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FREDHOLM THEORY OF TOEPLITZ OPERATORS ON DOUBLING FOCK HILBERT SPACES

AAMENA AL-QABANI, TITUS HILBERDINK and JANI A. VIRTANEN

Abstract

We study the Fredholm properties of Toeplitz operators acting on doubling Fock Hilbert spaces, and describe their essential spectra for bounded symbols of vanishing oscillation. We also compute the index of these Toeplitz operators in the special case when $\varphi(z) = |z|^\beta$ with $\beta > 0$. Our work extends the recent results on Toeplitz operators on the standard weighted Fock spaces to the setting of doubling Fock spaces.

1. Introduction

A positive Borel measure μ on the complex plane $\mathbb{C}$ is said to be a *doubling measure* if there is a positive constant C such that

$$\mu(D(z, 2r)) \le C\mu(D(z, r)),$$

for all $z \in \mathbb{C}$ and $r > 0$, where

$$D(z, r) = \{w \in \mathbb{C} : |w - z| < r\}.$$

We denote by dA the standard Lebesgue area measure on $\mathbb{C}$. Let φ be a subharmonic non-harmonic real-valued function of class C^2 on the complex plane $\mathbb{C}$ such that $\Delta\varphi\, dA$ is a doubling measure, where $\Delta\varphi$ is the Laplacian of the function φ defined by

$$\Delta\varphi = \varphi_{xx} + \varphi_{yy}.$$

The *doubling Fock space* F^2_φ is defined by

$$F^2_\varphi = \left\{ f \in H(\mathbb{C}) : \|f\|^2_\varphi = \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)} dA(z) < \infty \right\},$$

where $H(\mathbb{C})$ is the set of all entire functions. These spaces were introduced in [6], where their sampling and interpolating sequences were described.

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It is well known that F^2_φ is a Hilbert space with inner product given by

$$\langle f, g\rangle = \int_{\mathbb{C}} f(w)\overline{g(w)}e^{-2\varphi(w)}\,dA(w).$$

We also note that the doubling Fock spaces include the standard weighted Fock spaces [14], the Fock-Sobolev space [3], the Fock spaces with weights $\varphi(z) = |z|^\beta$, where $\beta > 0$ (see [10]), and the generalized Fock spaces [11] with weights φ satisfying $0 < C_1 \le \Delta\varphi(z) \le C_2$, for all $z \in \mathbb{C}$, where C_1 and C_2 are positive constants.

When $\Delta\varphi\, dA$ is doubling, we say that a measurable function $f : \mathbb{C} \to \mathbb{C}$ is in L^2_φ if

$$\|f\|^2_\varphi = \int_{\mathbb{C}} |f(z)|^2 e^{-2\varphi(z)}\,dA(z) < \infty.$$

We denote by P the orthogonal projection of L^2_φ onto F^2_φ. It has the integral representation

$$Pf(z) = \int_{\mathbb{C}} f(w)\overline{K_z(w)}e^{-2\varphi(w)}\,dA(w),$$

where K_z is the reproducing kernel function; that is, for each $z \in \mathbb{C}$, K_z is the unique function in F^2_φ for which

$$f(z) = \langle f, K_z\rangle,$$

for all $f \in F^2_\varphi$. Note that K_z depends on φ and we write

$$K_z(w) = K_\varphi(w, z) = K(w, z),$$

for $z, w \in \mathbb{C}$. For $z \in \mathbb{C}$, the *normalized reproducing kernel function* at z is defined by

$$k_z(w) = \frac{K_\varphi(w, z)}{\left\|K_\varphi(z, z)\right\|_\varphi},$$

for $w \in \mathbb{C}$.

For $f \in L^\infty = L^\infty(\mathbb{C})$, the *Toeplitz operator* T_f on F^2_φ with *symbol* f is defined by

$$T_f(g) = P(fg).$$

It is clearly a bounded linear operator on F^2_φ and $\|T_f\| \le \|f\|_\infty$.

We say that a bounded linear operator T on a Banach space X is *Fredholm* if $\ker T$ and $X/T(X)$ are both finite-dimensional. In this case, the *index* of T is defined to be

$$\operatorname{ind} T = \dim\ker T - \dim(X/T(X)).$$

The *essential spectrum* $\sigma_{\text{ess}}(T)$ is defined by

$$\sigma_{\text{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\},$$

where I denotes the identity operator on X. Clearly $\sigma_{\text{ess}}(T)$ is contained in the spectrum $\sigma(T)$ of T.

