) contains a strictly increasing sequence \((r_n)_{n \geq 1}\) of rationals in \((0,1)\) such that \(\sup r_n = 1\). Let \( g \in S(k,G) \) be the characteristic function of \(\{r_n: n \geq 1\}\). Suppose \( \bigcup_{n=1}^{\infty} S(k,G_n) \) were dense in \( S(k,G) \). Then for some \( m \geq 1 \) and some \( f \in S(k,G_m) \), \( w(g-f) > 1 \), whence \( f(r_n) = g(r_n) = 1 \) for all \( n \geq 1 \). But as \( G_m \) is cyclic, some \( r_n \notin G_m \), whence \( f \notin S(k,G_m) \), a contradiction.

REFERENCES