EXACT GREEN'S FORMULA FOR THE FRACTIONAL LAPLACIAN AND PERTURBATIONS

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Abstract

Let Ω be an open, smooth, bounded subset of $\mathbb{R}^n$. In connection with the fractional Laplacian $(-\Delta)^a$ $(a > 0)$, and more generally for a $2a$-order classical pseudodifferential operator (ψdo) P with even symbol, one can define the Dirichlet value $\gamma_0^{a-1}u$, resp. Neumann value $\gamma_1^{a-1}u$ of $u(x)$, as the trace, resp. normal derivative, of u/d^{a-1} on $\partial\Omega$, where $d(x)$ is the distance from $x \in \Omega$ to $\partial\Omega$; they define well-posed boundary value problems for P.

A Green's formula was shown in a preceding paper, containing a generally nonlocal term $(B\gamma_0^{a-1}u, \gamma_0^{a-1}v)_{\partial\Omega}$, where B is a first-order ψdo on $\partial\Omega$. Presently, we determine B from L in the case $P = L^a$, where L is a strongly elliptic second-order differential operator. A particular result is that $B = 0$ when $L = -\Delta$, and that B is multiplication by a function (is local) when L equals $-\Delta$ plus a first-order term. In cases of more general L, B can be nonlocal.

1. Introduction

The fractional Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, $a > 0$, is currently receiving much attention because of its great interest for applications in both probability, finance, mathematical physics and differential geometry. (References to many important contributions through the years are given e.g. in our preceding papers [8], [9], [10], [11].) $(-\Delta)^a$ can be defined as a pseudodifferential operator (ψdo), or equivalently as a singular integral operator:

$$(-\Delta)^a u = \mathrm{Op}(|\xi|^{2a})u = \mathscr{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)),$$

$$(-\Delta)^a u(x) = c_{n,a}\,\mathrm{PV}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2a}}\,dy, \tag{1.1}$$

where $\mathscr{F}$ denotes Fourier transformation $\hat{u}(\xi) = \mathscr{F}u = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,dx$. Since the operator is nonlocal for noninteger a, it is not obvious how to define its action over a subset Ω of $\mathbb{R}^n$, and there are several ways to define operators on Ω representing homogeneous boundary value problems for it (see e.g. the overview in §6 of [11]).

A much studied case is the so-called restricted Dirichlet problem

$$((-\Delta)^a u)|_\Omega = f, \quad \operatorname{supp} u \subset \overline{\Omega},$$

considered for functions u and f with a certain regularity.

One can also impose *nonhomogeneous* boundary conditions. We are particularly interested in *local* boundary operators (i.e, operators that can be defined pointwise at $\partial\Omega$). It was shown in [9, §5], for smooth open sets Ω, that the local operators

$$\gamma_j^{a+k} u = c_{akj} \gamma_j(u/d^{a+k}), \quad j \in \mathbb{N}_0, k \in \mathbb{Z} \text{ with } a + k > -1, \qquad (1.2)$$

$c_{akj} = \Gamma(a + k + j + 1)$, have a meaning in connection with $(-\Delta)^a$; here $d = \operatorname{dist}(x, \partial\Omega)$, and $\gamma_j v$ is the standard normal derivative $(\partial_\nu^j v)|_{\partial\Omega}$. In particular, defining the

Dirichlet trace $\gamma_0^{a-1} u = \Gamma(a)\gamma_0(u/d^{a-1})$,

Neumann trace $\gamma_1^{a-1} u = \Gamma(a+1)\gamma_1(u/d^{a-1})$,

one obtains well-posed nonhomogeneous boundary value problems on Ω for $(-\Delta)^a$ and more general operators; see [9] for the Dirichlet condition, and [8], [13] for the Neumann condition. The solutions are found to lie in so-called μ-transmission spaces (recalled in §2 below) with $\mu = a - 1$ or a.

When $0 < a < 1$, the solutions u with nonzero $\gamma_0^{a-1} u$ have an unbounded singularity like d^{a-1} at the boundary (also studied in Abatangelo [1]). However if $\gamma_0^{a-1} u = 0$, u behaves like d^a at the boundary, and $\gamma_1^{a-1} u$ coincides with $\gamma_0^a u$.

Recently, Abatangelo, Jarohs and Saldaña in [3], with further coauthors in [2], have studied nonhomogeneous boundary value problems for $(-\Delta)^a$ involving the trace operators (1.2), on the unit ball resp. halfspace in $\mathbb{R}^n$, with detailed calculations.

Formulas for integration by parts were first shown for functions with $\gamma_0^{a-1} u = 0$ by Ros-Oton and Serra [18], [19] (and jointly with Valdinoci for more general singular integral operators [20]) and Abatangelo [1], leading to Pohozaev identities important for uniqueness questions in nonlinear applications. In [10], we extended the formulas to general x-dependent $2a$-order pseudodifferential operators P satisfying the a-transmission condition at $\partial\Omega$.

More recently in [13] we obtained a general Green's formula for functions u, v in $(a - 1)$-transmission spaces, allowing both $\gamma_0^{a-1} u$ and $\gamma_1^{a-1} u$ to be nonzero:

$$(Pu, v)_\Omega - (u, P^* v)_\Omega$$
$$= (s_0 \gamma_1^{a-1} u + B\gamma_0^{a-1} u, \gamma_0^{a-1} v)_{\partial\Omega} - (s_0 \gamma_0^{a-1} u, \gamma_1^{a-1} v)_{\partial\Omega}.$$

Here $s_0(x)$ is a function defined from the principal symbol of P, and B is a first-order ψdo on $\partial\Omega$ depending on the first two terms in the symbol of P. It is nonlocal in general.

In the present paper we investigate how B looks in particular cases. We show that for $(-\Delta)^a$ itself, the operator B is zero:

$$((-\Delta)^a u, v)_\Omega - (u, (-\Delta)^a v)_\Omega = (\gamma_1^{a-1} u, \gamma_0^{a-1} v)_{\partial\Omega} - (\gamma_0^{a-1} u, \gamma_1^{a-1} v)_{\partial\Omega},$$

and for operators $(-\Delta + \boldsymbol{c}(x) \cdot \nabla + c_0(x))^a$, B is the multiplication by a function derived from $\boldsymbol{c}$ (Theorem 5.1). In these cases, B is local.

More generally, we investigate powers L^a of a general second-order strongly elliptic partial differential operator L, finding formulas for B in local coordinates (Theorem 4.2). It is seen here that when the normal component of the principal part of L varies along $\partial\Omega$, B can be nonlocal (Remark 4.3).

Plan of the paper. In §2 we list some prerequisites and recall the definition and properties of the μ-transmission spaces that play an important role as domains. In §3 we find the symbol of the fractional power L^a with two leading terms, when L is a strongly elliptic differential operator $-\sum_{j,k \leq n} a_{jk} \partial_j \partial_k + \boldsymbol{b} \cdot \nabla + b_0$. In §4 we determine the contribution from L^a to the symbol of B, in the case $\Omega = \mathbb{R}^n_+$. In §5 we apply this to the case of general smooth bounded sets Ω when L has principal part $-\Delta$, showing that B is the multiplication by a certain function, which vanishes when the first-order part is zero. The Appendix gives an analysis of Green's formula for $-\Delta$, connecting the formula for the general set Ω with the localized case and providing some ingredients for the treatment of $(-\Delta)^a$. Some misprints in [13] are listed at the end.

2. Notation and preliminaries, the μ-transmission spaces

Our notation has already been explained in several preceding papers [8]–[13], so we shall only recall the most important concepts needed here.

Multi-index notation is used for differentiation (and also for polynomials): $\partial = (\partial_1, \ldots, \partial_n)$, and $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$ for $\alpha \in \mathbb{N}_0^n$, with $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$. $D = (D_1, \ldots, D_n)$ with $D_j = -i\partial_j$. The function $\langle\xi\rangle$ stands for $(1 + |\xi|^2)^{1/2}$.

Operators are considered acting on functions or distributions on $\mathbb{R}^n$, and on subsets $\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n \mid x_n \gtrless 0\}$ (where $(x_1, \ldots, x_{n-1}) = x'$), and bounded C^∞-subsets Ω with boundary $\partial\Omega$, and their complements. Restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$ (or from $\mathbb{R}^n$ to Ω resp. $\complement\overline{\Omega}$) is denoted $r^\pm$, extension by zero from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$ (or from Ω resp. $\complement\overline{\Omega}$ to $\mathbb{R}^n$) is denoted $e^\pm$. Restriction from $\overline{\mathbb{R}}^n_+$ or $\overline{\Omega}$ to $\partial\mathbb{R}^n_+$ resp. $\partial\Omega$ is denoted γ_0.

We denote by $d(x)$ a function of the form $d(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$, x near $\partial\Omega$, extended to a smooth positive function on Ω; $d(x) = x_n$ in the case of $\mathbb{R}^n_+$. Then we define the spaces

$$\mathscr{E}_\mu(\overline{\Omega}) = e^+\big\{u(x) = d(x)^\mu v(x) \mid v \in C^\infty(\overline{\Omega})\big\}, \qquad (2.1)$$

for $\operatorname{Re}\mu > -1$; for other μ, cf. [9].

A *pseudodifferential operator* (ψdo) P on $\mathbb{R}^n$ is defined from a symbol $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$Pu = p(x, D)u = \operatorname{Op}(p(x, \xi))u$$
$$= (2\pi)^{-n} \int e^{ix\cdot\xi} p(x, \xi)\hat{u}\,d\xi = \mathscr{F}^{-1}_{\xi\to x}(p(x, \xi)\hat{u}(\xi)),$$

using the Fourier transform $\mathscr{F}$, cf. (1.1)ff. We refer to textbooks such as Hörmander [16], Taylor [23], Grubb [7] for the rules of calculus. p belongs to the symbol space $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, consisting of C^∞-functions $p(x, \xi)$ such that $\partial^\beta_x \partial^\alpha_\xi p(x, \xi)$ is $O(\langle\xi\rangle^{m-|\alpha|})$ for all α, β, for some $m \in \mathbb{R}$ (global estimates); then P (and p) has order m. P (and p) is said to be *classical* when p moreover has an asymptotic expansion $p(x, \xi) \sim \sum_{j\in\mathbb{N}_0} p_j(x, \xi)$ with p_j homogeneous in ξ of degree $m - j$ for $|\xi| \geq 1$, all j, and $p(x, \xi) - \sum_{j<J} p_j(x, \xi) \in S^{m-J}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for all J.