Our main results are concerned with the Fredholm properties and computation of the index of Toeplitz operators acting on doubling Fock spaces F^2_φ with bounded symbols of vanishing oscillation. Before introducing the notion of vanishing oscillation, we make a few useful observations. Since $d\mu = \Delta\varphi\, dA$ is a doubling measure, by [12, §I.8.6], for any $z \in \mathbb{C}$ and $r > 0$, we have that

$$\mu(\partial(D(z,r))) = \mu(\{z\}) = 0$$

and $\mu(D(z,2r)) \geq C\mu(D(z,r))$, where $C > 1$ is a constant. Moreover, our assumption that φ is non-harmonic implies that μ is a locally finite non-zero doubling measure on $\mathbb{C}$, so $0 < \mu(D(z,r)) < \infty$, for any $z \in \mathbb{C}$ and $r > 0$. It follows that, for each $z \in \mathbb{C}$, $\mu(D(z,r)) \to \infty$ as $r \to \infty$, and the function $r \mapsto \mu(D(z,r))$ is an increasing homeomorphism from the interval $(0,\infty)$ onto itself. Hence, for every $z \in \mathbb{C}$, there is a unique radius $\rho(z)$ such that

$$\mu(D(z,\rho(z))) = 1.$$

For $z \in \mathbb{C}$ and $r > 0$, we denote by $D^r(z)$ the disk $D(z, r\rho(z))$. Then we say that f is of *vanishing oscillation* and write $f \in \mathrm{VO}$ if f is a continuous function on $\mathbb{C}$ and

$$\omega_r(f)(z) = \sup_{w \in D^r(z)} |f(z) - f(w)| \to 0$$

as $z \to \infty$. Note that VO is independent of the choice of r.

2. Main results

The Fredholm properties of Toeplitz operators on the unweighted Fock space F^2 were described by Berger and Coburn [2] in 1987 using heavy machinery of C^*-algebras suitable for operators on Hilbert spaces. In 1992, Stroethoff [13] provided elementary proofs of their results, and very recently the theory was extended to the standard weighted Fock spaces F^p_α using elementary methods [1] and limit operator techniques [4]. In the following theorem, we extend the theory to doubling Fock Hilbert spaces F^2_φ, which also sets the stage for further extensions to other doubling Fock spaces F^p_φ.

THEOREM 2.1. *Let $f \in L^\infty \cap \mathrm{VO}$. Then the Toeplitz operator T_f on F^2_φ is Fredholm if and only if there is a radius $R > 0$ such that f is bounded away from zero on $\mathbb{C} \setminus D(0, R)$, that is,* $\inf_{|z|\geq R} |f(z)| > 0$. *In this case,*

$$\sigma_{\mathrm{ess}}(T_f) = \bigcap_{R>0} \mathrm{cl}\, f(\mathbb{C} \setminus D(0, R)),$$

where cl *stands for the closure of the given set.*

The difficulty with computing the index of Toeplitz operators on F^2_φ is that doubling Fock spaces do not have simple bases unlike the standard weighted Fock spaces where index formulas can be obtained more easily; see [2] for Toeplitz operators on F^2 and [1] for these operators on F^p_α. The following result gives an index formula in the special case $\varphi(z) = \frac{1}{2}|z|^\beta$ with $\beta > 0$, which was introduced in [9] to study the properties of Hankel operators. For convenience, we write $F^2_{|z|^\beta}$ for $F^2_{\frac{1}{2}|z|^\beta}$. It is not difficult to see that $F^2_{|z|^\beta}$ is a doubling Fock space (and not a standard weighted Fock space).

THEOREM 2.2. *Let $f \in L^\infty \cap \mathrm{VO}$. Then the Toeplitz operator T_f on $F^2_{|z|^\beta}$ is Fredholm if and only if there is a radius $R > 0$ such that f is bounded away from zero on $\mathbb{C} \setminus D(0, R)$. In this case,*

$$\mathrm{ind}(T_f) = -\,\mathrm{wind}(f|_{\{|z|=R\}}),$$

where $\mathrm{wind}(f|_{\{|z|=R\}})$ *is the winding number of the curve $f(\{|z| = R\})$ around the origin.*

The proofs of our main theorems are given in Section 4 below.

3. Preliminaries

The following estimate for the Bergman kernel for F^2_φ plays an important role in the study of concrete operators (such as Toeplitz and Hankel operators) on doubling Fock spaces (see [5], [8]).

