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ON GENERALIZED DIRICHLET PROBLEMS

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Introduction.

Let $G \subset \mathbf{R}^n$ be an open bounded set and

$$(1) \qquad L(.,D) = \sum_{\sigma,\tau \in \Gamma} D^\sigma(a_{\sigma\tau}(.)D^\tau)$$

a differential operator in generalized divergence form. Here Γ denotes a finite set of multiindices and $a_{\sigma\tau} \in L^\infty(G)$ for each pair $(\sigma,\tau) \in \Gamma \times \Gamma$. For a differential operator in generalized divergence form a sesquilinear form B can be defined on $C_0^\infty(G) \times C_0^\infty(G)$ by

$$(2) \qquad B(\varphi,\psi) = \sum_{\sigma,\tau \in \Gamma} \int_G a_{\sigma\tau}(x) D^\tau\varphi(x) \overline{D^\sigma\psi(x)}\, dx .$$

We also associate with the operator $L(.,D)$ the polynomial

$$(3) \qquad L(.,\xi) = \sum_{\sigma,\tau \in \Gamma} a_{\sigma\tau}(.)\xi^\sigma\xi^\tau .$$

We define a subset Γ^* of Γ such that

$$(4) \qquad \sum_{\sigma,\tau \in \Gamma^*} a_{\sigma\tau}(.)\xi^\sigma\xi^\tau$$

can be regarded as a generalized principal part of $L(.,\xi)$.

Under further assumptions on the principal part we will prove for the sesquilinear form B a Gårding inequality in a suitable Hilbert space $H_0^{\Gamma^*}(G)$.

By using the generalized Gårding inequality and a theorem which insures the compactness of the imbedding of $H_0^{\Gamma^*}(G)$ into $L^2(G)$ we will prove the Fredholm alternative for a homogeneous Dirichlet problem. Besides strongly elliptic operators nonhypoelliptic operators, such as the operator considered in [1] and [2], and some of those in [5], belong to our class.

The space $H_0^\Gamma(G)$.

Let $\mathbf{N}_0^n$ be the set of all multiindices and for $\sigma \in \mathbf{N}_0^n$, $\sigma = (\sigma_1,\ldots,\sigma_n)$, define $|\sigma| = \sigma_1 + \ldots + \sigma_n$. If $\xi = (\xi_1,\ldots,\xi_n) \in \mathbf{R}^n$ we set for $\sigma \in \mathbf{N}_0^n$

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$$\xi^\sigma = \xi_1^{\sigma_1} \dots \xi_n^{\sigma_n}$$

and with $D_j = -i\partial/\partial x_j$, $i^2 = -1$, let $D^\sigma := D_1^{\sigma_1} \dots D_n^{\sigma_n}$. Furthermore for $\sigma, \tau \in \mathbf{N}_0^n$ we write $\sigma \leqq \tau$ if $\sigma_j \leqq \tau_j$ for $1 \leqq j \leqq n$ and $\sigma < \tau$ if $\sigma_j < \tau_j$ for $1 \leqq j \leqq n$.

Let Γ be a finite subset of $\mathbf{N}_0^n$ and $m = \max_{\sigma \in \Gamma} |\sigma|$. For an open bounded set $G \subset \mathbf{R}^n$ we denote with $C_*^m(G)$ the space of all complex-valued functions which are m-times continuously differentiable and which together with their partial derivatives up to order m are elements of $L^2(G)$. For $\varphi \in C_*^m(G)$ we denote by $\|\varphi\|_m$ the usual Sobolev norm and we set

$$\|\varphi\|_\Gamma^2 = \sum_{\sigma \in \Gamma} \int_G |D^\sigma \varphi(x)|^2 \, dx + \int_G |\varphi(x)|^2 \, dx \,. \tag{5}$$

The completion of $C_*^m(G)$ with respect to the norm $\|\cdot\|_m$ is denoted by $H^m(G)$ and that with respect to $\|\cdot\|_\Gamma$ by $H^\Gamma(G)$, respectively. Notice that both norms are obtained from scalar products, namely from the scalar products $(\varphi, \psi)_m$ and

$$(\varphi, \psi)_\Gamma = \sum_{\sigma \in \Gamma} \int_G D^\sigma \varphi(x) \overline{D^\sigma \psi(x)} \, dx + \int_G \varphi(x) \overline{\psi(x)} \, dx \,, \tag{6}$$

respectively. The closure of $C_0^\infty(G)$ with respect to the norm $\|\cdot\|_m$ is denoted by $H_0^m(G)$ and that with respect to the norm (5) by $H_0^\Gamma(G)$.