Recall in particular the composition rule: When $PQ = R$, then R has a symbol $r(x, \xi)$ with the following asymptotic expansion, called the Leibniz product:

$$r(x, \xi) \sim p(x, \xi) \# q(x, \xi) = \sum_{\alpha\in\mathbb{N}^n_0} \partial^\alpha_\xi p(x, \xi) D^\alpha_x q(x, \xi)/\alpha!. \qquad (2.2)$$

When P (and p) is classical, it is said to be *even*, when

$$p_j(x, -\xi) = (-1)^j p_j(x, \xi), \quad \text{for all } j. \qquad (2.3)$$

Then if P is of order $2a$, it satisfies the a-transmission condition defined in [9], with respect to any smooth subset Ω of $\mathbb{R}^n$. Even-order differential operators L have this evenness property, and so do the powers L^a (as constructed by Seeley [21]) when L is strongly elliptic.

When P is a ψdo on $\mathbb{R}^n$, $P_+ = r^+ P e^+$ denotes its truncation to $\mathbb{R}^n_+$, or to Ω, depending on the context.

The L_2-Sobolev spaces are defined for $s \in \mathbb{R}$ by

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) \mid \mathscr{F}^{-1}(\langle \xi \rangle^s \hat{u}) \in L_2(\mathbb{R}^n)\},$$

$$\dot{H}^s(\overline{\Omega}) = \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}, \quad \text{the supported space,}$$

$$\overline{H}^s(\Omega) = \{u \in \mathscr{D}'(\Omega) \mid u = r^+U \text{ for a } U \in H^s(\mathbb{R}^n)\}, \quad \text{the restricted space;}$$

here $\operatorname{supp} u$ denotes the support of u. The definition is also used with $\Omega = \mathbb{R}^n_+$. In most current texts, $\overline{H}^s(\Omega)$ is denoted $H^s(\Omega)$ without the overline (that was introduced along with the notation $\dot{H}$ in [15], [16]), but we keep it here since it is practical in indications of dualities, and makes the notation more clear in formulas where both types occur. We recall that $\overline{H}^s(\Omega)$ and $\dot{H}^{-s}(\overline{\Omega})$ are dual spaces with respect to a sesquilinear duality extending the $L_2(\Omega)$-scalar product, written e.g.

$$\langle f, g \rangle_{\overline{H}^s(\Omega), \dot{H}^{-s}(\overline{\Omega})}, \quad \text{or just } \langle f, g \rangle_{\overline{H}^s, \dot{H}^{-s}}.$$

There are many other interesting scales of spaces, the Bessel-potential spaces H^s_p, the Triebel-Lizorkin spaces $F^s_{p,q}$ and the Besov spaces B^s_p and $B^s_{p,q}$, where the problems can be studied; see details in [8], [9]. This includes the Hölder-Zygmund spaces $B^s_{\infty,\infty}$, also denoted C^s_*; they are interesting because $C^s_*(\mathbb{R}^n)$ equals the Hölder space $C^s(\mathbb{R}^n)$ when $s \in \mathbb{R}_+ \setminus \mathbb{N}$. The survey in [12, §3] recalls the theory in H^s_p-spaces. We continue here with $p = 2$.

A special role in the theory is played by the *order-reducing operators*. There is a simple definition of operators $\Xi^t_\pm$ on $\mathbb{R}^n$ for $t \in \mathbb{R}$,

$$\Xi^t_\pm = \operatorname{Op}(\chi^t_\pm), \quad \chi^t_\pm = (\langle \xi' \rangle \pm i\xi_n)^t;$$

they preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively. The functions $(\langle \xi' \rangle \pm i\xi_n)^t$ do not satisfy all the estimates required for the class $S^t_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, but the operators are useful for many purposes. There is a more refined choice $\Lambda^t_\pm$ [5], [9], with symbols $\lambda^t_\pm(\xi)$ that do satisfy all the estimates for $S^t_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$; here $\overline{\lambda^t_+} = \lambda^t_-$. The symbols have holomorphic extensions in ξ_n to the complex halfspaces $\mathbb{C}_\mp = \{z \in \mathbb{C} \mid \operatorname{Im} z \lessgtr 0\}$; it is for this reason that the operators preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively. Operators with that property are called "plus" resp. "minus" operators. There is also a pseudodifferential definition $\Lambda^{(t)}_\pm$ adapted to the situation of a smooth domain Ω, cf. [9].

It is elementary to see by the definition of the spaces $H^s(\mathbb{R}^n)$ in terms of Fourier transformation, that the operators define homeomorphisms for all s: $\Xi^t_\pm : H^s(\mathbb{R}^n) \xrightarrow{\sim} H^{s-t}(\mathbb{R}^n)$, $\Lambda^t_\pm : H^s(\mathbb{R}^n) \xrightarrow{\sim} H^{s-t}(\mathbb{R}^n)$. The special interest is that the "plus"/"minus" operators also define homeomorphisms related to

$\overline{\mathbb{R}}^n_+$ and $\overline{\Omega}$, for all $s \in \mathbb{R}$: $\Xi^t_+ \colon \dot{H}^s(\overline{\mathbb{R}}^n_+) \overset{\sim}{\to} \dot{H}^{s-t}(\overline{\mathbb{R}}^n_+)$, $r^+\Xi^t_- e^+ \colon \overline{H}^s(\mathbb{R}^n_+) \overset{\sim}{\to}$
$\overline{H}^{s-t}(\mathbb{R}^n_+)$, with similar statements for $\Lambda^{(t)}_\pm$ relative to Ω. Moreover, the operators Ξ^t_+ and $r^+\Xi^t_- e^+$ identify with each other's adjoints over $\overline{\mathbb{R}}^n_+$, because of the support preserving properties; there is a similar statement for $\Lambda^{(t)}_+$ and $r^+\Lambda^{(t)}_- e^+$ relative to the set Ω.

The special μ-*transmission spaces* were introduced by Hörmander [15] and redefined in [9] (we just recall them for real $\mu > -1$):

$$H^{\mu(s)}(\overline{\mathbb{R}}^n_+) = \Xi^{-\mu}_+ e^+ \overline{H}^{s-\mu}(\mathbb{R}^n_+) = \Lambda^{-\mu}_+ e^+ \overline{H}^{s-\mu}(\mathbb{R}^n_+), \quad s > \mu - \tfrac{1}{2},$$

$$H^{\mu(s)}(\overline{\Omega}) = \Lambda^{(-\mu)}_+ e^+ \overline{H}^{s-\mu}(\Omega), \quad s > \mu - \tfrac{1}{2};$$

they are the appropriate solution spaces for homogeneous Dirichlet problems for elliptic operators P having the μ-transmission property (cf. [9]). We also recall that r^+P maps $\mathscr{E}_\mu(\overline{\Omega})$ (cf. (2.1)) into $C^\infty(\overline{\Omega})$, and that $\mathscr{E}_\mu(\overline{\Omega})$ is the solution space for the homogeneous Dirichlet problem with data in $C^\infty(\overline{\Omega})$. $\mathscr{E}_\mu(\overline{\Omega})$ is dense in $H^{\mu(s)}(\overline{\Omega})$ for all s, and $\bigcap_s H^{\mu(s)}(\overline{\Omega}) = \mathscr{E}_\mu(\overline{\Omega})$. (For $\Omega = \mathbb{R}^n_+$, $\mathscr{E}_\mu(\overline{\mathbb{R}}^n_+) \cap \mathscr{E}'(\mathbb{R}^n)$ is dense in $H^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ for all s.)

One has that $H^{\mu(s)}(\overline{\Omega}) \supset \dot{H}^s(\overline{\Omega})$, and the elements are locally in H^s on Ω, but at the boundary they in general have a singular behavior (cf. [9, Theorem 5.4]):

$$H^{\mu(s)}(\overline{\Omega}) \begin{cases} = \dot{H}^s(\overline{\Omega}), & \text{if } s \in \,]\mu - \tfrac{1}{2}, \mu + \tfrac{1}{2}[, \\ \subset \dot{H}^s(\overline{\Omega}) + e^+ d^\mu \overline{H}^{s-\mu}(\Omega), & \text{if } s > \mu + \tfrac{1}{2}, s - \mu - \tfrac{1}{2} \notin \mathbb{N}. \end{cases} \tag{2.4}$$

The inclusion in the second line of (2.4) has recently been sharpened in [14] to a precise description: When $s \in \,]\mu + M - \tfrac{1}{2}, \mu + M + \tfrac{1}{2}[$, $M \in \mathbb{N}$, then

$$H^{\mu(s)}(\overline{\Omega}) = \dot{H}^s(\overline{\Omega}) + \widetilde{\mathscr{K}}^\mu_M \prod_{j=0}^{M-1} H^{s-\mu-j-\frac{1}{2}}(\partial\Omega),$$

where $\widetilde{\mathscr{K}}^\mu_M$ is d^μ times a system of Poisson operators in the Boutet de Monvel calculus constructed in a simple way from a Poisson operator $K_{(0)}$ solving the Dirichlet problem for $-\Delta$. For $M = 1$, $\widetilde{\mathscr{K}}^\mu_1$ is proportional to $d^\mu K_{(0)}$.

Analogous results hold in the other scales of function spaces (H^s_p, B^s_p, $F^s_{p,q}$, $B^s_{p,q}$) mentioned above. Let us in particular mention the Hölder-Zygmund spaces $C^s_* = B^s_{\infty,\infty}$ (coinciding with ordinary Hölder spaces for $s \in \mathbb{R}_+ \setminus \mathbb{N}$).

