BETTI NUMBERS OF MONOID ALGEBRAS.
APPLICATIONS TO 2-DIMENSIONAL
TORUS EMBEDDINGS

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Introduction.

The starting point of this paper is the rather elementary observation (1.2),
which leads to a formula (1.3) for the Betti numbers of a monoid algebra in
terms of the combinatorial properties of the monoid, see [2]. The rest of the
paper is concerned with the application of this formula to the case of 2-
dimensional torus embeddings, see [3]. More specifically: In section 1 we give
a method for computing the Betti numbers \( \beta_i = \dim_k \text{Tor}_i^A(k,k) \), when \( A \) is the
monoid algebra over \( k \) of a commutative monoid \( A \) with cancellation law, and
no non-trivial inverses. Proposition 1.3 relates the Betti numbers to the local
homology of the simplicial set associated to \( A_+ = A - \{1\} \) ordered such that
\( \lambda \leq \lambda \cdot \mu \), when \( \lambda, \mu \in A \).

In section 2 this is used to compute the Betti numbers of 2-dimensional torus
embeddings \( A \). In particular we prove that the Betti series

\[
B(t) = \sum_{n \geq 0} \beta_n t^n
\]

of \( A \) is a rational function \( P(t)/Q(t) \). The main result of this paper is, in fact, the
explicit computation of the denominator \( Q(t) \), see Corollary 2.20.

1. Betti numbers of monoid algebras.

Fix a field \( k \) and let \( A \) be a commutative monoid with cancellation law, i.e.
such that \( \lambda \cdot \mu = \lambda \cdot \mu' \) implies \( \mu = \mu' \). Let \( A = k(A) \) and put \( m = A_+ \cdot A \) where \( A_+ = A - \{1\} \). Assume \( A/m = k, \) that is assume \( A \) has no non-trivial subgroups. Put
\( \beta_i = \dim_k \text{Tor}_i^A(k,k) \), the \( i \)th Betti number of \( A \). Then the power series \( B(t) = \sum_{n \geq 0} \beta_n t^n \) is called the Betti series of \( A \). The purpose of this first paragraph is
to give a method for computing the Betti series of \( A \) using only combinatorial
properties of \( A_+ \).

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Let $A_+$ be ordered as follows: $\lambda_1 \leq \lambda_2$ if and only if there exists a $\mu \in A$ such that $\mu \cdot \lambda_1 = \lambda_2$. There is a natural presheaf (projective system)

$$F: A_+ \to \text{Ab}$$

defined by $F(\lambda) = A$, where $F(\lambda_1 \leq \lambda_2): F(\lambda_2) \to F(\lambda_1)$ is multiplication by $\mu = \lambda_2 / \lambda_1$.

**Lemma 1.1.**

$$\lim\limits_{A_+} F = (A_+) \cdot A = m.$$ 

**Proof.** For every $\lambda \in A_+$, consider the morphism $\eta_\lambda: F(\lambda) \to A$, the multiplication by $\lambda$. This defines a morphism

$$\eta: \lim\limits_{A_+} F \to m.$$ 

Given an element $\alpha \in m$, there is a unique representation $\alpha = \sum_{i=1}^{N} \alpha_i \cdot \lambda_i$; $\alpha_i \in k$, $\lambda_i \in A_+$. Consider $\alpha_i$ as an element of $F(\lambda_i)$ and let $\bar{\alpha}_i$ be the image of $\alpha_i$ in $\lim\limits_{A_+} F$. Define $\mu: m \to \lim\limits_{A_+} F$ by $\mu(\alpha) = \sum_{i=1}^{N} \bar{\alpha}_i$. Then $\mu$ is an inverse of $\eta$.

**Lemma 1.2.**

$$\lim\limits_{A_+} F = 0 \quad \text{for } n \geq 1.$$ 

**Proof.** By [1, (1, 1.4)] it is enough to show that $F$ is coflabby (coflasque). Let $\lambda \in A_+$ and put

$$\bar{\lambda} = \{ \lambda' \in A_+ \mid \lambda \leq \lambda' \}.$$ 

Suppose $A_1 \subseteq \bar{\lambda}$ is such that if $\lambda' \in A_1$ and $\lambda' \leq \lambda''$, then $\lambda'' \in A_1$. $F$ is coflabby if in this situation

$$\lim\limits_{A_1} F \to \lim\limits_{\bar{\lambda}} F = F(\lambda) = A$$

is an injection.

However, the proof of Lemma 1.1 applies to show that

$$\lim\limits_{A_1} F = \{ \lambda' / \lambda \mid \lambda' \in A_1 \} \cdot A$$

and that the morphism

$$\lim\limits_{A_1} F \to \lim\limits_{\bar{\lambda}} F = A$$

is the obvious inclusion. Therefore we are done.
Consider the resolving complex \( C_*(A_+; -) \) for \( \lim_{A_+} \), see [1, (1.2)]. By Lemma 1.2, \( C_*(A_+; F) \) is an \( A \)-free resolution of the maximal ideal \( m \) of \( A \). Therefore

\[
\text{Tor}_i^A(k, k) \cong \begin{cases} 
  k & i = 0 \\
  H_{i-1}(C_*(A_+; F) \otimes_A k) & i \geq 1
\end{cases}.
\]

Now \( C_*(A_+; F) \otimes_A k = C_*(A_+; F \otimes_A k) \), therefore

\[
H_{i-1}(C_*(A_+; F) \otimes_A k) = \lim_{A_+}^{(i-1)} (F \otimes_A k) .
\]

Observe that the projective system \( F \otimes_A k \) is isomorphic to \( \bigsqcup_{\lambda \in A_+} k(\lambda) \), where \( k(\lambda) \) is the projective system defined by

\[
k(\lambda)(\lambda') = \begin{cases} 
  0 & \text{if } \lambda' \neq \lambda \\
  k & \text{if } \lambda' = \lambda .
\end{cases}
\]

Put for any \( \lambda \in A_+ 
\)

\[
\hat{\lambda} = \{ \lambda' \in A_+ \mid \lambda' \leq \lambda \}
\]

\[
L(\lambda) = \{ \lambda' \in A_+ \mid \lambda' \leq \lambda, \lambda' \neq \lambda \} = \hat{\lambda} - \{ \lambda \} .
\]