THEOREM 3.1 ([7, Theorem 1.1 and (3)]). *There are constants $C > 0$ and $\epsilon > 0$ such that*

$$\left|K(z_1, z_2)\right| \leq C \frac{1}{\rho(z_1)\rho(z_2)} \frac{e^{\varphi(z_1)+\varphi(z_2)}}{\exp(d_\varphi(z_1, z_2)^\epsilon)},$$

for any $z_1, z_2 \in \mathbb{C}$, where d_φ is the distance on $\mathbb{C}$ induced by the metric $\varphi(z)^{-2} dz \otimes d\bar{z}$.

The previous theorem combined with the estimate (see [7, Proposition 2.10])

$$C^{-1}\frac{e^{\varphi(z)}}{\rho(z)} \le \left\| K_\varphi(\cdot, z) \right\|_\varphi \le C\frac{e^{\varphi(z)}}{\rho(z)}, \quad \text{for } z \in \mathbb{C}, \tag{3.1}$$

where C is a positive constant, gives the following result.

Lemma 3.2 ([8, Lemma 2.7]). *Let $r \ge 1$. For every $k \ge 0$ and $\epsilon > 0$ there exists a constant $C_{k,\epsilon}(r) > 0$ such that*

$$\int_{\mathbb{C}\setminus D^r(z_2)} \frac{|z_1 - z_2|^k}{\exp(d_\varphi(z_1, z_2)^\epsilon)} \frac{dA(z_1)}{\rho(z_1)^2} \le C_{k,\epsilon}(r)\rho(z_2)^k, \quad \textit{for any } z_2 \in \mathbb{C}.$$

Moreover, $C_{k,\epsilon}(r) \to 0$ as $r \to \infty$, for every $k \ge 0$ and $\epsilon > 0$.

For $f \in L^\infty$, the *Berezin transform* $\tilde{f}$ of f is defined by

$$\tilde{f}(z) = \langle f k_z, k_z \rangle = \int_{\mathbb{C}} \left|k_z(w)\right|^2 f(w) e^{-2\varphi(w)} dA(w).$$

Lemma 3.3. *Let $f \in L^\infty \cap \mathrm{VO}$. Then $\tilde{f}(z) - f(z) \to 0$ as $z \to \infty$.*

Proof. Let $\epsilon > 0$. By Theorem 3.1, (3.1) and Lemma 3.2, there is a radius $r \ge 1$ so that

$$\int_{\mathbb{C}\setminus D^r(z)} |k_z(w)|^2 e^{-2\varphi(w)} dA(w) < \frac{\epsilon}{\|f\|_\infty + 1},$$

for all $z \in \mathbb{C}$. Then

$$\begin{aligned} |\tilde{f}(z) - f(z)| &= \left| \int_{\mathbb{C}} (f(w) - f(z)) |k_z(w)|^2 e^{-2\varphi(w)} dA(w) \right| \\ &\le \omega_r(f)(z) + 2\|f\|_\infty \int_{\mathbb{C}\setminus D^r(z)} |k_z(w)|^2 e^{-2\varphi(w)} dA(w) \\ &\le \omega_r(f)(z) + 2\epsilon \le 3\epsilon, \end{aligned}$$

when $|z|$ is large enough. Thus, $\tilde{f}(z) - f(z) \to 0$ as $|z| \to \infty$, which completes the proof.

For $f \in L^\infty$, the *Hankel operator* H_f on F^2_φ with symbol f is defined by

$$H_f(g) = (I - P)(fg).$$

Note that H_f is a bounded linear operator from F^2_φ into L^2_φ and $\|H_f\| \le \|f\|_\infty$. A useful relationship between Toeplitz and Hankel operators is the well-known multiplication formula

$$T_f T_g = T_{fg} - H^*_{\bar{f}} H_g, \tag{3.2}$$

which holds for $f, g \in L^\infty$.

In order to characterize compact Hankel operators, we state the definition of functions of vanishing mean oscillation. We say that $f \in L^2_{\text{loc}}(\mathbb{C})$ is in VMO if

$$\lim_{z\to\infty} \text{MO}(f)(z) = 0,$$

where

$$\text{MO}(f)(z) = \left(\frac{1}{A(D^1(z))}\int_{D^1(z)} |f - \hat{f}(z)|^2 dA\right)^{1/2}$$

and

$$\hat{f}(z) = \frac{1}{A(D^1(z))}\int_{D^1(z)} f\, dA.$$

The following two results can be found in [5].