Here the μ-transmission spaces are defined by

$$C_*^{\mu(s)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-\mu} e^+ \overline{C}_*^{s-\mu}(\mathbb{R}_+^n) = \Lambda_+^{-\mu} e^+ \overline{C}_*^{s-\mu}(\mathbb{R}_+^n), \quad s > \mu - 1,$$

$$C_*^{\mu(s)}(\overline{\Omega}) = \Lambda_+^{(-\mu)} e^+ \overline{C}_*^{s-\mu}(\Omega), \quad s > \mu - 1.$$

Again, $C_*^{\mu(s)}(\overline{\Omega}) \supset \dot{C}_*^s(\overline{\Omega})$, and the elements are locally in C_*^s on Ω. More precisely, $C_*^{\mu(s)}(\overline{\Omega}) = \dot{C}_*^s(\overline{\Omega})$ if $s \in {]\mu - 1, \mu[}$, and when $s \in {]\mu + M - 1, \mu + M[}$ for an $M \in \mathbb{N}$:

$$C_*^{\mu(s)}(\overline{\Omega}) = \dot{C}_*^s(\overline{\Omega}) + \widetilde{\mathcal{K}}_M^\mu \prod_{j=0}^{M-1} C_*^{s-\mu-j}(\partial\Omega) \subset \dot{C}_*^s(\overline{\Omega}) + e^+ d^\mu \overline{C}_*^{s-\mu}(\Omega),$$

cf. [8], [14]. The spaces C_*^s are denoted Λ_s in Stein [22] and sequels.

In the present paper, we shall in particular work with the spaces where $\mu = a - 1$, which is negative in the important case where $0 < a < 1$. The results in cases where $a > 1$, for example for $(-\Delta)^{3/2} = |\nabla|^3$, should also be of interest.

Note that we always have $\mathscr{E}_\mu(\overline{\Omega})$ as a dense subset.

3. Powers of a second-order elliptic differential operator

The following result was shown in [13]:

THEOREM 3.1. *Let P be a classical ψdo on $\mathbb{R}^n$ of order $2a > 0$ (not necessarily elliptic), with even symbol, cf. (2.3), and let Ω equal $\mathbb{R}_+^n$ or a smooth bounded subset of $\mathbb{R}^n$. The following Green's formula holds for $u, v \in H^{(a-1)(s)}(\overline{\Omega})$ when $s > a + \frac{1}{2}$ with $s \geq 2a$:*

$$\int_\Omega (Pu\bar{v} - u\overline{P^*v}) \, dx = (s_0 \gamma_1^{a-1} u + B\gamma_0^{a-1} u, \gamma_0^{a-1} v)_{L_2(\partial\Omega)}$$
$$- (s_0 \gamma_0^{a-1} u, \gamma_1^{a-1} v)_{L_2(\partial\Omega)}.$$

(When only $s > a + \frac{1}{2}$, the formula holds with the left-hand side interpreted as dualities.) Here $s_0(x) = p_0(x, \nu(x))$ at boundary points x with interior normal $\nu(x)$, and B is a first-order ψdo on $\partial\Omega$. In the case $\Omega = \mathbb{R}_+^n$, the symbol of B equals the jump at $z_n = 0$ of the distribution $\mathcal{F}_{\xi_n \to z_n}^{-1} q(x', 0, \xi)$, where q is the symbol of $Q = \Xi_-^{-a} P \Xi_+^{-a}$ (the case of curved Ω is derived from this).

Since $C_*^s \subset H^{s-\varepsilon}$ (any $\varepsilon > 0$), the formula is in particular valid when $u, v \in C_*^{(a-1)(s)}(\overline{\Omega})$ for some $s > a + \frac{1}{2}$ with $s > 2a$; then Pu and P^*v are continuous functions on $\overline{\Omega}$.

We now want to describe B more precisely in interesting special cases. A natural class of operators P satifying the hypotheses arises from taking a'th powers of second-order differential operators; it will be studied in the following.

Consider a general second-order strongly elliptic partial differential operator given on $\mathbb{R}^n$ or on an open subset containing the set $\overline{\Omega}$ we are interested in,

$$L = -\sum_{j,k \leq n} a_{jk}\partial_j\partial_k + \boldsymbol{b}(x) \cdot \nabla + b_0(x) = L_0 + L_1 + L_2, \qquad (3.1)$$

with symbols $\ell = \ell_0 + \ell_1 + \ell_2$,

$$\ell_0(\xi) = \sum_{j,k \leq n} a_{jk}\xi_j\xi_k, \quad \ell_1(x,\xi) = \boldsymbol{b}(x) \cdot i\xi, \quad \ell_2(x) = b_0(x),$$

where the $a_{jk}(x)$, $\boldsymbol{b}(x) = (b_1(x), \ldots, b_n(x))$ and $b_0(x)$ are bounded complex C^∞-functions. The strong ellipticity means that

$$\operatorname{Re} \sum_{j,k \leq n} a_{jk}(x)\xi_j\xi_k \geq c|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n,$$

with $c > 0$.

We can describe the fractional powers by use of Seeley's analysis [21]. Assume that the functions $a_{jk}(x), b_j(x), b_0(x)$ have been extended to all of $\mathbb{R}^n$, such that L equals $1 - \Delta$ outside a large ball. The resolvent of L is the inverse of $L - \lambda$, defined when λ is in the resolvent set; it includes a truncated sector $V = \{\lambda \in \mathbb{C} \mid |\arg\lambda - \pi| \leq \frac{\pi}{2} + \delta, |\lambda| \geq R\}$ for some large R and small δ. If the matrix $(a_{j,k})_{j,k \leq n}$ is real (or hermitian symmetric), V can be taken as $\{\lambda \in \mathbb{C} \mid |\arg\lambda - \pi| \leq \pi - \delta, |\lambda| \geq R\}$ for some large R and small δ. The resolvent symbol $\tilde{\ell}_\lambda$ is constructed by use of the Leibniz product formula (2.2) from the symbol $\ell - \lambda$ of $L - \lambda$.

It is known from [21] that the resolvent symbol has an expansion in symbols $\tilde{\ell}_{l,\xi}$ homogeneous of degree $-2 - l$ in $(\xi, |\lambda|^{1/2})$,

$$\tilde{\ell}_\lambda \sim \sum_{l=0,1,2,\ldots} \tilde{\ell}_{l,\lambda},$$

with

$$\tilde{\ell}_{0,\lambda} = (\ell_0 - \lambda)^{-1}, \qquad \tilde{\ell}_{l,\lambda} = \sum_{l/2 \leq k \leq 2l} c_{l,k}(x,\xi)\tilde{\ell}_{0,\lambda}^{-k-1} \quad \text{for } l = 1, 2, \ldots,$$

the $c_{l,k}(x,\xi)$ being polynomials in ξ of degree $2k - l$.

Let us work out the construction in exact form up to the second homogeneous term (homogeneous of degree -3 with respect to (ξ, μ), $\mu = (-\lambda)^{1/2}$), with the subsequent terms grouped together under the indication *l.o.t.* (lower order terms). We use *l.o.t.* to denote terms of order at least two integers lower than the principal term, in each step in the deduction (this precision is all we need for the discussion of Green's formula).

The principal term in the resolvent symbol is $\tilde{\ell}_{0,\lambda} = (\ell_0(x,\xi) - \lambda)^{-1}$, as noted. Now

$$(\ell - \lambda) \,\#\, \tilde{\ell}_{0,\lambda} = (\ell_0 - \lambda)\tilde{\ell}_{0,\lambda} + \ell_1 \tilde{\ell}_{0,\lambda} + \sum_{j=1}^{n} \frac{1}{i} \partial_{\xi_j} \ell_0 \, \partial_{x_j} \tilde{\ell}_{0,\lambda} + l.o.t.$$

$$= 1 + i\boldsymbol{b}(x) \cdot \xi \, \tilde{\ell}_{0,\lambda} + \sum_{j=1}^{n} i \partial_{\xi_j} \ell_0 \, \partial_{x_j} \ell_0 \, \tilde{\ell}_{0,\lambda}^2 + l.o.t.$$

$$= 1 + r,$$

where

$$r = i\boldsymbol{b}(x) \cdot \xi \, \tilde{\ell}_{0,\lambda} + \sum_{j=1}^{n} i \partial_{\xi_j} \ell_0 \, \partial_{x_j} \ell_0 \, \tilde{\ell}_{0,\lambda}^2 + l.o.t.$$

Since $(1 + r) \,\#\, (1 - r) = 1 - r \,\#\, r$ with $r \,\#\, r$ of order -2, it follows that

$$(\ell - \lambda) \,\#\, \tilde{\ell}_{0,\lambda} \,\#\, (1 - r) = 1 + l.o.t.,$$

so $\ell - \lambda$ has a right parametrix

$$\tilde{\ell}_\lambda = \tilde{\ell}_{0,\lambda} \,\#\, (1 - r) + l.o.t. = \tilde{\ell}_{0,\lambda} - i\boldsymbol{b}(x) \cdot \xi \, \tilde{\ell}_{0,\lambda}^2 - \sum_{j=1}^{n} i \partial_{\xi_j} \ell_0 \, \partial_{x_j} \ell_0 \, \tilde{\ell}_{0,\lambda}^3 + l.o.t.$$

One finds similarly a left parametrix, and concludes (by a standard argument in elliptic theory) that $\tilde{\ell}_\lambda$ is a two-sided parametrix.

Now the fractional powers are constructed by use of Cauchy integral formulas:

We can describe L^a approximately as

$$L^a = \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^a (L - \lambda)^{-1} \, d\lambda,$$

with some interpretation: The curve $\mathscr{C}$ is chosen to encircle the spectrum of L in the positive direction, except possibly for a finite set of eigenvalues of finite multiplicity (it can for example consist of the rays $\left\{ \lambda = r e^{i(\frac{\pi}{2} - \delta)} \mid \infty > r \geq r_0 \right\}$ and $\left\{ \lambda = r e^{i(\frac{3\pi}{2} + \delta)} \mid r_0 \leq r < \infty \right\}$ connected by a small curve going clockwise around zero $\left\{ \lambda = r_0 e^{i\theta} \mid \frac{\pi}{2} - \delta < \theta < \frac{3\pi}{2} + \delta \right\}$). The integral

converges when $a < 0$; for positive a one can involve recomposition with integer powers of L.