THEOREM 1. *If $u \in H_0^\Gamma(G)$ and $\sigma \in \Gamma$, then there exists the strong L^2-derivative $D^\tau u$ for all $\tau \in \mathbf{N}_0^n$ such that $\tau \leqq \sigma$.*

For a proof of Theorem 1 see [4, Satz 1.2].

THEOREM 2. *Let $G \subset \mathbf{R}^n$ be an open bounded set and $k \in \mathbf{N}$ be a fixed integer. If for each $\tau \in \mathbf{N}_0^n$, $|\tau| \leqq k$, there exists a $\sigma_\tau \in \Gamma$ such that, $\tau \leqq \sigma_\tau$, then we have a continuous imbedding of $H_0^\Gamma(G)$ into $H_0^k(G)$. Furthermore for each $m < k$ the imbedding of $H_0^\Gamma(G)$ into $H_0^m(G)$ is compact.*

PROOF. The first part of the theorem is obvious by Theorem 1 and the second part follows from the compactness of the imbedding of $H_0^k(G)$ into $H_0^m(G)$ for $m < k$.

Finally we would like to mention that the space of $H_0^\Gamma(G)$ could be regarded as a subspace of $H^\Gamma(G)$ consisting of all those elements which have generalized homogeneous boundary-data, [4, (Satz 1.5)].

A generalized Gårding inequality.

In this section we prove a generalized Gårding inequality for a class of differential operators of the form (1).

Therefore we need

LEMMA 1. *Let $G \subset \mathbf{R}^n$ be a bounded open set and for $\sigma, \tau \in \mathbf{N}_0^n$, $\sigma \neq 0$, we assume $\sigma < \tau$. Then for each $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that*

$$\|D^\sigma \varphi\|_0^2 \leqq \varepsilon \|D^\tau \varphi\|_0^2 + c(\varepsilon)\|\varphi\|_0^2 \tag{7}$$

holds for all $\varphi \in C_0^\infty(G)$.

PROOF. Since $0 \leqq \sigma < \tau$ and $\sigma \neq 0$ there exists $\sigma_1 \in \mathbf{N}_0^n$ such that $0 \neq \sigma_1 \leqq \sigma$ and $\sigma + \sigma_1 \leqq \tau$. For $\varphi \in C_0^\infty(G)$ it follows that

$$\begin{aligned} \|D^\sigma \varphi\|_0^2 &= (D^\sigma \varphi, D^\sigma \varphi)_0 = (D^{\sigma+\sigma_1}\varphi, D^{\sigma-\sigma_1}\varphi)_0 \\ &\leqq \eta_1 \|D^{\sigma+\sigma_1}\varphi\|_0^2 + (1/\eta_1)\|D^{\sigma-\sigma_1}\varphi\|_0^2 \end{aligned}$$

for an arbitrary positive η_1. Using Theorem 1 we obtain with a suitable cohstant c_1

$$\|D^\sigma \varphi\|_0^2 \leqq c_1\eta_1 \|D^\tau \varphi\|_0^2 + (1/\eta_1)\|D^{\sigma-\sigma_1}\varphi\|_0^2 . \tag{8}$$