It is easy to see that there are isomorphisms

\[
\lim_{\lambda \in A_+} k(\lambda) \cong \lim_{\lambda \in \hat{\lambda}} k(\lambda) \quad \text{for } n \geq 0 .
\]

In fact this follows from the existence of a \( \bigoplus \)-projective resolution of \( k(\lambda) \), trivial outside of \( \hat{\lambda} \), see [1, (1.2)]. Let \( k_\lambda \) be the constant projective system on \( \hat{\lambda} \) defined by \( k_\lambda(\lambda') = k \), and let \( k'_\lambda \) be the subprojective system of \( k_\lambda \) defined by \( k'_\lambda(\lambda') = 0 \) if \( \lambda' = \lambda \) and \( k'_\lambda(\lambda') = k \) if \( \lambda' \neq \lambda \). Then there is an exact sequence of projective systems on \( \hat{\lambda} \)

\[
0 \to k'_\lambda \to k_\lambda \to k(\lambda) \to 0 .
\]

As

\[
\lim_{\lambda \in \hat{\lambda}} k_\lambda = \begin{cases} 
  k & \text{for } n = 0 \\
  0 & \text{for } n \geq 1
\end{cases}
\]

and since

\[
\lim_{\lambda \in \hat{\lambda}} k'_\lambda \cong \lim_{L(\lambda)} k = H_n(E(\lambda); k) \quad n \geq 0
\]

where \( k \) is the constant projective system \( k \) on \( L(\lambda) \), and where we denote by \( E(\lambda) \) the simplicial set defined by the ordered set \( L(\lambda) \), see [1, (1.1)], we obtain an exact sequence
\[ 0 \rightarrow \lim_{\lambda} k(\lambda) \rightarrow \lim_{\lambda} k'_{\lambda} \rightarrow k \rightarrow \lim_{\lambda} k(\lambda) \rightarrow 0 \]

and isomorphisms
\[ \lim_{\lambda} k(\lambda) \cong H_{n-1}(E(\lambda); k) \quad n \geq 2. \]

Notice that \( \lim_{\lambda} k(\lambda) = 0 \) unless \( \lambda \) is minimal in \( \Lambda_+ \), in which case \( \lim_{\lambda} k(\lambda) \equiv k \), and \( \lim_{\lambda} k(\lambda) = 0 \).

If \( \lambda \) is not minimal, then
\[ \lim_{\lambda} k(\lambda) \cong \check{H}_0(E(\lambda); k) \]

where \( \check{H} \) is the augmented homology.

Summing up we have proved the following

**Proposition 1.3.**

\[ \mathrm{Tor}^A_n(k, k) \cong \begin{cases} k & n = 0 \\ k^q & n = 1 \\ \bigoplus_{\lambda \in \Lambda_+} \check{H}_{n-2}(E(\lambda); k) & n \geq 2 \end{cases} \]

where \( q \) is the number of minimal elements of \( \Lambda_+ \).

2. Application to 2-dimensional Torus embeddings.

Let \( \Lambda' \) be a submonoid of \( \mathbb{Z}_+^2 \), satisfying the following condition:

There exist \((m_i, n_i) \in \Lambda', i = 1, 2\) with \( m_i \) and \( n_i \) relatively prime such that

(i) \( \Lambda' = \{(m_0, n_0) \in \mathbb{Z}_+^2 \mid \exists t_j \in \mathbb{Z}_+, j = 0, 1, 2 \text{ such that} \]
\[ t_0(m_0, n_0) = t_1(m_1, n_1) + t_2(m_2, n_2) \}

(ii) \( m_1 \cdot n_2 - m_2 \cdot n_1 = p > 0. \)

Any such monoid will be referred to as a saturated rational (sub)monoid (of \( \mathbb{Z}_+^2 \), see Fig. 1.

There is a one-to-one correspondence between saturated rational submonoids \( \Lambda' \) of \( \mathbb{Z}_+^2 \) and affine 2-dimensional normal torus imbeddings \( A \), see \([3]\), such that \( A = k(\Lambda') \).

By (1.3) we know that
\[ \mathrm{Tor}^A_n(k, k) \cong \bigoplus_{\lambda \in \Lambda'} \check{H}_{n-2}(E(\lambda), k), \quad n \geq 2. \]
The purpose of the rest of this paper is to establish a recursion formula for computing $\tilde{H}_r(E(\lambda), k)$, $\lambda \in A'_+, r \geq 0$, see (2.17), from which we easily deduce the rationality of the Betti series

$$B(t) = \sum_{n=0}^{\infty} \beta_n t^n$$

where

$$\beta_n = \dim_k \text{Tor}_n^{k(A')} (k, k).$$

We are therefore interested in the simplicial structure of $E(\lambda)$, $\lambda \in A'_+$, which is determined by the structure of the ordered set $L(\lambda)$.

Consequently we shall have to study the ordered sets $L(\lambda)$ for arbitrary $\lambda \in A'_+$, see Fig. 1 and 2.

Consider first the unique linear transformation $T: \mathbb{Z}^2 \to \mathbb{Z}^2$ mapping $(m_1, n_1)$ to $(p, 0)$ and $(m_2, n_2)$ to $(0, p)$. $T$ is represented by the $2 \times 2$ matrix

$$\begin{bmatrix} n_2 & -m_2 \\ -n_1 & m_1 \end{bmatrix}$$

Put $A = T(A')$. Notice that $A$ is submonoid of $\mathbb{Z}_+^2$ but no longer a saturated rational submonoid, see Fig. 2. Nevertheless $T$ defines an isomorphism $A' \cong A$, and we shall, from now on, find it more convenient to work with $A$. We may assume $p \geq 2$, since otherwise $A = k(A)$ is a polynomial algebra in 2 variables, for which the Betti series is well known.