The symbol p of $P = L^a$ then satisfies

$$p(x, \xi) = \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^a \Big((\ell_0 - \lambda)^{-1} - i\boldsymbol{b} \cdot \xi (\ell_0 - \lambda)^{-2} \\ + \sum_{j=1}^{n} i\partial_{\xi_j} \ell_0\, \partial_{x_j} \ell_0 (\ell_0 - \lambda)^3 \Big)\, d\lambda + l.o.t.;$$

here the formula holds as it stands when $a < 0$, and since the integration curve can for each (x, ξ) be replaced by a closed curve $\mathscr{C}_0$ around $\ell_0(x, \xi)$, the formula generalizes to all a.

The first term gives, by Cauchy's formula, that the principal symbol of P is $p_0 = \ell_0^a$. The next terms give

$$p_1 = -i\boldsymbol{b} \cdot \xi\, \frac{i}{2\pi} \int_{\mathscr{C}_0} \lambda^a (\ell_0 - \lambda)^{-2}\, d\lambda$$

$$- \sum_{j=1}^{n} i\partial_{\xi_j} \ell_0\, \partial_{x_j} \ell_0\, \frac{i}{2\pi} \int_{\mathscr{C}_0} \lambda^a (\ell_0 - \lambda)^3\, d\lambda$$

$$= -i\boldsymbol{b} \cdot \xi\, \frac{i}{2\pi} \int_{\mathscr{C}_0} \lambda^a \frac{d}{d\lambda} (\ell_0 - \lambda)^{-1}\, d\lambda$$

$$- \sum_{j=1}^{n} i\partial_{\xi_j} \ell_0\, \partial_{x_j} \ell_0\, \frac{i}{2\pi} \int_{\mathscr{C}_0} \lambda^a \frac{1}{2} \frac{d^2}{d\lambda^2} (\ell_0 - \lambda)^{-1}\, d\lambda$$

$$= i\boldsymbol{b} \cdot \xi\, \frac{i}{2\pi} \int_{\mathscr{C}_0} \Big(\frac{d}{d\lambda} \lambda^a \Big) (\ell_0 - \lambda)^{-1}\, d\lambda$$

$$- \sum_{j=1}^{n} i\partial_{\xi_j} \ell_0\, \partial_{x_j} \ell_0\, \frac{i}{2\pi} \int_{\mathscr{C}_0} \Big(\frac{d^2}{d\lambda^2} \lambda^a \Big) \frac{1}{2} (\ell_0 - \lambda)^{-1}\, d\lambda$$

$$= i\boldsymbol{b} \cdot \xi\, \frac{i}{2\pi} \int_{\mathscr{C}_0} a\lambda^{a-1} (\ell_0 - \lambda)^{-1}\, d\lambda$$

$$- \sum_{j=1}^{n} i\partial_{\xi_j} \ell_0\, \partial_{x_j} \ell_0\, \frac{i}{2\pi} \int_{\mathscr{C}_0} \frac{1}{2} a(a-1) \lambda^{a-2} (\ell_0 - \lambda)^{-1}\, d\lambda$$

$$= i\boldsymbol{b} \cdot \xi\, a\ell_0^{a-1} - \binom{a}{2} \sum_{j=1}^{n} i\partial_{\xi_j} \ell_0\, \partial_{x_j} \ell_0\, \ell_0^{a-2},$$

evaluated for $\xi \neq 0$ so that the negative powers of ℓ_0 make sense. Thus we

find that the symbol of P satisfies (for $\xi \neq 0$):

$$p(x, \xi) = p_0(x, \xi) + p_1(x, \xi) + l.o.t.,$$

$$p_0 = \ell_0^a, \quad p_1 = \ell_0^a \left(iab \cdot \xi \ell_0^{-1} - \binom{a}{2} \sum_{j=1}^{n} i \partial_{\xi_j} \ell_0 \, \partial_{x_j} \ell_0 \, \ell_0^{-2} \right). \tag{3.2}$$

4. The boundary term B in Green's formula

We know from [13, Theorem 4.1] that the symbol of the ψdo B entering in Green's formula for P in the half-space situation equals (for $|\xi'| \geq 1$):

$$b(x', \xi') = \lim_{z_n \to 0+} \check{q}(x', 0, \xi', z_n) - \lim_{z_n \to 0-} \check{q}(x', 0, \xi', z_n)$$

$$\equiv \text{jump } \mathscr{F}_{\xi_n \to z_n}^{-1} q(x', 0, \xi), \tag{4.1}$$

where $q(x, \xi)$ is the symbol of $Q = \Xi_-^{-a} P \Xi_+^{-a}$, and $\check{q}(x, \xi', z_n) = \mathscr{F}_{\xi_n \to z_n}^{-1} q$. For $P = L^a$ as above we shall first describe q with two precise terms.

Recalling that $\Xi_\pm^{-a}$ are the (generalized) ψdo's with symbols $\chi_\pm^{-a} = (\langle \xi' \rangle \pm i\xi_n)^{-a}$, we have that the symbol of Q satisfies, by (3.2) and the Leibniz product formula (2.2),

$$q(x, \xi) = \chi_-^{-a} \# p \# \chi_+^{-a} = \chi_-^{-a} \# p \chi_+^{-a}$$

$$= \langle \xi \rangle^{-2a} (p_0 + p_1) + \sum_{j \leq n} \frac{1}{i} \partial_{\xi_j} \chi_-^{-a} \, \partial_{x_j} p_0 \, \chi_+^{-a} + l.o.t., \tag{4.2}$$

using that $\chi_-^{-a} \chi_+^{-a} = \langle \xi \rangle^{-2a}$. Here

$$\langle \xi \rangle^{-2a} (p_0 + p_1) = \left(\frac{\ell_0}{\langle \xi \rangle^2} \right)^a \left(1 + iab \cdot \xi \ell_0^{-1} - \binom{a}{2} \sum_{j=1}^{n} i \partial_{\xi_j} \ell_0 \, \partial_{x_j} \ell_0 \, \ell_0^{-2} \right), \tag{4.3}$$

and, since $\partial_{\xi_j} \langle \xi' \rangle = \xi_j \langle \xi' \rangle^{-1}$ for $j < n$,

$$\sum_{j \leq n} \frac{1}{i} \partial_{\xi_j} \chi_-^{-a} \, \partial_{x_j} p_0 \, \chi_+^{-a}$$

$$= \sum_{j < n} ia\chi_-^{-a-1} \partial_{\xi_j} \langle \xi' \rangle \, a\ell_0^{a-1} \partial_{x_i} \ell_0 \, \chi_+^{-a}$$

$$+ ia\chi_-^{-a-1}(-i)a\ell_0^{a-1} \partial_{x_n} \ell_0 \, \chi_+^{-a} \tag{4.4}$$

$$= ia^2 \left(\frac{\ell_0}{\langle \xi \rangle^2} \right)^a \chi_-^{-1} \ell_0^{-1} \left(\sum_{j < n} \xi_j \langle \xi' \rangle^{-1} \partial_{x_j} \ell_0 - i \partial_{x_n} \ell_0 \right).$$

We observe that the resulting expressions have the form of a product of $(\ell_0 l \langle\xi\rangle^2)^a$ with a rational function of ξ_n that is $O(\xi_n^{-1})$ for $|\xi_n| \to \infty$. This prepares the way for evaluating $b(x', \xi')$ in (4.1), but we first have to deal also with the factor $(\ell_0/\langle\xi\rangle^2)^a$. Write

$$\frac{\ell_0}{\langle\xi\rangle^2} = \frac{a_{nn}\langle\xi\rangle^2 + \sum'_{jk} a_{jk}\xi_j\xi_k - a_{nn}\langle\xi'\rangle^2}{\langle\xi\rangle^2} = a_{nn}\left(1 + \frac{c(x,\xi)}{\langle\xi\rangle^2}\right),$$

where $\sum'_{jk}$ denotes the sum omitting the term with $j = k = n$, and

$$c(x,\xi) = a_{nn}^{-1}\left(\sum_{j<n}(a_{jn} + a_{nj})\xi_j\xi_n + \sum_{j,k<n} a_{jk}\xi_j\xi_k\right) - \langle\xi'\rangle^2. \qquad (4.5)$$

Note that $c(x, \xi)$ is a second-order polynomial in ξ of order 1 in ξ_n. Then, by Taylor expansion of $(1 + t)^a$,

$$\begin{aligned}
\left(\frac{\ell_0}{\langle\xi\rangle^2}\right)^a &= a_{nn}^a\left(1 + \frac{c(x,\xi)}{\langle\xi\rangle^2}\right)^a \\
&= a_{nn}^a\left(1 - a\frac{c(x,\xi)}{\langle\xi\rangle^2} + \binom{a}{2}\frac{c(x,\xi)^2}{\langle\xi\rangle^4} + O(\xi_n^{-3})\right),
\end{aligned} \qquad (4.6)$$

for $|\xi_n| \to \infty$. (Only the expansion up to $O(\xi_n^{-2})$ is used in the following.)