Now, if $\sigma - \sigma_1 = 0$, then the lemma is proved, otherwise we have $\sigma - \sigma_1 < \tau$ and therefore there exists $\sigma_2 \in \mathbf{N}_0^n$ such that $0 \neq \sigma_2 \leqq \sigma - \sigma_1$ and $\sigma - \sigma_1 + \sigma_2 \leqq \tau$. We conclude as before

$$\|D^{\sigma-\sigma_1}\varphi\|_0^2 \leqq c_2\eta_2 \|D^\tau \varphi\|_0^2 + (1/\eta_2)\|D^{\sigma-\sigma_1-\sigma_2}\varphi\|_0^2 \tag{9}$$

for $\varphi \in C_0^\infty(G)$ and an arbitrary positive η_2 and a suitable c_2. If $\sigma - \sigma_1 - \sigma_2 = 0$, the lemma follows from (8) and (9), otherwise we repeat the conclusion made above. Hence after a finite number k of steps we have

$$\|D^\sigma \varphi\|_0^2 \leqq \sum_{j=1}^{k} q_j \|D^\tau \varphi\|_0^2 + q\|\varphi\|_0^2$$

where

$$q_j = \frac{c_j \eta_j}{\prod_{l=0}^{j-1} \eta_l}, \quad \eta_0 = 1, \quad \text{and} \quad q = \prod_{j=1}^{k} (1/\eta_j) .$$

Now given $\varepsilon > 0$, then take $\eta_1 = \varepsilon / k c_1$ and

$$\eta_j = (\varepsilon / k c_j) \prod_{l=0}^{j-1} \eta_l, \quad j = 2, \ldots, k .$$

It follows that

$$\sum_{j=1}^{k} q_j = \sum_{j=1}^{k} \varepsilon/k = \varepsilon$$

and therefore the lemma is proved.

Now we will define the subset Γ^* of Γ mentioned in the introduction. We say $\sigma \in \Gamma$ is a maximal element in Γ with respect to the relation $<$ if and only if there is no $\tau_0 \in \Gamma$ such that $\sigma < \tau_\sigma$. The set of all maximal elements of Γ is denoted by Γ^*. In other words, if $\Gamma' = \Gamma \setminus \Gamma^*$, then $\sigma \in \Gamma'$ if and only if there exists $\tau_\sigma \in \Gamma^*$ such that $\sigma < \tau_\sigma$. Notice that an element $\sigma \in \Gamma$ which can not be compared with any other element of Γ belongs to Γ^*.

We pose the following generalized ellipticity condition on $L(.,D)$: For at least one $x_0 \in G$, there exists two constants $c_0 > 0$ and $R \geqq 0$ such that for all $\xi \in \mathbb{R}^n$, $|\xi| > R$,

$$\text{(10)} \qquad \operatorname{Re} \sum_{\sigma,\tau \in \Gamma^*} a_{\sigma\tau}(x_0)\xi^\sigma \xi^\tau \geqq c_0 \sum_{\sigma \in \Gamma^*} \xi^{2\sigma}$$

holds. Let B be the sesquilinear form (2) associated with the differential operator (1). We claim

THEOREM 3. *If the differential operator* (1) *satisfies condition* (10) *and if for all* $(\sigma,\tau) \in \Gamma^* \times \Gamma^*$

$$\text{(11)} \qquad \sup_{x \in G} |a_{\sigma\tau}(x) - a_{\sigma\tau}(x_0)| \leqq c_0/2k$$

holds, where k is the number of elements of Γ^*, then there exist a nonnegative constant c_1 and a constant $c_0' > 0$ such that for all $\varphi \in C_0^\infty(G)$ we have

$$\text{(12)} \qquad \operatorname{Re} B(\varphi,\varphi) \geqq c_0' \|\varphi\|_{\Gamma^*}^2 - c_1 \|\varphi\|_0^2 \,.$$

PROOF. A. We consider first the case where $a_{\sigma\tau} = 0$ for all $(\sigma,\tau) \in (\Gamma \times \Gamma) - (\Gamma^* \times \Gamma^*)$. For $\varphi \in C_0^\infty(G)$ it follows

$$\begin{aligned} \operatorname{Re} B(\varphi,\varphi) &= \operatorname{Re} \sum_{\sigma,\tau \in \Gamma^*} \int_G \overline{a_{\sigma\tau}(x) D^\tau \varphi(x)} D^\sigma \varphi(x)\, dx \\ &= \operatorname{Re} \sum_{\sigma,\tau \in \Gamma^*} \overline{a_{\sigma\tau}(x_0)} \int_G \overline{D^\tau \varphi(x)} D^\sigma \varphi(x)\, dx \\ &\quad - \operatorname{Re} \sum_{\sigma,\tau \in \Gamma^*} \int_G \overline{(a_{\sigma\tau}(x_0) - a_{\sigma\tau}(x)) D^\tau \varphi(x)} D^\sigma \varphi(x)\, dx \,. \end{aligned}$$