Let, for $n \geq 1$, $A_n = \{(n, r) \in A \mid r \in \mathbb{Z}_+\}$. Then the following lemma holds:
Lemma 2.1. There exists a unique \( \xi \in \mathbb{Z}_+ \), with \( 0 < \xi < p \), such that

\[
\Lambda_1 = \{(1, \xi + \eta \cdot p) \mid \eta \in \mathbb{Z}_+\}
\]

\[
\Lambda_a = \{(n, n \cdot \xi + \eta \cdot p) \mid \eta \in \mathbb{Z}, n \cdot \xi + \eta \cdot p \geq 0\}.
\]

Proof: Since \( (m_2, n_2) = 1 \), there exists an integer pair \((x_0, y_0) \in \mathbb{Z}^2\) such that

\[
T(x_0, y_0) = (n_2 x_0 - m_2 y_0, -n_1 x_0 + m_1 y_0) \in (\{1\} \times \mathbb{Z}).
\]

The set \(\{(m_1, n_1), (m_2, n_2)\}\) forms a basis for \(\mathbb{Q}^2\), and there exist \(\alpha, \beta \in \mathbb{Q}\) such that

\[
(*) \quad (x_0, y_0) = \alpha(m_1, n_1) + \beta_0(m_2, n_2).
\]

But \(T\) is a linear map so we have

\[
T(x_0, y_0) = \alpha \cdot T(m_1, n_1) + \beta_0 \cdot T(m_2, n_2)
\]

\[
= \alpha \cdot (p, 0) + \beta_0 \cdot (0, p) \in (\{1\} \times \mathbb{Z}).
\]

This implies \(\alpha = 1/p\) and from equation \((*)\) and the fact \((m_1, n_1) = 1\) we deduce that \(\beta_0 \notin \mathbb{Z}\). So there exists an integer \(\mu \in \mathbb{Z}\) such that \(0 < \beta_0 + \mu < 1\) and

\[
T((x_0, y_0) + \mu(m_2, n_2)) = \alpha \cdot (p, 0) + (\beta_0 + \mu)(0, p) \in (\{1\} \times [0, p]).
\]

Put \(\beta = \beta_0 + \mu\) and \((x, y) = (x_0, y_0) + \mu(m_2, n_2) \in \mathbb{Z}_+^2\), and let \(\gamma\) be the product of the denominators of \(\alpha\) and \(\beta\). The numbers \(\gamma \cdot \alpha, \gamma \cdot \beta\) are integers, and

\[
\gamma \cdot (x, y) \in \Lambda'.
\]

Since the monoid \(\Lambda'\) is saturated, it follows that \((x, y) \in \Lambda'.\) Let \(\xi = \beta \cdot p\). Then

\[
T(n \cdot (x, y)) = (n, n \cdot \xi).
\]

Now consider the equivalence

\[
n \cdot \xi + \eta \cdot p = n \cdot \beta \cdot p + \eta \cdot p
\]

\[
= (n \cdot \beta + \eta) \cdot p \geq 0
\]

\[
\iff n \cdot \beta + \eta \geq 0.
\]

If \(n \cdot \xi + \eta \cdot p \geq 0\) then we have

\[
(n, n \cdot \xi + \eta \cdot p) = T(n(x, y) + \eta(m_2, n_2))
\]

\[
= T(n \cdot \alpha(m_1, n_1) + (n \cdot \beta + \eta)(m_2, n_2))
\]

and \((n, n \cdot \xi + \eta \cdot p) \in \Lambda\). This follows from the fact that an integer pair, positively generated by \((m_1, n_1)\) and \((m_2, n_2)\) is element of \(\Lambda'\).

Suppose \((x, y), (x', y') \in \Lambda'\) satisfy \(T(x, y) \in \Lambda_a, T(x', y') \in \Lambda_a\) for some \(a \in \mathbb{Z}_+\). Then we have
\[ n_2 \cdot x - m_2 \cdot y = n_2 \cdot x' - m_2 \cdot y' \]

or equivalently
\[ n_2(x - x') = m_2(y - y') \]

Since \((m_2, n_2) = 1\) this is equivalent to
\[ x - x' = c \cdot m_2, \quad y - y' = c \cdot n_2 \]

for some \(c \in \mathbb{Z}\). But then we have
\[
-n_1 \cdot x + m_1 \cdot y = -n_1(c \cdot m_2 + x') + m_1(y' + c \cdot n_2) \\
= -n_1 \cdot x' + m_1 \cdot y' - c(n_1 \cdot m_2 - m_1 \cdot n_2) \\
= -n_1 \cdot x' + m_1 \cdot y' + c \cdot p
\]

It is easy to see that this proves the lemma.

Thus we have a complete description of \(A\) given by
\[
A = \{(a, b) \in \mathbb{Z}_+^2 \mid a \cdot \xi \equiv b (\text{mod } p)\}.
\]

If we interchange \((m_1, n_1)\) and \((m_2, n_2)\) and apply the proof of Lemma 2.1 we get a number \(\eta \in \mathbb{Z}_+\) satisfying

i) \(0 < \eta < p\)

ii) \(\eta \cdot \xi \equiv 1 \pmod{p}\)

The use of this will appear later.

**Remark 2.2.** One of the advantages with this description of \(A\) is the following property of \(A\): If \(\lambda = (a, b), \lambda' = (a', b') \in A\) and if \(\lambda' - \lambda = (a' - a, b' - b) \in \mathbb{Z}_+^2\), then \(\lambda' - \lambda \in \Lambda\).

In fact since for \((a, b), (a', b') \in A; b \equiv a \cdot \xi \pmod{p}, b' \equiv a' \cdot \xi \pmod{p}, \) and \(a' - a \geq 0, b' - b \geq 0\), we find \(b' - b = (a' - a) \cdot \xi \pmod{p}\) therefore \((a' - a, b' - b) \in \Lambda\). Notice that this implies that the order relation on \(A\) (see section 1) induced by the order relation on \(A'\) is the restriction of the ordinary order relation on \(\mathbb{Z}_+^2\).

**Definition 2.3.** Let \(P \in \mathbb{Z}_+^2\). Define the ordered set \(\hat{P}\) associated with \(P\) by \(\hat{P} = \{\lambda \in \Lambda \mid \lambda \leq P\} \subseteq \Lambda\). The associated simplicial set will also be denoted by \(\hat{P}\).

Correspondingly we shall let \(L(P) = \{\lambda \in \Lambda \mid \lambda \preceq P\}\) also denote the associated simplicial set. (When \(P \in \Lambda\), this is precisely the set \(E(P)\) of section 1, see Fig. 2, 3, and 4.)
Remark 2.4. Notice that for $P \in \mathbb{Z}_+^2 - \Lambda$ we have $L(P) = \tilde{P}$.