This leads to:

THEOREM 4.1. *Let $P = L^a$ on $\mathbb{R}^n$, where L is a second-order strongly elliptic differential operator (3.1), and let q be the symbol of $Q = \Xi_-^{-a} P \Xi_+^{-a}$, defined relative to the halfspace $\mathbb{R}^n_+$. Let $|\xi'| \geq 1$. As a function of ξ_n, q satisfies:*

$$a_{nn}^{-a}q(x,\xi) = 1 - \frac{aa_{nn}^{-1}\sum_{j<n}(a_{jn} + a_{nj})\xi_j\xi_n}{\langle\xi\rangle^2} + iab_n\xi_n\ell_0^{-1} - \binom{a}{2}i\xi_n\partial_{x_n}a_{nn}\,\ell_0^{-1}$$

$$+ a^2 a_{nn}^{-1}\chi_-^{-1}\left(\sum_{j<n} i\xi_j\langle\xi'\rangle^{-1}\partial_{x_j}a_{nn} + \partial_{x_n}a_{nn}\right) + O(\xi_n^{-2}). \quad (4.7)$$

PROOF. In the various expressions we absorb terms that are $O(\xi_n^{-2})$ in the remainder. For (4.6) we have, using (4.5):

$$\left(\frac{\ell_0}{\langle\xi\rangle^2}\right)^a = a_{nn}^a\left(1 - \frac{aa_{nn}^{-1}\xi_n \sum_{j<n}(a_{jn} + a_{nj})\xi_j}{\langle\xi\rangle^2} + O(\xi_n^{-2})\right).$$

Now consider the terms in (4.3). For the first term, we note:

$$iab \cdot \xi \ell_0^{-1} = iab_n \xi_n \ell_0^{-1} + O(\xi_n^{-2}).$$

For the second term we shall use that $\ell_0 = a_{nn}\xi_n^2 + O(\xi_n)$ implies

$$\xi_n^2 \ell_0^{-1} = (a_{nn}^{-1}\ell_0 + O(\xi_n))\ell_0^{-1} = a_{nn}^{-1} + O(\xi_n^{-1}), \tag{4.8}$$

in the calculation

$$-\binom{a}{2} \sum_{j=1}^{n} i\partial_{\xi_j}\ell_0\, \partial_{x_j}\ell_0\, \ell_0^{-2} = -\binom{a}{2} i a_{nn}\xi_n \partial_{x_n}\ell_0\, \ell_0^{-2} + O(\xi_n^{-2})$$

$$= -\binom{a}{2} i\xi_n^3 a_{nn}\partial_{x_n}a_{nn}\, \ell_0^{-2} + O(\xi_n^{-2})$$

$$= -\binom{a}{2} i\xi_n \partial_{x_n}a_{nn}\, \ell_0^{-1} + O(\xi_n^{-2}).$$

The expression in (4.4) satisfies

$$ia^2\chi_-^{-1}\ell_0^{-1}\left(\sum_{j<n}\xi_j\langle\xi'\rangle^{-1}\partial_{x_j}\ell_0 - i\partial_{x_n}\ell_0\right)$$

$$= ia^2\xi_n^2\chi_-^{-1}\ell_0^{-1}\left(\sum_{j<n}\xi_j\langle\xi'\rangle^{-1}\partial_{x_j}a_{nn} - i\partial_{x_n}a_{nn}\right) + O(\xi_n^{-2})$$

$$= a^2 a_{nn}^{-1}\chi_-^{-1}\left(\sum_{j<n}i\xi_j\langle\xi'\rangle^{-1}\partial_{x_j}a_{nn} + \partial_{x_n}a_{nn}\right) + O(\xi_n^{-2}),$$

again using (4.8). This gives (4.7), when the terms are collected in (4.2).

To find $b(x', \xi')$, the jump of $\check{q}$ at $z_n = 0$, we appeal to a little of the knowledge used in the Boutet de Monvel calculus. Recall from [6], [7] that the space $\mathscr{H}^+ = \mathscr{F}(e^+r^+\mathscr{S}(\mathbb{R}))$ consists of functions of $\xi_n \in \mathbb{R}$ that are $O(\xi_n^{-1})$ at infinity and extend holomorphically into the lower halfplane $\mathbb{C}_-$ (with further estimates), and that there is a similar space $\mathscr{H}_{-1}^- = \mathscr{F}(e^-r^-\mathscr{S}(\mathbb{R}))$ consisting of functions that extend holomorphically into the upper halfplane $\mathbb{C}_+$; it is the conjugate space of $\mathscr{H}^+$. All we shall use here is that the fractional terms in q can (for $|\xi'| \geq 1$) be decomposed into parts in $\mathscr{H}^+$ and $\mathscr{H}_{-1}^-$ with respect to

ξ_n, in view of the formulas

$$\langle\xi\rangle^{-2} = \frac{1}{\langle\xi'\rangle^2 + \xi_n^2} = \frac{1}{2\langle\xi'\rangle}\left(\frac{1}{\langle\xi'\rangle + i\xi_n} + \frac{1}{\langle\xi'\rangle - i\xi_n}\right),$$

$$\frac{-2i\xi_n}{\langle\xi\rangle^2} = \frac{1}{\langle\xi'\rangle + i\xi_n} - \frac{1}{\langle\xi'\rangle - i\xi_n},$$

$$\chi_-^{-1} = \frac{1}{\langle\xi'\rangle - i\xi_n},$$

$$\ell_0^{-1} = \frac{1}{\sum a_{jk}\xi_j\xi_k} = \frac{1}{a_{nn}(\sigma_+ + i\xi_n)(\sigma_- - i\xi_n)} \qquad (4.9)$$

$$= \frac{1}{a_{nn}(\sigma_+ + \sigma_-)}\left(\frac{1}{\sigma_+ + i\xi_n} + \frac{1}{\sigma_- - i\xi_n}\right),$$

$$-i\xi_n\ell_0^{-1} = \frac{-i\xi_n}{a_{nn}(\sigma_+ + \sigma_-)}\left(\frac{1}{\sigma_+ + i\xi_n} + \frac{1}{\sigma_- - i\xi_n}\right)$$

$$= \frac{1}{a_{nn}(\sigma_+ + \sigma_-)}\left(\frac{\sigma_+}{\sigma_+ + i\xi_n} - \frac{\sigma_-}{\sigma_- - i\xi_n}\right).$$

Here $\pm i\sigma_\pm$ are the roots of $\sum_{jk} a_{jk}\xi_j\xi_k$ in $\mathbb{C}_\pm$ with respect to ξ_n, respectively (then $\mathrm{Re}\,\sigma_\pm > 0$). When $\mathrm{Re}\,\sigma > 0$, $(\sigma - i\xi_n)^{-1} \in \mathscr{H}_{-1}^-$ and $(\sigma + i\xi_n)^{-1} \in \mathscr{H}^+$, and (with H equal to the Heaviside function $1_{\overline{\mathbb{R}}_+}$)

$$\mathscr{F}_{\xi_n\to z_n}^{-1}\frac{1}{\sigma + i\xi_n} = H(z_n)e^{-\sigma z_n}, \quad \mathscr{F}_{\xi_n\to z_n}^{-1}\frac{1}{\sigma - i\xi_n} = H(-z_n)e^{\sigma z_n};$$

these functions have the limit 1 for $z_n \to 0+$, resp. $z_n \to 0-$.

Then from (4.9) follows for example:

$$\begin{aligned}
&\text{(i)} \quad \text{jump } \mathscr{F}_{\xi_n\to z_n}^{-1}\langle\xi\rangle^{-2} = 0,\\
&\text{(ii)} \quad \text{jump } \mathscr{F}_{\xi_n\to z_n}^{-1}\xi_n\langle\xi\rangle^{-2} = i,\\
&\text{(iii)} \quad \text{jump } \mathscr{F}_{\xi_n\to z_n}^{-1}\chi_-^{-1} = -1,\\
&\text{(iv)} \quad \text{jump } \mathscr{F}_{\xi_n\to z_n}^{-1}\xi_n\ell_0^{-1} = ia_{nn}^{-1}.
\end{aligned} \qquad (4.10)$$

This leads to:

THEOREM 4.2. *Assumptions as in Theorem 4.1. The symbol $b(x', \xi')$ defined*

by (4.1) *satisfies:*

$$b(x', \xi') = -aa_{nn}^{a-1}\sum_{j<n}(a_{jn} + a_{nj})i\xi_j - aa_{nn}^{a-1}b_n + \binom{a}{2}a_{nn}^{a-1}\partial_{x_n}a_{nn}$$
$$- a^2 a_{nn}^{a-1}\left(\sum_{j<n}i\xi_j\langle\xi'\rangle^{-1}\partial_{x_j}a_{nn} + \partial_{x_n}a_{nn}\right),$$

(4.11)

all coefficients evaluated at $x_n = 0$.

PROOF. Consider $q(x', 0, \xi', \xi_n)$ described by (4.7) multiplied by $a_{nn}(x', 0)^a$. To evaluate the inverse Fourier transform from ξ_n to z_n, we begin by noting that the first term contributes with $a_{nn}(x', 0)^a\delta(z_n)$, supported in $\{z_n = 0\}$, which disappears when the limits in (4.1) are calculated. Moreover we will use that, as already noted, symbols that are $O(\xi_n^{-2})$ at infinity transform to continuous functions of z_n, hence have jump 0.

Now consider the second term in the right-hand side of (4.7). Here we find by use of (4.10)(ii) that the jump, it contributes, equals

$$-aa_{nn}^{a-1}\sum_{j<n}(a_{jn} + a_{nj})i\xi_j.$$

The third term is found by use of (4.10)(iv) to contribute with

$$-aa_{nn}^{a-1}b_n.$$

The fourth term gives in view of (4.10)(iv) the contribution

$$\binom{a}{2}a_{nn}^{a-1}\partial_{x_n}a_{nn}.$$

The fifth term gives by use of (4.10)(iii) the contribution

$$-a^2 a_{nn}^{a-1}\left(\sum_{j<n}i\xi_j\langle\xi'\rangle^{-1}\partial_{x_j}a_{nn} + \partial_{x_n}a_{nn}\right).$$

The contributions are collected in (4.11).

REMARK 4.3. Observe that the only possibly nonlocal contributions to $B = \mathrm{Op}(b(x', \xi'))$ come from the terms with $\langle\xi'\rangle^{-1}\partial_{x_j}a_{nn}(x', 0)$, $j < n$. So if the first tangential derivatives of a_{nn} vanish on the boundary, B is local, and otherwise it can be nonlocal.