We denote by $(F\varphi)(\xi)$ the Fourier transform of φ. By using Plancherel's theorem we get

$$\operatorname{Re} B(\varphi,\varphi) \geqq \operatorname{Re} \sum_{\sigma,\tau\in\Gamma^*} \int_{\mathbb{R}^n} a_{\sigma\tau}(x_0)\overline{(F(D^\tau\varphi))(\xi)}(F(D^\sigma\varphi))(\xi)\,d\xi -$$
$$- \sum_{\sigma,\tau\in\Gamma^*} \int_G |a_{\sigma\tau}(x_0) - a_{\sigma\tau}(x)||D^\tau(x)||D^\sigma\varphi(x)|\,dx\,.$$

Furthermore it follows

$$\operatorname{Re} B(\varphi,\varphi) \geqq \operatorname{Re} \int_{|\xi|>R} \sum_{\sigma,\tau\in\Gamma^*} a_{\sigma\tau}(x_0)\xi^\sigma\xi^\tau|(F\varphi)(\xi)|^2\,d\xi +$$
$$+ \operatorname{Re} \int_{|\xi|\leqq R} \sum_{\sigma,\tau\in\Gamma^*} a_{\sigma\tau}(x_0)\xi^\sigma\xi^\tau|(F\varphi)(\xi)|^2\,d\xi -$$
$$- \sum_{\sigma,\tau\in\Gamma^*} \int_G |a_{\sigma\tau}(x_0) - a_{\sigma\tau}(x)\|D^\tau\varphi(x)\|D^\sigma\varphi(x)\|\,dx\,.$$

By (10) and (11) we have

$$\operatorname{Re} B(\varphi,\varphi) \geqq c_0 \sum_{\sigma\in\Gamma^*} \int_{\mathbb{R}^n} \xi^\sigma\overline{(F\varphi)(\xi)}\xi^\sigma(F\varphi)(\xi)\,d\xi - c\|\varphi\|_0^2 -$$
$$- (c_0/2k) \sum_{\sigma\in\Gamma^*} \|D^\sigma\varphi\|_0 \sum_{\tau\in\Gamma^*} \|D^\tau\varphi\|_0$$

with a suitable constant c. Hence we obtain

$$\operatorname{Re} B(\varphi,\varphi) \geqq c_0\|\varphi\|_{\Gamma^*}^2 - (c_0/2k) \sum_{\sigma\in\Gamma^*} \|D^\sigma\varphi\|_0 \sum_{\tau\in\Gamma^*} \|D^\tau\varphi\|_0 -$$
$$- (c_0 + c)\|\varphi\|_0^2\,.$$

By the Cauchy–Schwarz inequality we have

$$\sum_{\sigma\in\Gamma^*} \|D^\sigma\varphi\|_0 \leqq k^{1/2}\|\varphi\|_{\Gamma^*} \quad \text{and} \quad \sum_{\tau\in\Gamma^*} \|D^\tau\varphi\|_0 \leqq k^{1/2}\|\varphi\|_{\Gamma^*}\,,$$

respectively. Hence

$$\operatorname{Re} B(\varphi,\varphi) \geqq c_0\|\varphi\|_{\Gamma^*}^2 - (c_0/2k)\|\varphi\|_{\Gamma^*}k^{1/2}\|\varphi\|_{\Gamma^*}k^{1/2} - (c_0 + c)\|\varphi\|_0^2$$

or

$$\operatorname{Re} B(\varphi,\varphi) \geqq (c_0/2)\|\varphi\|_{\Gamma^*} - (c_0 + c)\|\varphi\|_0^2\,.$$