Lemma 2.5. Let $\xi$ and $\eta$ be defined as above. Let $U \subseteq \mathbb{Z}_+^2$ be the set defined by

$$U = \{(a, b) \in \mathbb{Z}_+^2 \mid b > p + a \cdot \xi \text{ or } a > p + b \cdot \eta\}.$$

Then for any $P \in U$

$$\tilde{H}_n(L(P)) = 0 \quad n \geq 0.$$

Proof. It is obviously sufficient to prove the lemma in the case where $P = (a, b)$ satisfies the condition $b > p + a \cdot \xi$. Given $P = (a, b) \in \mathbb{Z}_+^2$, and suppose $b > p + a \cdot \xi$. Then there exist integers $\alpha, \beta \in \mathbb{Z}$ such that

$$b - a \cdot \xi = \alpha \cdot p + \beta$$

with $0 < \beta \leq p$ and $\alpha \geq 1$. We shall prove the lemma by induction on the integer $a$.

Suppose $a = 0$. Then $L(P)$ has a final object and the homology vanishes.

Suppose $a > 0$. Let $P = (a, b) \in U$, and suppose the formula is valid for all $(m, c) \in U$ with $m < a$. Notice that Lemma 2.1 implies $(a, b - \beta) = (a, a \cdot \xi + \alpha \cdot p) \in \Lambda$.

Now it is easy to see that

i) $L(P) = (a - 1, b) \cup (a, b - \beta)$

ii) $(a - 1, b - \beta) = (a - 1, b) \cap (a, b - \beta)$.

Apply the Mayer-Vietoris sequence and obtain the long exact sequence
\[ \ldots \to \tilde{H}_n(a, b - \beta) \to \tilde{H}_n(a, b - \beta) \oplus \tilde{H}_n(a - 1, b) \to \]
\[ \to \tilde{H}_n(L(P)) \to \tilde{H}_{n-1}(a - 1, b - \beta) \to \ldots \]

where \( \tilde{H}_n(P) \) is the homology of the ordered set associated with \( P \). But now we have \( b > p + a \cdot \xi > p + (a - 1) \cdot \xi \) and \( b - \beta = a \cdot p + a \cdot \xi \geq p + a \cdot \xi > p + (a - 1) \cdot \xi \), so \( (a - 1, b) \in U \) and \( (a - 1, b - \beta) \in U \). The induction hypothesis implies
\[ \tilde{H}_n(a - 1, b - \beta) = \tilde{H}_n(a - 1, b) = 0 \quad \forall n \geq 0. \]

\( (a, b - \beta) \in \Lambda \) and \( (a, b - \beta) \) has a final object; therefore
\[ \tilde{H}_n(a, b - \beta) = 0 \quad \forall n \geq 0. \]

Thus, using the exactness of the above sequence, we get
\[ \tilde{H}_n(P) = 0 \quad \forall n \geq 0 \]

which proves the lemma.

**Definition 2.6.** Let \( P \in \mathbb{Z}^2_+ \). The maximal polygon associated with \( P, M(P) \) is the set of maximal elements of the convex hull of \( L(P) \) in \( \mathbb{R}^2_+ \), see Fig. 3 and 4.

Put \( M_0(P) = L(P) \cap M(P) \). Then the following lemma holds.

**Lemma 2.7.** \( M_0(P) \) is the set of maximal elements of \( L(P) \).

**Proof.** Let \( \text{max} \ L(P) \) be the set of maximal elements of \( L(P) \). Obviously \( M_0(P) \subseteq \text{max} \ L(P) \). Assume \( \lambda \in \text{max} \ L(P) \) and \( \lambda \notin M_0(P) \). \( M(P) \) is a convex polygon and \( \lambda \) has to sit strictly below some edge \( e \). Pick vertices of \( e \), \( \mu, \mu' \in M_0(P) \), \( \mu \neq \mu' \), and consider the element \( \eta = \mu + \mu' - \lambda \). Since \( \eta \in \mathbb{Z}^2_+ \) we have seen (Remark 2.2) that \( \eta \in \Lambda \). An easy argument then shows that \( \eta \in L(P) \) and that \( \eta \) is above the edge \( e \), a contradiction.

It is easily seen that \( M(P) \) must lie inside a square, \( p \times p \), with \( P \) as the maximal point.

**Lemma 2.8.** For every \( P \in \mathbb{Z}^2_+ \) with \( P \geq (p, p) \), and every \( \lambda \in \Lambda \)
\[ M(P + \lambda) = M(P) + \lambda. \]

**Proof.** It is enough to show the equality \( M_0(P + \lambda) = M_0(P) + \lambda \). So let \( \mu \in M_0(P) \). Then \( \lambda \leq \mu + \lambda < P + \lambda \). Now choose \( \eta \in M_0(P + \lambda) \) such that \( \lambda \leq \mu + \lambda \leq \eta < P + \lambda \). Then we have \( \mu \leq \eta - \lambda < P \). Since \( \mu, \eta, \lambda \in \Lambda \), the remark (2.2) implies \( \eta - \lambda \in \Lambda \), thus we get \( \mu = \eta - \lambda \) or \( \eta = \mu + \lambda \). Consequently \( \mu + \lambda \in \mathbb{Z}^2_+ \).
$M_0(P + \lambda)$ and $M_0(P) + \lambda \subseteq M_0(P + \lambda)$. To prove the inverse inclusion, we first notice that if $\mu \in M_0(P + \lambda)$, then $\mu \geq \lambda$. This follows from the fact that $P \subseteq (p, p)$ and that $M_0(P + \lambda)$ sits inside a square $p \times p$ with $P + \lambda$ as the maximal point.

So let $\mu \in M_0(P + \lambda)$. Then $\mu \leq P + \lambda$ or $\mu - \lambda \leq P$. Choose $\eta \in M_0(P)$ such that $\mu - \lambda \leq \eta < P$. This implies $\mu \leq \eta + \lambda < P + \lambda$. But $\mu \in M_0(P + \lambda)$ so the last equation implies $\mu = \eta + \lambda$, which proves the lemma.

**Definition 2.9.** Let $P \in \mathbb{Z}_+^2$ and denote by

$$\{V_{i,j}(P) \mid i = 1, 2, \ldots, n; \ j = 1, 2, \ldots, m_i\}$$

the lattice points on $M(P)$ where $i$ is the number of the edge counted from right, and $j$ is the number of the lattice point on the edge, also counted from right. (See Fig. 2, 3, and 4.)

Put $V_i = V_{i,1}$ for $i = 1, 2, \ldots, n$ and $V_{n+1} = V_{n,m}$. Notice that for $i = 1, 2, \ldots, n$ we have $m_i \geq 2$ and $V_{i,m_i} = V_{i+1}$.