A special case is where L stems from the Laplacian. In the reduction of the Laplacian to local coordinates described in the Appendix, we arrive at an

operator of the form (cf. (A.4))

$$L = -\underline{\Delta} = -\underline{\Delta}'(y', y_n, \partial_{y'}) - g(y')\partial_{y_n} - \partial_{y_n}^2. \qquad (4.12)$$

In comparison with the general expression (3.1), we here have

$$a_{nj}(y) = a_{jn}(y) \equiv 0 \text{ for } j < n, \quad a_{nn}(y) \equiv 1, \quad b_n(y) = -g(y'),$$

since $\underline{\Delta}'$ differentiates in the y'-variables only. The derivatives of the functions a_{nj}, a_{jn}, a_{nn} are zero. Hence (4.7) gives a much simplified expression for q. We find, as special cases of Theorems 4.1 and 4.2:

COROLLARY 4.4. *When $P = L^a$ with $L = -\underline{\Delta}$ in (4.12), as obtained by reduction of the Laplacian to local coordinates in the Appendix, then*

$$q(y, \xi) = 1 - iag(y')\xi_n \ell_0^{-1} + O(\xi_n^{-2}),$$

where $\ell_0 = \ell_0'(y, \xi') + \xi_n^2$, ℓ_0' being the principal symbol of $-\underline{\Delta}'$. The symbol $b(y', \xi')$ of B is

$$b(y', \xi') = ag(y').$$

As a slightly more general case, let $P = (-\Delta + \boldsymbol{c}(x) \cdot \nabla + c_0(x))^a$ on $\mathbb{R}^n$. On $\partial\Omega$, $-\Delta + \boldsymbol{c}(x) \cdot \nabla + c_0(x)$ may be written in the form, cf. the Appendix:

$$-\Delta + \boldsymbol{c}(x) \cdot \nabla + c_0(x) = -\Delta_S - \mathscr{G}\frac{\partial}{\partial n} - \frac{\partial^2}{\partial n^2} + c_\nu\frac{\partial}{\partial n} + T + c_0, \quad (4.13)$$

where $\mathscr{G} = \operatorname{div} \nu$, $c_\nu(x) = \boldsymbol{c}(x) \cdot \nu(x)$ and T is a first-order differential operator acting along $\partial\Omega$. In fact this decomposition extends to a tubular neighborhood of each coordinate patch for $\partial\Omega$, as described in the Appendix for $-\Delta$. When $c_\nu(x)$ on $\partial\Omega$ carries over to $\underline{c}_\nu(y')$ at $t = 0$, we extend it as constant in t on the neighborhood. Then P has the form L_1^a in the local coordinates, where

$$L_1 = -\underline{\Delta}'(y', y_n, \partial_{y'}) - \partial_{y_n}^2 - (g(y') - \underline{c}_\nu(y'))\partial_{y_n} + \underline{T} + \underline{c}_0. \qquad (4.14)$$

Here we find:

COROLLARY 4.5. *When $P = L_1^a$ with L_1 as in (4.14), obtained by reduction to local coordinates of the perturbed Laplacian $-\Delta + \boldsymbol{c}(x) \cdot \nabla + c_0(x)$ (decomposed on $\partial\Omega$ as in (4.13)), then the corresponding symbol q satisfies*

$$q(y, \xi) = 1 - ia(g(y') - \underline{c}_\nu(y'))\xi_n \ell_0^{-1} + O(\xi_n^{-2}),$$

with ℓ_0 as in Corollary 4.4. The symbol $b(y', \xi')$ of B is a function of y',

$$b(y', \xi') = a(g(y') - \underline{c}_\nu(y')).$$

5. Green's formula for the fractional Laplacian and its perturbations

The above considerations in local coordinates will now be applied to find Green's formula in the curved situation for the powers of the perturbed Laplacian, in particular for the fractional Laplacian $(-\Delta)^a$ itself.

THEOREM 5.1. *Let Ω be a smooth bounded subset of $\mathbb{R}^n$, and let $P = (-\Delta + \mathbf{c} \cdot \nabla + c_0)^a$, $a > 0$. Let $u, v \in H^{(a-1)(s)}(\overline{\Omega})$, $s > a + \frac{1}{2}$. When $s \geq 2a$,*

$$\int_\Omega (Pu\bar{v} - u\overline{P^*v})\, dx$$
$$= (\gamma_1^{a-1} u - ac_v \gamma_0^{a-1} u, \gamma_0^{a-1} v)_{L_2(\partial\Omega)} - (\gamma_0^{a-1} u, \gamma_1^{a-1} v)_{L_2(\partial\Omega)}$$

$$\equiv \Gamma(a)\Gamma(a+1)$$
$$\cdot \int_{\partial\Omega} \left(\gamma_1 \left(\frac{u_k}{t^{a-1}} \right) - c_v \gamma_0 \left(\frac{u_k}{t^{a-1}} \right) \right) \gamma_0 \left(\frac{\bar{v}_l}{t^{a-1}} \right) - \gamma_0 \left(\frac{u_k}{t^{a-1}} \right) \gamma_1 \left(\frac{\bar{v}_l}{t^{a-1}} \right) d\sigma,$$
(5.1)

where $c_v(x) = \mathbf{c}(x) \cdot v(x)$. The formula extends to general s with $s > a + \frac{1}{2}$, when the left-hand side is replaced by

$$\langle r^+ Pu, v \rangle_{\overline{H}^{-a+\frac{1}{2}+\varepsilon}(\Omega), \dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\Omega})} - \langle u, r^+ P^*v \rangle_{\dot{H}^{a-\frac{1}{2}-\varepsilon}(\overline{\Omega}), \overline{H}^{-a+\frac{1}{2}+\varepsilon}(\Omega)}.$$

In particular, the fractional Laplacian $(-\Delta)^a$ satisfies

$$\int_\Omega \left((-\Delta)^a u\bar{v} - u\overline{(-\Delta)^a v} \right) dx$$
$$= (\gamma_1^{a-1} u, \gamma_0^{a-1} v)_{L_2(\partial\Omega)} - (\gamma_0^{a-1} u, \gamma_1^{a-1} v)_{L_2(\partial\Omega)}$$

$$\equiv \Gamma(a)\Gamma(a+1) \int_{\partial\Omega} \gamma_1 \left(\frac{u}{t^{a-1}} \right) \gamma_0 \left(\frac{\bar{v}}{t^{a-1}} \right) - \gamma_0 \left(\frac{u}{t^{a-1}} \right) \gamma_1 \left(\frac{\bar{v}}{t^{a-1}} \right) d\sigma.$$

PROOF. It is shown in [13, Theorem 4.4] for operators satisfying the a-transmission condition how the formula for a general domain Ω is deduced from the knowledge of Green's formula in flat cases $\Omega = \mathbb{R}^n_+$, by use of local coordinates. We shall follow that construction for our special operators, and rather than taking up space by repeating the whole proof, we shall just explain the needed ingredients.

The general transformation rule is (A.7). We first note that when L corresponds to $\underline{L}$, then in view of Seeley's analysis of a'th powers of ψdo's by passage via the resolvent and a Cauchy integral formula (recalled in §3), the terms in the symbol of $\underline{P} = (\underline{L^a})$, carried over from $P = L^a$ by the coordinate change, are consistent with the terms in the symbol of $(\underline{L})^a$. This will be

used with $L = -\Delta + c(x) \cdot \nabla + c_0(x)$, reduced to the form (4.14) in a local coordinate system.

The set $\overline{\Omega}$ is covered by a system of bounded open sets $U_0, \dots, U_{I_1}$, with diffeomorphisms $\kappa_i: U_i \rightarrow V_i \subset \mathbb{R}^n$ such that $U_i^+ = U_i \cap \Omega$ is mapped to $V_i^+ = V_i \cap \mathbb{R}_+^n$, and $U_i' = U_i \cap \partial\Omega$ is mapped to $V_i' = V_i \cap \mathbb{R}^{n-1}$ ($\mathbb{R}^{n-1} = \partial\overline{\mathbb{R}}_+^n$), the restriction of κ_i to U_i' denoted κ_i'. The diffeomorphism is chosen such that the interior normal $\nu(x')$ at $x' \in \partial\Omega$ defines a normal coordinate near $\partial\Omega$:

$$\kappa_i \quad \text{maps} \quad x' + t\nu(x') \in U_i \quad \text{to} \quad (y', t) \in V_i \subset \mathbb{R}^{n-1} \times \mathbb{R}$$

near $\partial\Omega$ (with $y' = \kappa_i'(x')$). We shall denote the inverses $\kappa_i^{-1} = \tilde{\kappa}_i$, $(\kappa_i')^{-1} = \tilde{\kappa}_i'$.

There is a partition of unity ϱ_k, $k = 0, \dots, J_0$, with $\sum_k \varrho_k = 1$ on a neighborhood of $\overline{\Omega}$, *subordinate* to the covering, in the sense that for any two functions ϱ_k, ϱ_l there is an $i = i(k, l)$ in $\{0, \dots, I_1\}$ such that $\operatorname{supp} \varrho_k \cup \operatorname{supp} \varrho_l \subset U_{i(k,l)}$. Moreover, nonnegative functions ψ_k and $\zeta_k \in C_0^\infty(U_i)$ are chosen such that $\zeta_k \varrho_k = \varrho_k$ and $\psi_k \zeta_k = \zeta_k$.

Now a given $u \in H^{(a-1)(s)}(\overline{\Omega})$ can be decomposed in this space as a sum $u = \sum_{k \leq J_0} u_k + r$, where $\operatorname{supp} u_k \subset \operatorname{supp} \zeta_k \subset U_i$, and $r \in \dot{C}^\infty(\overline{\Omega})$ does not contribute to the boundary integrals. There is a similar decomposition $v = \sum_{l \leq J_0} v_l + r'$ of $v \in H^{(a-1)(s)}(\overline{\Omega})$. The operators P and P^* can in their action on u_k and v_l in the scalar products be replaced by

$$P_{kl} = \psi_l P \psi_k, \quad P_{kl}^* = \psi_k P^* \psi_l.$$

As earlier noted, the action of the operators in local coordinates follows the rule recalled in (A.7); we indicate localized operators and functions by underlines. It is shown in Theorem 4.4 of [13] how the contribution from u_k, v_l is reformulated and worked out as

$$
\begin{aligned}
(P_{kl}u_k, & \, v_l)_{\Omega_i^+} - (u_k, P_{kl}^* v_l)_{\Omega_i^+} \\
&= (r^+ \underline{P}_{kl}\underline{u}_k, J\underline{v}_l)_{\mathbb{R}_+^n} - (\underline{u}_k, r^+(\underline{P}_{kl})^{(*)} J\underline{v}_l)_{\mathbb{R}_+^n} \\
&= (\underline{s}_{kl,0}\gamma_1^{a-1}\underline{u}_k, \gamma_0^{a-1} J\underline{v}_l)_{\mathbb{R}^{n-1}} - (\underline{s}_{kl,0}\gamma_0^{a-1}\underline{u}_k, \gamma_1^{a-1} J\underline{v}_l)_{\mathbb{R}^{n-1}} \\
&\quad + (\underline{B}_{kl}\gamma_0^{a-1}\underline{u}_k, \gamma_0^{a-1} J\underline{v}_l)_{\mathbb{R}^{n-1}},
\end{aligned}
\tag{5.2}
$$

where $J = |\partial x / \partial(y', t)|$, the absolute value of the functional determinant of $\tilde{\kappa}$ going from the local coordinates (y', t) to the given coordinates x. (We omit marking the operators with (i) as in [13] indicating the dependence on the coordinate patch.)