PROPOSITION 3.4. *We have that*

$$\text{VMO} = \text{VO} + \text{VA},$$

where

$$\text{VA} = \{h \in L^2_{\text{loc}}(\mathbb{C}) : \lim_{z\to\infty} \widetilde{|h|^2}(z) = 0\} = \{h \in L^2_{\text{loc}}(\mathbb{C}) : \lim_{z\to\infty} \widehat{|h|^2}(z) = 0\}.$$

THEOREM 3.5. *Let $f \in L^\infty$. Then H_f and $H_{\overline{f}}$ are both compact operators from F^2_φ into L^2_φ if and only if $f \in$ VMO.*

We finish this section with some basic properties of the Fock spaces $F^2_{|z|^\beta}$, where $\beta > 0$. For any non-negative integer n, we define e_n by

$$e_n(z) = \frac{z^n}{C_n}, \tag{3.3}$$

where

$$C_n^2 = \langle z^n, z^n\rangle_\beta = \int_{\mathbb{C}} |z^n|^2 e^{-|z|^\beta} dA(z) = \frac{2\pi}{\beta}\Gamma\left(\frac{2n+2}{\beta}\right). \tag{3.4}$$

Then $\{e_n\}_{n\geq 0}$ is an orthonormal basis for $F^2_{|z|^\beta}$, and it follows that the reproducing kernel function is given by

$$K_\beta(z, w) = \sum_{n=0}^{\infty} \frac{(z\overline{w})^n}{C_n^2},$$

for $z, w \in \mathbb{C}$, and the orthogonal projection has the following integral representation

$$\begin{aligned} P_\beta f(z) &= \int_{\mathbb{C}} f(w) K_\beta(z, w) e^{-|w|^\beta} dA(w) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{C_n^2} \int_{\mathbb{C}} f(w) \overline{w}^n e^{-|w|^\beta} dA(w). \end{aligned} \tag{3.5}$$

4. Proofs of the main results

For the proof of the following approximation result, see [2, Lemma 17] and [1, Proposition 9].

PROPOSITION 4.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function in A, where $A = L^\infty(\mathbb{C}) \cap \mathrm{VO}$ or $A = L^\infty(\mathbb{C}) \cap \mathrm{VMO}$. Then f is bounded away from zero on $\mathbb{C} \setminus D(0, R)$, for some $R > 0$, if and only if there is a continuous function $g \in A$ such that $f(z)g(z) \to 1$ as $z \to \infty$.*

THEOREM 4.2. *If $f \in C(\mathbb{C})$ and $f(z) \to 0$ as $z \to \infty$, then the Toeplitz operator T_f is compact on F^2_φ.*

PROOF. It directly follows from the proof of (2) ⇒ (1) in [8, Theorem 5.4].

PROOF OF THEOREM 2.1. Let $f \in L^\infty(\mathbb{C}) \cap \mathrm{VO}$ and suppose that there are constants $R > 0$ and $m > 0$ such that $|f(z)| \geq m$ when $|z| > R$. By Proposition 4.1, there exists $g \in L^\infty(\mathbb{C}) \cap \mathrm{VO}$ such that $fg - 1 \to 0$ as $|z| \to \infty$. Then

$$T_f T_g = I + T_{fg-1} - P M_f H_g,$$

where M_f is the multiplication operator on L^2_φ defined by $M_f(h) = fh$. By Theorems 3.5 and 4.2, both T_{fg-1} and H_g are compact. Therefore, $T_f T_g = I + K$ for some compact operator K. Similarly, $T_g T_f = I + K_1$, where K_1 is a compact operator. Thus, T_f has a regularizer and hence it is Fredholm.

Conversely, suppose that there are $z_j \in \mathbb{C}$ such that $|z_j| \to \infty$ and $|f(z_j)| \to 0$. Then, by Lemma 3.3, we have that $\widetilde{|f|}(z_j) \to 0$. Since

$$\begin{aligned} \widetilde{|f|^2}(z) &= \int_{\mathbb{C}} |f(w)|^2 |k_z(w)|^2 e^{-2\varphi(w)} dA(w) \\ &\leq \|f\|_\infty \widetilde{|f|}(z), \end{aligned}$$

for $z \in \mathbb{C}$, we get that $\widetilde{|f|^2}(z_j) \to 0$. Now (3.2) gives that

$$\begin{aligned} \widetilde{|f|^2}(z_j) &= \langle T_{|f|^2} k_{z_j}, k_{z_j} \rangle = \langle (T_{\bar{f}} T_f + H_f^* H_f) k_{z_j}, k_{z_j} \rangle \\ &= \langle T_f k_{z_j}, T_f k_{z_j} \rangle + \langle H_f k_{z_j}, H_f k_{z_j} \rangle \\ &= \|T_f k_{z_j}\|_\varphi^2 + \|H_f k_{z_j}\|_\varphi^2. \end{aligned}$$

Moreover, since $k_{z_j} \to 0$ weakly in F^2_φ and H_f is compact on F^2_φ (by Theorem 3.5), $\|H_f k_{z_j}\|^2_\varphi \to 0$. Therefore, $\|T_f k_{z_j}\|^2_\varphi \to 0$, and hence T_f is not invertible modulo compact operators, or equivalently, T_f is not Fredholm, completing the proof.