Denote by

$$\{e_{i,j}(P) \mid i = 1, 2, \ldots, n; \ j = 1, \ldots, m_i\}$$

the edges between $V_{i,j}(P)$ and $V_{i,j+1}(P)$. For $i = 1, \ldots, n$,

$$e_i(P) = \bigcup_{j=1}^{m_i-1} e_{i,j}(P)$$

are then the edges of $M(P)$.

Let $\{S_i(P)\}_{i=1, \ldots, n}$ be the absolute values of the slopes of the $e_i(P)$'s and let finally

$$\{X_i(P)\}_{i=1, \ldots, n} \quad X_i = X(V_{i,2}) - X(V_{i,1})$$

and

$$\{Y_i(P)\}_{i=1, \ldots, n} \quad Y_i = Y(V_{i,2}) - Y(V_{i,1})$$

be the differences in the values of the coordinates of $V_{i,1}(P)$ and $V_{i,2}(P)$.

It is clear that $M(P)$ is determined by these families of numbers. Moreover, we deduce the following

$$Y_i(P) = S_i(P) \cdot X_i(P) \quad i = 1, \ldots, n.$$

Put, as a shorthand, $\alpha_i(P) = X(P) - X(V_i(P))$ and $\beta_i(P) = Y(P) - Y(V_i(P))$, and notice that $\alpha_{i+1}(P) > \alpha_i(P)$, $\beta_{i+1}(P) < \beta_i(P)$.

For every pair $(i, j)$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ the proof of Lemma 2.7 gives the existence of unique points

$$Q_{i,j}(P) = (X(V_{i,j}(P)), Y(V_{i,j+1}(P)))$$

.
and
\[ P_{i,j}(P) = (X(V_{i,j+1}(P), Y(V_{i,j}(P))) \]

with the properties
\[
L(Q_{i,j}(P)) = V_{i,j}(P) \cap V_{i,j+1}(P) \]
\[
P_{i,j}(P) = V_{i,j}(P) \cap V_{i,j+1}(P) \]

(See Fig. 3 and 4.)

**Definition 2.10.** Denote by \( P_i \) the unique element of \( \mathbb{Z}^2_+ \) such that \( P^* = \bigcap_{j=1}^{n_i} P_{i,j} \).

Let \( \lambda \in \Lambda \) and let \( n \) be the number of edges of \( M(\lambda) \). The next lemma will show that \( M(P_i(\lambda)) \) is congruent to the polygon \( M(\lambda) \) with the \( i \)-th edge removed. We shall therefore index the vertices and the edges etc. of \( M(P_i(\lambda)) \) by restricting the corresponding indexing of \( M(\lambda) \). Thus \( e_i(P_i(\lambda)) \) does not exist and, modulo translation, \( e_j(P_i(\lambda)) \) is congruent to \( e_j(\lambda) \) whenever \( i \neq j \). Likewise \( V_i(P_i(\lambda)) \) does not exist and \( V_{i-1,m-1}(P_i(\lambda)) = V_{i+1}(P_i(\lambda)) \). Notice that the intersection points \( P_j(P_i(\lambda)) \) and \( P_i(P_j(\lambda)) \) are, in general, different when \( i \neq j \). Let \( P_{(i,j)}(\lambda) \) denote their intersection, i.e. the unique element of \( \mathbb{Z}^2_+ \) such that

\[
P_{(i,j)}(\lambda) = P_i(P_j(\lambda)) \cap P_j(P_i(\lambda)) \]

In general we make the following definition (\( \lambda \gg 0 \) means \( X(\lambda), Y(\lambda) \gg 0 \)).

**Definition 2.11.** Let \( \lambda \in \Lambda \) and \( M(\lambda) \) as above, \( \lambda \gg 0 \). Let \( I \subseteq \{1, 2, \ldots, n\} \) be a set of integers different from the empty set. Define \( P_I(\lambda) \) recursively via the intersection property

\[
P_I(\lambda) = \bigcap_{i \in I} P_i(P_{\{i\}}(\lambda)) \]

where \( P_{\emptyset}(\lambda) = \lambda \).

Lemma 2.12 will show that \( M(P_{(i,j)}(\lambda)) \) is congruent to \( M(\lambda) \) with the \( i \)-th and the \( j \)-th edge removed, and that in general \( M(P_i(\lambda)) \) is congruent to \( M(\lambda) \) with the \( i \)-th edge removed for every \( i \in I \subseteq \{1, 2, \ldots, n\} \).

**Lemma 2.12.** Let \( \lambda, M(\lambda) \) be as above and let \( I \subseteq \{1, 2, \ldots, n\} \) be a set of integers, the empty set included.

i) The maximal polygon \( M(P_i(\lambda)) \) of the set \( P_I(\lambda) \) is congruent to the maximal polygon \( M(\lambda) \) of \( \lambda \) with the \( i \)-th edge removed for every \( i \in I \).
ii) Let for \( i=1,2,\ldots,n \), \( r_i=(\alpha_i,\beta_i) \). Then for every \( j \notin I \)

\[
P_j(P_1(\lambda)) = \lambda - \sum_{i \in I} r_i - r_{n+1} - \sum_{h \notin I, h \geq j} e_h - (\alpha_{j+1} - \alpha_j, 0),
\]

where \( e_h \) is the vector \( \vec{V}_h \vec{V}_{h+1} \) associated to the edge \( e_h(\lambda) \), and \( \alpha_i = \alpha_i(\lambda), \beta_i = \beta_i(\lambda) \).