The effect of J in the boundary values with respect to $\underline{v}_l$ is as follows:

$$\gamma_0^{a-1}(J\underline{v}_l) = \Gamma(a)\gamma_0\left(\frac{J\underline{v}_l}{t^{a-1}}\right) = \gamma_0(J)\gamma_0^{a-1}(\underline{v}_l),$$

$$\gamma_1^{a-1}(J\underline{v}_l) = \Gamma(a+1)\gamma_0\left(\frac{\partial}{\partial t}\frac{J\underline{v}_l}{t^{a-1}}\right)$$

$$= \Gamma(a+1)\gamma_0(J)\gamma_1\left(\frac{\underline{v}_l}{t^{a-1}}\right) + a\Gamma(a)\gamma_1(J)\gamma_0\left(\frac{\underline{v}_l}{t^{a-1}}\right)$$

$$= \gamma_0(J)\gamma_1^{a-1}(\underline{v}_l) + a\gamma_1(J)\gamma_0^{a-1}(\underline{v}_l).$$

Here we recall from the Appendix that $\gamma_0(J) = J_0$, the factor entering in integration formulas over $\partial\Omega$, and

$$\gamma_1(J) = J_1 = J_0 g, \quad g = \operatorname{div} v.$$

Hence

$$\gamma_0^{a-1}(J\underline{v}_l) = J_0\gamma_0^{a-1}(\underline{v}_l), \quad \gamma_1^{a-1}(J\underline{v}_l) = J_0\gamma_1^{a-1}(\underline{v}_l) + aJ_0 g\gamma_0^{a-1}(\underline{v}_l).$$
$$(5.3)$$

Now we apply Theorem 4.1 of [13], using the formula for the localized version of $P = (-\Delta + \boldsymbol{c} \cdot \nabla + c_0)^a$ described in §4. Since the cutoff functions $\underline{\psi}_k, \underline{\psi}_l$ are 1 on supp $\underline{u}_k$ resp. supp $\underline{v}_l$, they can be disregarded in the formulas.

As shown in Corollary 4.5, $\underline{B}$ for $\underline{P}$ is the multiplication by $a(g(y')-\underline{c}_\nu(y'))$. Then (5.2) takes the form, in view of (5.3),

$$(r^+\underline{P}_{kl}\underline{u}_k, J\underline{v}_l)_{\mathbb{R}^n_+} - (\underline{u}_k, r^+(\underline{P}_{kl})^{(*)}(J\underline{v}_l))_{\mathbb{R}^n_+}$$

$$= (\underline{s}_{kl,0}\gamma_1^{a-1}\underline{u}_k, J_0\gamma_0^{a-1}\underline{v}_l)_{\mathbb{R}^{n-1}}$$

$$\quad - (\underline{s}_{kl,0}\gamma_0^{a-1}\underline{u}_k, J_0\gamma_1^{a-1}\underline{v}_l + aJ_0 g\gamma_0^{a-1}\underline{v}_l)_{\mathbb{R}^{n-1}}$$

$$\quad + (a(g - \underline{c}_\nu)\gamma_0^{a-1}\underline{u}_k, J_0\gamma_0^{a-1}\underline{v}_l)_{\mathbb{R}^{n-1}} \qquad\qquad (5.4)$$

$$= (\underline{s}_{kl,0}\gamma_1^{a-1}\underline{u}_k, J_0\gamma_0^{a-1}\underline{v}_l)_{\mathbb{R}^{n-1}} - (\underline{s}_{kl,0}\gamma_0^{a-1}\underline{u}_k, J_0\gamma_1^{a-1}\underline{v}_l)_{\mathbb{R}^{n-1}}$$

$$\quad - (a\underline{c}_\nu\gamma_0^{a-1}\underline{u}_k, J_0\gamma_0^{a-1}\underline{v}_l)_{\mathbb{R}^{n-1}},$$

where the terms with *ag cancelled out*! Here $p(y',0,0,1) = (\ell_0(y',0,0,1))^a = 1$ (since the coefficient of $\partial_{y_n}^2$ in ℓ_0 is 1), so the factor $\underline{s}_{kl,0}$ is simply $\underline{\psi}_k\underline{\psi}_l$, which is 1 on supp $\underline{u}_k \cap$ supp $\underline{v}_l$, and the last display in (5.4) simplifies to

$$(\gamma_1^{a-1}\underline{u}_k, J_0\gamma_0^{a-1}\underline{v}_l)_{\mathbb{R}^{n-1}} - (\gamma_0^{a-1}\underline{u}_k, J_0\gamma_1^{a-1}\underline{v}_l)_{\mathbb{R}^{n-1}}$$

$$- (a\underline{c}_\nu\gamma_0^{a-1}\underline{u}_k, J_0\gamma_0^{a-1}\underline{v}_l)_{\mathbb{R}^{n-1}}.$$

Expressed in x-coordinates, this gives

$$(\gamma_1^{a-1} u_k, \gamma_0^{a-1} v_l)_{L_2(\partial\Omega)} - (\gamma_0^{a-1} u_k, \gamma_1^{a-1} v_l)_{L_2(\partial\Omega)} - (ac_v \gamma_0^{a-1} u_k, \gamma_0^{a-1} v_l)_{L_2(\partial\Omega)}$$

$$\equiv \Gamma(a)\Gamma(a+1)$$

$$\cdot \int_{\partial\Omega} \gamma_1\Big(\frac{u_k}{t^{a-1}}\Big) \gamma_0\Big(\frac{\bar{v}_l}{t^{a-1}}\Big) - \gamma_0\Big(\frac{u_k}{t^{a-1}}\Big) \gamma_1\Big(\frac{\bar{v}_l}{t^{a-1}}\Big) - c_v \gamma_0\Big(\frac{u_k}{t^{a-1}}\Big) \gamma_0\Big(\frac{\bar{v}_l}{t^{a-1}}\Big) \, d\sigma,$$

and a summation over k, l leads to (5.1).

The validity on lower-order function spaces is accounted for in [13], and the formula for $(-\Delta)^a$ is a special case where $c_v = 0$.

Appendix A. Localization of the Laplacian

The basic arguments in [13] depend on a study of pseudodifferential boundary value problems, reduced from the general situation where $\partial\Omega$ is a hypersurface in $\mathbb{R}^n$ to the situation where $\partial\Omega$ equals the boundary $\mathbb{R}^{n-1}$ of the half-space $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$ (where $(x_1, \ldots, x_{n-1}) = x'$). As a preparation for seeing how such coordinate changes affect $(-\Delta)^a$, we here investigate their effect on $-\Delta$ itself (the case $a = 1$).

In the following, S is a smooth hypersurface in $\mathbb{R}^n$ (e.g. a piece of $\partial\Omega$) defined as

$$S = \chi(V'), \quad V' \text{ open} \subset \mathbb{R}^{n-1}$$

where $\chi = (\chi_1(y'), \ldots, \chi_n(y'))$ is a smooth injective mapping. With $\nu(\chi(y'))$ denoting a unit normal to S at each point $\chi(y') \in S$ (orthogonal to the tangent vectors $\partial\chi/\partial y_j$, $j = 1, \ldots, n-1$, its orientation depending continuously on y'), we parametrize a tubular neighborhood U of S by a diffeomorphism

$$\tilde{\kappa}: y = (y', t) \mapsto x = \chi(y') + t\nu(\chi(y')), \tag{A.1}$$

from $V = V' \times]-\delta, \delta[$ to U (possibly after replacing U, V and S by smaller sets). The functional matrix is

$$\frac{\partial x}{\partial y} = \frac{\partial \tilde{\kappa}}{\partial(y', t)} = \Big(\frac{\partial\chi}{\partial y'} + t\frac{\partial\nu(\chi)}{\partial y'} \quad \nu(\chi) \Big),$$

written as an $n \times (n-1)$-block next to an $n \times 1$-block. We can view this as $M + tN$, where

$$M = \Big(\frac{\partial\chi}{\partial y'} \quad \nu(\chi) \Big) = \frac{\partial x}{\partial y}\Big|_{t=0}, \quad N = \Big(\frac{\partial\nu(\chi)}{\partial y'} \quad 0 \Big)$$

The Jacobian is the absolute value of the functional determinant

$$J = \left| \det \frac{\partial x}{\partial y} \right| = \left| \det(M + tN) \right|.$$

To fix the ideas, assume that $\det M > 0$, so that $J = \det(M + tN)$ for small t.

In comparison with the notation in [13], we are leaving out the indexation by i in Remark 4.3 there, $\tilde{\kappa}$ is the inverse of the diffeomorphism denoted κ_i there, and χ is the inverse of the mapping κ_i'.