We now switch our focus and consider Toeplitz operators on $F^2_{|z|^\beta}$ with $\beta > 0$. To prove the index formula of Theorem 2.2, we consider first Toeplitz operators with symbols of the simple form $(z/|z|)^m$.

LEMMA 4.3. *For $m \in \mathbb{N}$, let $\psi_m(z) = (z/|z|)^m$. Then T_{ψ_m} is an m-weighted shift operator on $F^2_{|z|^\beta}$, that is, there are numbers $\alpha_{m,n} > 0$ such that $T_{\psi_m} e_n = \alpha_{m,n} e_{n+m}$ for all $n \geq 0$, where $\{e_n\}_{n\geq 0}$ is defined by* (3.3). *Moreover, for any $m \in \mathbb{N}$, $\alpha_{m,n} \to 1$ as $n \to \infty$.*

PROOF. By (3.5),

$$P_\beta(\psi_m e_n)(z) = \frac{1}{C_n} \sum_{k=0}^{\infty} \frac{z^k}{C_k^2} \int_{\mathbb{C}} \overline{w}^k \frac{w^{n+m}}{|w|^m} e^{-|w|^\beta} dA(w)$$
$$= \frac{1}{C_n} \frac{z^{n+m}}{C_{n+m}^2} \int_{\mathbb{C}} |w|^{2n+m} e^{-|w|^\beta} dA(w).$$

Therefore,

$$P_\beta(\psi_m e_n) = \frac{e_{n+m}}{C_n C_{n+m}} \int_{\mathbb{C}} |w|^{2n+m} e^{-|w|^\beta} dA(w)$$
$$= \frac{e_{n+m}}{C_n C_{n+m}} \frac{2\pi}{\beta} \Gamma\left(\frac{2n+m+2}{\beta}\right),$$

and so $T_{\psi_m} e_n = \alpha_{m,n} e_{n+m}$ with

$$\alpha_{m,n} = \frac{1}{C_n C_{n+m}} \frac{2\pi}{\beta} \Gamma\left(\frac{2n+m+2}{\beta}\right).$$

Then, by (3.4), Stirling's formula shows that, for any m, $\alpha_{m,n} \to 1$ as $n \to \infty$, and the proof is complete.

PROPOSITION 4.4. *For every $m \in \mathbb{N}$, the Toeplitz operator $T_{\overline{\psi_m}}$ is a Fredholm operator on $F^2_{|z|^\beta}$ of index m.*

PROOF. Let

$$f(z) = \sum_{n=0}^{\infty} \lambda_n e_n$$

be a function in $F^2_{|z|^\beta}$. By Lemma 4.3,

$$T_{\psi_m} f = \sum_{n=0}^{\infty} \lambda_n T_{\psi_m} e_n = \sum_{n=0}^{\infty} \lambda_n \alpha_{m,n} e_{n+m},$$

where $\alpha_{m,n} \neq 0$, for any $n \geq 0$, and $\alpha_{m,n} \to 1$ as $n \to \infty$. It follows that $\dim \ker T_{\psi_m} = 0$ and $\dim(F^2_{|z|^\beta}/T_{\psi_m}(F^2_{|z|^\beta})) = m$. Therefore, T_{ψ_m} is a Fredholm operator of index $-m$, and hence $T_{\overline{\psi}_m} = T^*_{\psi_m}$ is Fredholm of index m.

We can now prove the index formula of Toeplitz operators on $F^2_{|z|^\beta}$ with symbols in the class $L^\infty \cap \mathrm{VO}$.

PROOF OF THEOREM 2.2. Taking into account Proposition 4.1 and its proof, Theorem 4.2, (3.2), and Proposition 4.4, this proof is mutatis mutandis the proof of [1, Theorem 20].

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DEPARTMENT OF MATHEMATICS
AND COMPUTER APPLICATIONS
AL-NAHRAIN UNIVERSITY
BAGHDAD
IRAQ
E-mail: aminarasim@hotmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF READING
READING RG6 6AX
ENGLAND
E-mail: t.w.hilberdink@reading.ac.uk
j.a.virtanen@reading.ac.uk