**Proof.** We shall prove the lemma by induction on the number of elements of \( I, \# I = k \).

The case \( k=0 \) is vacuous; just notice that \( e_h = r_h - r_{h+1} \) so

\[
\lambda - r_{n+1} - \sum_{h \notin I, h \geq j} e_h = \lambda - r_j.
\]

Suppose the lemma holds for \( \# I = k - 1, 0 < k \leq n \), and let \( I \subseteq \{1, \ldots, n\} \) with \( \# I = k \). To simplify notation, write for every \( i \in I \); \( P_{I,i}(\lambda) = P_i(P_{I-\{i\}}(\lambda)) \). Obviously

\[
P_I(\lambda) = \bigcap_{i \in I} P_{I,i}(\lambda) = \left( \min_{i \in I} X(P_{I,i}(\lambda)), \min_{i \in I} Y(P_{I,i}(\lambda)) \right)
\]

so we have to study the relation between the intersection points \( P_{I,i}(\lambda) \). The induction hypothesis gives

\((**)
\]

\[
P_{I,j}(\lambda) = \lambda - \sum_{i \in I - \{j\}} r_i - r_{n+1} - \sum_{h \notin I - \{j\}, h \geq j} e_h - (\alpha_{j+1} - \alpha_j, 0)
\]

\[
= \lambda - \sum_{i \in I} r_i + \sum_{h \notin I, h > j} e_h - (\alpha_{j+1} - \alpha_j, 0).
\]

Consider the last part of the above sum, \( \sum_{h \notin I, h > j} e_h + (X(e_j), 0) \). The fact that \( \alpha_{j+1} > \alpha_j \) and \( \beta_{j+1} < \beta_j \) shows that the \( X \)-value of this vector increases and the \( Y \)-value decreases with increasing \( j \in I \). So it follows that

\[
P_I(\lambda) = P_{I,i_k}(\lambda) \cap P_{I,i_k}(\lambda) \]

\[
= (X(P_{I,i_k}(\lambda)), Y(P_{I,i_k}(\lambda)))
\]

where \( I = \{i_1 < i_2 < \ldots < i_k\} \). From \((**)) we deduce that \( X(P_I(\lambda)) = X(P_{I,i_1}(\lambda)) = X(\lambda - \sum_{i \in I} r_{i+1}) \) and \( Y(P_I(\lambda)) = Y(P_{I,i_k}(\lambda)) = Y(\lambda - \sum_{i \in I} r_i) \). In addition we get the two inequalities

\[
P_{I,i_1}(\lambda) < -\sum_{i \in I} r_{i+1}
\]

\[
P_{I,i_k}(\lambda) < \lambda - \sum_{i \in I} r_i.
\]
Obviously \( \lambda - \sum_{i \in I} r_i \geq \lambda - \sum_{i \in I} r_{i+1} - r_1 \) and \( \lambda - \sum_{i \in I} r_{i+1} \geq \lambda - \sum_{i \in I} r_i - r_{n+1} \) and therefore

\[
\lambda - \sum_{i \in I} r_{i+1} - r_1 < P_I(\lambda) \quad \text{and} \quad \lambda - \sum_{i \in I} r_i - r_{n+1} < P_I(\lambda).
\]

Thus \( \lambda - \sum_{i \in I} r_{i+1} - r_1 \) and \( \lambda - \sum_{i \in I} r_i - r_{n+1} \) are the "endpoints" of the maximal polygon of \( P_I(\lambda) \).

Using the fact that \( \sum_{h=1}^{n} e_h = r_1 - r_{n+1} \) we have the equalities

\[
\lambda - \sum_{i \in I} r_{i+1} - r_1 = \lambda - \sum_{i \in I} r_{i+1} - r_{n+1} - \sum_{h=1}^{n} e_h
\]

\[
= \lambda - \sum_{i \in I} (r_{i+1} - r_i) - \sum_{i \in I} r_i - r_{n+1} - \sum_{h=1}^{n} e_h
\]

\[
= \lambda - \sum_{i \in I} r_i - r_{n+1} + \sum_{h \in I} e_h
\]

This proves part i).

To prove ii) observe that i) implies

\[
X(P_J(P_I(\lambda))) = X\left( \lambda - \sum_{i \in I} r_{i+1} - r_1 + \sum_{h \in I, h \leq j} e_h \right)
\]

\[
= X\left( \lambda - \sum_{i \in I} r_i - r_{n+1} - \sum_{h \in I} e_h + e_j \right).
\]

We already know

\[
Y(P_J(P_I(\lambda))) = Y\left( \lambda - \sum_{i \in I} r_i - r_{n+1} - \sum_{h \in I} e_h \right)
\]

and therefore

\[
P_J(P_I(\lambda)) = \lambda - \sum_{i \in I} r_i - r_{n+1} - \sum_{h \in I, h \geq j} e_h + (X(e_j), 0),
\]

which is the claimed equation for \( P_J(P_I(\lambda)), \# I = k \).

**Corollary 2.13.** \( P_I(\lambda) \in \Lambda \) if and only if \( I = \{1, 2, \ldots, n\} \) or \( I = \emptyset \).

**Proof.** \( 0 \leq \sum_{i \in I} \alpha_{i+1} - \alpha_i \leq p \) with equality on the left or right if and only if \( I = \emptyset \), respectively \( I = \{1, 2, \ldots, n\} \).
In the next few lemmas we shall relate the homology of \( L(P) \) to the homology of ordered sets connected with \( M(P) \). Let \( P \in \mathbb{Z}_+^2 \) and assume \( P \gg 0 \). Put \( M = M(P) \), \( V_i = V_i(P) \), etc.

**Lemma 2.14.** In the situation above we have an isomorphism for every \( r \geq 0 \)

\[
\bigoplus_{j=1}^{m_i-1} \tilde{H}_r(P_{i,j}) \cong \bigoplus_{j=2}^{m_i-1} \tilde{H}_r(L(V_{i,j})) \oplus \tilde{H}_r(P_i).
\]

**Proof.** Define \( V = V_{i-r_{i+1}} \in A \). Then for \( j = 1, 2, \ldots, m_i \)

\[
P_{i,j} = (X(P_{i,j}), Y(V)) \cup (X(V), Y(P_{i,j}))
\]

\[
V^\sim = (X(P_{i,j}), Y(V)) \cap (X(V), Y(P_{i,j})).
\]

The proof of this is left to the reader; an argument analogous to the proof of Lemma 2.7 will give the result.