Denote $J|_{t=0} = J_0$, note that it equals $\det M$. It is well-known that integration over S can be described via the local coordinates as follows: When $\varphi(x)$ is a function on S, denote by $\underline{\varphi}(y')$ the corresponding function on V', that is,

$$\varphi(\chi(y')) = \underline{\varphi}(y').$$

The rule for integration is then

$$\int_S \varphi(x)\, d\sigma = \int_{V'} \underline{\varphi}(y') J_0 \, dy' \tag{A.2}$$

(the appropriate "area-element" is $J_0\, dy'$). This is found in introductory textbooks on differential geometry; note that J_0 can also be described by

$$J_0 = \sqrt{\det(M^t M)} = \sqrt{\det\left(\frac{\partial \chi}{\partial y_j} \cdot \frac{\partial \chi}{\partial y_k} \right)_{j,k \leq n-1}}.$$

For Green's formula we shall moreover need the value of the t-derivative of J at $t = 0$, that we calculate here for completeness:

LEMMA A.1. *Assume that* $\det M > 0$. *The value of* $J_1 = \partial_t J|_{t=0}$ *at the points of S is*

$$J_1 = J_0 \operatorname{div} \nu.$$

PROOF. Fix $x \in S$. Since $J(t) = \det(M + tN)$ is a polynomial of degree n in t, J_1 is the coefficient of the first power t. Now with $s = 1/t$,

$$\det(M + tN) = t^n \det M \det(s + NM^{-1})$$
$$= t^n \det M (s^n + \operatorname{trace}(NM^{-1}) s^{n-1} + \cdots + \det(NM^{-1})),$$

so the coefficient of t is $\det M \operatorname{trace}(NM^{-1})$. Here

$$NM^{-1} = \left(\frac{\partial \nu(\chi)}{\partial y'} \quad 0 \right) \frac{\partial y}{\partial x} = \frac{\partial \nu(\chi)}{\partial y} \frac{\partial y}{\partial x} = \frac{\partial \nu(\chi)}{\partial x}.$$

The trace of this matrix equals

$$\text{trace}(NM^{-1}) = \partial_{x_1} v_1 + \cdots + \partial_{x_n} v_n = \text{div } v.$$

Since $\det M = J_0$, it follows that $\det M \text{trace}(NM^{-1}) = J_0 \text{div } v$.

The function $\text{div } v$ on S represents the mean curvature, modulo a dimensional factor.

It is known that Δu on S has the form

$$\Delta u = \Delta_S u + \mathscr{G}\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2} \quad \text{on } S, \tag{A.3}$$

cf. e.g. Hsiao and Wendtland [17] (with reference to Leis 1967) and Duduchava, Mitrea and Mitrea [4] (with reference to Günther 1934). Here Δ_S is the Laplace-Beltrami operator on S, $\partial u/\partial n$ is the normal derivative $\partial u/\partial n = \sum_{j=1}^{n} v_j \partial_{x_j} u$, and $\mathscr{G} = \text{div } v$. In our local coordinates, $\partial/\partial n$ corresponds to $\partial/\partial t$, and Δ_S corresponds to an operator $\underline{\Delta}'(y', \partial_{y'}, 0)$ acting with respect to y' (we do not need its exact form at present).

For $|\varepsilon| < \delta$, the parallel surfaces S_ε represented by

$$\chi_\varepsilon(y') = \chi(y') + \varepsilon v(\chi(y')), \quad y' \in V',$$

again have normals $v(\chi(y'))$. Indeed, if we denote $v(\chi(y')) = \tilde{v}(y')$, we have since $\|\tilde{v}(y')\| = 1$ for $y' \in V'$, that the vectors $\partial\tilde{v}/\partial y_j$ are orthogonal to $\tilde{v}$ at the point, hence $\tilde{v}$ is orthogonal to $\partial(\chi + \varepsilon\tilde{v})/\partial y_j$ at the point, for $j = 1, \ldots n-1$. So (A.1) is also a parametrization of a neighborhood of S_ε (for t near ε), with $v(\chi(y'))$ as normal at the point $\chi(y') + \varepsilon v(\chi(y'))$. On S_ε there is a formula like (A.3),

$$\Delta u = \Delta_{S_\varepsilon} u + \mathscr{G}_\varepsilon \frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial n^2} \quad \text{on } S_\varepsilon,$$

where $\mathscr{G}_\varepsilon$ at $\chi(y') + \varepsilon\tilde{v}(y')$ is the same as $\mathscr{G}$ at $\chi(y')$, and $\partial/\partial n$ corresponds to $\partial/\partial t$.

We conclude that in the local coordinates (y', t), when $u(x)$ corresponds to $\underline{u}(y)$ (i.e., $u(\tilde{\kappa}(y', t)) = \underline{u}(y', t)$), Δu takes the form

$$\Delta u = \underline{\Delta}\,\underline{u} \equiv \Delta'(y', t, \partial_{y'})\underline{u} + g(y')\partial_t \underline{u} + \partial_t^2 \underline{u}, \quad \text{for } (y', t) \in V, \tag{A.4}$$

where Δ' is a second-order operator differentiating only in the y'-variables, and

$$g(y') = \mathscr{G}(\chi(y')).$$

We can now compare Green's formulas worked out in the different coordinates.

When S is a piece of the boundary $\partial\Omega$ of a smooth open set $\Omega \subset \mathbb{R}^n$ with ν as the interior normal, and u and v are supported in U, then, as is very well known,

$$(-\Delta u, v)_{U^+} - (u, -\Delta v)_{U^+} = (\gamma_1 u, \gamma_0 v)_S - (\gamma_0 u, \gamma_1 v)_S, \qquad (A.5)$$

here $U^+ = U \cap \Omega$, $\gamma_0 u = u|_{\partial\Omega}$, and $\gamma_1 u = \gamma_0(\partial u/\partial n)$.

For the operator in (A.4) we have for $\underline{u}$ and $\underline{w}$ supported in V, denoting $V \cap \mathbb{R}^n_+ = V^+$,

$$
\begin{aligned}
(-\underline{\Delta}\,\underline{u}, \underline{w})_{V^+} &- (\underline{u}, -\underline{\Delta}^{(*)}\underline{w})_{V^+} \\
&= (-\Delta'\underline{u} - g\partial_t\underline{u} - \partial_t^2\underline{u}, \underline{w})_{V^+} - (\underline{u}, (-\Delta')^{(*)}\underline{w} + \partial_t(g\underline{w}) - \partial_t^2\underline{w})_{V^+} \\
&= (\gamma_1\underline{u}, \gamma_0\underline{w})_{V'} - (\gamma_0\underline{u}, \gamma_1\underline{w})_{V'} + (g\gamma_0\underline{u}, \gamma_0\underline{w})_{V'};
\end{aligned}
$$
$$(A.6)$$

here the star in parentheses indicates the adjoint with respect to y-coordinates, to distinguish it from the adjoint in x-coordinates, and $\gamma_1\underline{w} = \gamma_0(\partial\underline{w}/\partial t)$ (consistently with the normal derivative).

It may seem surprising at first, that the two formulas (A.5) and (A.6) are so different, in that the latter has the extra term with g. However, they are consistent, as we shall now show by deducing (A.5) from (A.6).

Recall, as also accounted for in [13], that when the operator P in x-coordinates corresponds to $\underline{P}$ in y-coordinates:

$$\underline{P}\,\underline{u} = P(\underline{u} \circ \tilde\kappa^{-1}) \circ \tilde\kappa = (Pu) \circ \tilde\kappa, \qquad (A.7)$$

with $\underline{u} = u \circ \tilde\kappa$, then
$$(Pu, v)_U = (\underline{P}\underline{u}, J\underline{v})_V,$$

and the formal adjoint of $\underline{P}$ in y-coordinates satisfies

$$(\underline{P})^{(*)} = J(\underline{P^*})J^{-1}.$$

Thus (for sufficiently smooth u, v supported in U)

$$
\begin{aligned}
(-\Delta u, v)_{U^+} &- (u, -\Delta v)_{U^+} \\
&= (-\underline{\Delta}\,\underline{u}, J\underline{v})_{V^+} - (\underline{u}, -\underline{\Delta}^{(*)}J\underline{v})_{V^+} \\
&= (\gamma_1\underline{u}, \gamma_0(J\underline{v}))_{V'} - (\gamma_0\underline{u}, \gamma_1(J\underline{v}))_{V'} + (g\gamma_0\underline{u}, \gamma_0(J\underline{v}))_{V'},
\end{aligned}
$$

by (A.6). Here

$$\gamma_0(J\underline{v}) = J_0\gamma_0\underline{v}, \quad \gamma_1(J\underline{v}) = J_0\gamma_1\underline{v} + J_1\gamma_0\underline{v},$$

where $J_0 = \gamma_0 J$. $J_1 = \gamma_0(\partial_t J)$ as defined above. In view of Lemma A.1, $J_1 = J_0 g$. As a result,

$$
\begin{aligned}
(-\Delta u, &v)_{U^+} - (u, -\Delta v)_{U^+} \\
&= (\gamma_1 \underline{u}, J_0 \gamma_0 \underline{v})_{V'} - (\gamma_0 \underline{u}, J_0 \gamma_1 \underline{v} + J_0 g \gamma_0 \underline{v})_{V'} + (g \gamma_0 \underline{u}, J_0 \gamma_0 \underline{v})_{V'} \\
&= (\gamma_1 \underline{u}, J_0 \gamma_0 \underline{v})_{V'} - (\gamma_0 \underline{u}, J_0 \gamma_1 \underline{v})_{V'} \\
&= (\gamma_1 u, \gamma_0 v)_S - (\gamma_0 u, \gamma_1 v)_S,
\end{aligned}
$$

where the terms with *g cancelled out*. Thus (A.5) follows from (A.6). In the last step we used (A.2).

REMARK A.2 (Corrections to a preceding paper). A few misprints in the paper [13], that were not eliminated during the typesetting, are listed here. Page 750, line 4 from below, "p_j" should be lifted. Page 752, line 24, "derived from P^*" should be "derived from P^+". Page 756, line 9, "for $p > -1/\mu$" should be "for $p < -1/\mu$". Page 757, line 8 from below, "$x_n^a \partial_{n_{\acute{w}}}^e(\xi', 0)$" should be "$x_n^a \partial_n \acute{w}_e(\xi', 0)$". Page 762, line 8, "$aB$" should be "$a^{-1}B$". Page 768, line 3, "Let P is" should be "Let P be"; line 13, "aB" should be "$a^{-1}B$". Page 769, line 8 from below, replace "$\underline{u}k$" by "$\underline{u}_k$".

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