Applying the reduced Mayer-Vietoris sequence, and using the fact that \( V^\sim \) has a final object, we get an isomorphism for \( j = 1, 2, \ldots, m_i - 1 \) and \( r \geq 0 \)

\[
(\ast) \quad \tilde{H}_r(P_{i,j}) \cong \tilde{H}_r(\{P_{i,j}\}, Y(V)) \oplus \tilde{H}_r(\{V\}, Y(P_{i,j})).
\]

But we also have for \( j = 2, 3, \ldots, m_i - 1 \)

\[
L(V_{i,j}) = (X(P_{i,j-1}), Y(V)) \cup (X(V), Y(P_{i,j}))
\]

\[
V^\sim = (X(P_{i,j-1}), Y(V)) \cap (X(V), Y(P_{i,j})).
\]

So for every \( r \geq 0 \)

\[
(\ast\ast) \quad \tilde{H}_r(L(V_{i,j})) \cong \tilde{H}_r(\{P_{i,j-1}\}, Y(V)) \oplus \tilde{H}_r(\{V\}, Y(P_{i,j})).
\]

Summing over \( j = 1, 2, \ldots, m_i - 1 \) the isomorphisms \((\ast)\), changing parentheses, and using \((\ast\ast)\) we get

\[
\bigoplus_{j=1}^{m_i-1} \tilde{H}_r(P_{i,j}) \cong \bigoplus_{j=2}^{m_i-1} \tilde{H}_r(L(V_{i,j})) \oplus \tilde{H}_r((V_{i,j}), Y(V), Y(P_{j,1})) \oplus \tilde{H}_r(X(P_{i,m_i-1}, Y(V))
\]

\[
\cong \bigoplus_{j=2}^{m_i-1} \tilde{H}_r(L(V_{i,j})) \oplus \tilde{H}_r(P_i) \quad \forall r \geq 0.
\]

The next lemma gives the relation between the homology of \( L(P) \) and the homology of the intersection points \( P_i \).

**Lemma 2.15.** Let the symbols \( P, M, V_{i,j} \) be as above; \( n \) is the number of edges of \( M \). There is an isomorphism for every \( r > 0 \)

\[
\tilde{H}_r(L(P)) \cong \left[ \bigoplus_{i=1}^{n} \bigoplus_{j=2}^{m_i-1} \tilde{H}_r(L(V_{i,j})) \right] \oplus \left[ \bigoplus_{i=1}^{n} \tilde{H}_r-1(P_i) \right].
\]
PROOF. As a consequence of Lemma 2.7 we have
\[ L(P) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i-1} Q_i,j \]
where \( Q_{i,j} = Q_{i,j}(P) \) and the intersections \( Q_i,j \cap Q_i,j+1 \) and \( Q_i,m_i-1 \cap Q_i+1,1 \) always are ordered sets with \( V_{i,j+1} \), respectively \( V_{i+1,1} \), as final elements. Using the Mayer-Vietoris sequence repeatedly we find
\[ \check{H}_r(L(P)) \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i-1} \check{H}_r(Q_{i,j}) . \]
Apply the Mayer-Vietoris sequence once more to the system \( (\widehat{Q}_i,j, V_i,j, V_i,j+1, P_i,j) \). Since \( V_i,j \) has a final element we obtain an isomorphism for every \( r > 0 \)
\[ \check{H}_r(Q_{i,j}) \cong \check{H}_{r-1}(P_{i,j}) \]
where \( i = 1, \ldots, n, j = 1, \ldots, m_i-1 \). Using Lemma 2.14 the lemma follows immediately.

**LEMMA 2.16.** Let \( \lambda \in A \) and let \( I \subseteq \{1, 2, \ldots, n\} \). Suppose \( 2 \leq \#I = k \leq n \). Let \( P_I = P_I(\lambda) \) and \( P_{i,I} = P_i(P_{I-\{i\}}(\lambda)) \). Then for every \( r \geq 0 \) we have an isomorphism
\[ \bigoplus_{i \in I} \check{H}_r(P_{i,I}) \cong \bigoplus_{i \in I} \check{H}_r(L(V_i(P_{I-\{i\}}))) \oplus \check{H}_r(P_I) , \]
where \( I = \{i_1 < \ldots < i_k\} \).

**PROOF.** Define \( P_{i,j} \) via the intersection property
\[ \check{P}_{i,j} = \check{P}_{i,j} \cap \check{P}_{i,j} \]
for every pair \( i,j \in I \). From the proof of Lemma 2.12 we deduce \( \check{P}_{i,i,j} \)
\[ = \check{P}_{i,i,j} \cap P_I,j \] for every \( j = 2, \ldots, k \). For \( j = 1, \ldots, k - 1 \) we have the inequalities
\[ \check{P}_{i,i,j} < P_{i,j} < V_j(P_{I-\{ij\}}) \]
and from Lemma 2.12 the equality
\[ (***) \quad P_{i,j+1} = V_j(P_{I-\{ij\}}) - (0, \beta_{ij+1} - \beta_{ij+1+1} ) . \]
Thus \( P_{i,j+1} < V_j(P_{I-\{ij\}}) \). In addition we have the inequality \( V_j(P_{I-\{ij\}}) - r_{ij+1} < P_{i,i,j+1} \). The last statement is an immediate consequence of the two relations
\[ V_j(P_{I-\{ij\}}) - r_{ij+1} < P_{i,j+1} , \quad V_j(P_{I-\{ij\}}) - r_{ij+1} < P_{i,j} . \]
The first follows from equation (***) , the other is easily deduced from Lemma 2.12 using the analytic formula for \( P_{i,i} \). Thus we have
\( V_i(P_{I-i}) - r_{i+1} \leq P_{I-i} < V_i(P_{I-i}) \)
\( V_i(P_{I-i}) - r_{i+1} \leq P_{I-i+1} < V_i(P_{I-i}) \)
\( X(P_{I,i+1}) = X(V_i(P_{I-i})) \)
\( Y(P_{I,i+i}) = Y(V_i(P_{I-i})) \).

Applying the Mayer-Vietoris sequence three times we obtain for every \( r \geq 0 \) an isomorphism
\[
\tilde{H}_r(P_{I,i,i}) \oplus \tilde{H}_r(P_{I,i+i}) \cong \tilde{H}_r(L(V_i(P_{I-i}))) \oplus \tilde{H}_r(P_{I,i+i+i})
\]
But \( P_{I,i,i} = P_I \) so an iterated use of the described process will give the lemma.

We are now in position to state and prove the main result of this paragraph.

**Theorem 2.17.** Let \( \lambda \in \Lambda, \lambda \gg 0 \) and \( P_I = P_I(\lambda) \), as above. Let \( n \) be the number of edges of \( M(\lambda) \). Then for every integer \( r \geq n \) there is an isomorphism
\[
\tilde{H}_r(L(\lambda)) = \left( \bigoplus_{k=1}^{n} \bigoplus_{I \subseteq k-1} \bigoplus_{j=2}^{m_{i-1}} \tilde{H}_{r-k}(V_{i,j}(P_{I_i})) \right) \oplus \left( \bigoplus_{k=2}^{n} \bigoplus_{I \subseteq k} \bigoplus_{i \subseteq I} \tilde{H}_{r-k}(L(V_{i}(P_{I-i}))) \right)
\]
where \( P_\emptyset = \lambda \) and \( I = \{i_1 < \ldots < i_k\} \).

**Proof.** This is just an iterated use of Lemma 2.15 and Lemma 2.16, where we for each step increase the order of \( I \). Remember that if \( I \neq \emptyset \), \( P_I \in \Lambda \) if and only if \( I = \{1, \ldots, n\} \). Therefore the process stops when \( \#I = n \). Moreover, for \( \#I < n \) we have \( L(P_I) = P_I \).

Now go back to the calculation of the right-hand side of the equation in Proposition 1.3. In Theorem 2.17 we made the assumption \( \lambda \gg 0 \). In fact it suffices to know that \( \lambda > \sum_{i=1}^{n+1} r_i \). This is to ensure that all the points needed in Lemma 2.16 really are elements of \( \Lambda \).

Put
\[
Z = \left\{ \lambda \in \Lambda \mid \lambda > \sum_{i=1}^{n+1} r_i \right\}
\]
and recall the definition of
\[
U = \{(a,b) \in Z_+^2 \mid b > p + a \cdot \xi \text{ or } a > p + b \cdot \xi \},
\]
see (2.5). Put

$$W = (A - Z) \cap (A - U).$$

$W$ is a finite set containing all $\lambda \in A - Z$ with the property $\tilde{H}_l(\lambda) > 0$. Since for each $\lambda \in A$, $L(\lambda)$ is a finite ordered set, there exist $N'$ such that $\tilde{H}_m(L(\lambda)) = 0$ for all $m \geq N'$. Since $W$ is finite we may choose $N'$ such that $\tilde{H}_m(L(\lambda)) = 0$ for all $m \geq N'$ and all $\lambda \in W$. Putting $h_m(L(\lambda)) = \dim_k \tilde{H}_m(L(\lambda))$ we have thus proved

$$\sum_{\lambda \in Z} h_m(L(\lambda)) = \sum_{\lambda \in A} h_m(L(\lambda))$$

for every $m \geq N'$. Going back to Theorem 2.17 we see that the problem is to calculate the number $\sum_{\lambda \in Z} h_m - k(L(V_{i,j}(P_{1}L(\lambda))))$. So we need a lemma.

**Lemma 2.18.** Let $Z \subseteq A$ and $N'$ be defined as above. Let $N = N' + n$. Pick $m \geq N$ and let $(k, l, i, j)$ be a quadruple which occurs in Theorem 2.17. Then we have the equality

$$\sum_{\lambda \in Z} h_{m-k}(L(V_{i,j}(P_{1}L(\lambda)))) = \sum_{\lambda \in Z} h_{m-k}(L(\lambda)) .$$

**Proof.** The map $\lambda \mapsto V_{i,j}(P_{1}L(\lambda))$ from $Z$ into $A$, is obviously a rigid translation. Of course we have $\lambda \geq V_{i,j}(P_{1}L(\lambda))$ so

$$Z \subseteq \{ \lambda \in A \mid \exists \lambda' \in Z \text{ with } \lambda = V_{i,j}(P_{1}L(\lambda')) \} .$$

Let $\lambda' \in Z$ with $V_{i,j}(P_{1}L(\lambda')) \notin Z$. We have $m - k \geq N - k \geq N'$ and by definition of $N'$; $h_{m-k}(L(V_{i,j}(P_{1}L(\lambda')))) = 0$. Since

$$\sum_{\lambda \in Z} h_{m-k}(L(V_{i,j}(P_{1}L(\lambda)))) = \sum_{\lambda \in Z} h_{m-k}(L(\lambda)) + \sum_{\lambda' \in Z'} h_{m-k}(L(V_{i,j}(P_{1}L(\lambda'))))$$

where $Z' \equiv \{ \lambda' \in Z \mid V_{i,j}(P_{1}L(\lambda')) \notin Z \}$, we have proved the lemma.

**Theorem 2.19.** Let the number $N$ be as above. Let for every $m \geq N$, $\gamma_m = \sum_{\lambda \in A} h_m L(\lambda)$. Then there exists a recursion in the $\gamma$'s: $\gamma_m = \sum_{k=1}^{n} R_k \cdot \gamma_{m-k}$ given by

$$R_k = \binom{n-1}{k-1} \cdot S + \binom{n}{k} (k-1) \quad k = 1, 2, \ldots, n ,$$

where $n$ is the number of edges of the maximal polygon $M(\lambda)$ of $\lambda$, $\lambda \gg 0$, and $S = \sum_{i=1}^{n} (m_i - 2)$, where $m_i$ is the number of lattice points on the $i$th edge of $M(\lambda)$.

**Proof.** Due to Lemma 2.18 and Theorem 2.17 the only problem is to calculate the sums $(I = \{ i_1 < \ldots < i_k \})$.
\[ S_1 = \sum_{\# I = k-1} \sum_{i \notin I} \sum_{j=1}^{m_i-1} \gamma_{m-k} \]

\[ S_2 = \sum_{\# I = k} \sum_{i \in I \atop i \neq l_k} \gamma_{m-k} \cdot \]

This is a purely combinatorial problem and it is easy to show that

\[ S_1 = \binom{n-1}{k-1} \cdot S \cdot \gamma_{m-k} \]

\[ S_2 = \binom{n}{k} \cdot (k-1) \cdot \gamma_{m-k} \]

which proves the theorem.

**Corollary 2.20.** Let \( A' \subseteq \mathbb{Z}_+^2 \) be a saturated rational monoid, and let \( k[A'] \) be the associated monoid algebra. Consider the corresponding isolated singularity of the affine scheme \( X = \text{Spec} \ k[A'] \). The Betti series \( B(t) = \sum_{n \geq 0} \beta_n t^m \) of the local ring of this singularity is rational with denominator

\[-1 + \sum_{k=1}^{n} \left[ \binom{n-1}{k-1} \cdot S + \binom{n}{k} (k-1) \right] t^k .\]

**Proof.** Follows immediately from Theorem 2.17 and the formula of Proposition 1.3 implying \( \beta_m = \gamma_{m-2} \) for \( m \gg 0 \).

**BIBLIOGRAPHY**


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