B. Now we prove the general case. For $\varphi \in C_0^\infty(G)$ we have

$$\operatorname{Re} B(\varphi, \varphi) = \operatorname{Re} \sum_{\sigma, \tau \in \Gamma^*} \int \overline{a_{\sigma\tau}(x) D^\tau \varphi(x)} D^\sigma \varphi(x)\, dx +$$

$$\operatorname{Re} \sum_{(\sigma,\tau) \in (\Gamma \times \Gamma) - (\Gamma^* \times \Gamma^*)} \int_G \overline{a_{\sigma\tau}(x) D^\tau \varphi(x)} D^\sigma \varphi(x)\, dx .$$

With the abbreviation

$$\sum := \sum_{(\sigma,\tau) \in (\Gamma \times \Gamma) - (\Gamma^* \times \Gamma^*)} \int_G |a_{\sigma\tau}(x)| |D^\tau \varphi(x)| |D^\sigma \varphi(x)|\, dx$$

we have

$$\operatorname{Re} B(\varphi, \varphi) \geqq \operatorname{Re} \sum_{\sigma, \tau \in \Gamma^*} \int_G \overline{a_{\sigma\tau}(x) D^\tau \varphi(x)} D^\sigma \varphi(x)\, dx - \sum .$$

Applying part A of our proof we obtain

$$\operatorname{Re} B(\varphi, \varphi) \geqq (c_0/2) \|\varphi\|_{\Gamma^*}^2 - (c_0 + c) \|\varphi\|_0^2 - \sum . \tag{13}$$

We will now estimate $\sum$. By the assumption on the coefficients $a_{\sigma\tau}$ it follows with a suitable constant c_2

$$\sum \leqq c_2 \sum_{(\sigma,\tau) \in (\Gamma \times \Gamma) - (\Gamma^* \times \Gamma^*)} \int_G |D^\tau \varphi(x)| |D^\sigma \varphi(x)|\, dx . \tag{14}$$

As before let $\Gamma' = \Gamma - \Gamma^*$. We split the right hand side of (14) into three sums

$$\sum\nolimits_1 := \sum_{\sigma \in \Gamma^*, \tau \in \Gamma'} \int_G |D^\tau \varphi(x)| |D^\sigma \varphi(x)|\, dx ,$$

$$\sum\nolimits_2 := \sum_{\sigma \in \Gamma', \tau \in \Gamma^*} \int_G |D^\tau \varphi(x)| |D^\sigma \varphi(x)|\, dx ,$$

$$\sum\nolimits_3 := \sum_{\sigma, \tau \in \Gamma'} \int_G |D^\tau \varphi(x)| |D^\sigma \varphi(x)|\, dx .$$

For an arbitrary $\varepsilon > 0$ it follows

$$\sum\nolimits_1 \leqq \sum_{\sigma \in \Gamma^*, \tau \in \Gamma'} (\varepsilon_\sigma \|D^\sigma \varphi\|_0^2 + (1/\varepsilon_\sigma) \|D^\tau \varphi\|_0^2) .$$

For $\tau \in \Gamma'$ there exists $\sigma_\tau \in \Gamma^*$ such that $\tau < \sigma_\tau$ and by Lemma 1 we have for an arbitrary $\eta > 0$ the estimate

$$\|D^\tau \varphi\|_0^2 \leqq \eta \|\varphi\|_{\Gamma^*}^2 + c(\eta) \|\varphi\|_0^2$$

with a suitable constant $c(\eta)$. Hence

$$\sum\nolimits_1 \leqq \sum_{\sigma \in \Gamma^*, \tau \in \Gamma} (\varepsilon_\sigma \|\varphi\|_{\Gamma^*}^2 + (\eta/\varepsilon_\sigma) \|\varphi\|_{\Gamma^*}^2 + (c(\eta)/\varepsilon_\sigma) \|\varphi\|_0^2) .$$

Given c_0 and c_2 we now choose ε_σ and η such that

$$\sum_{\sigma\in\Gamma^*,\tau\in\Gamma'} \varepsilon_\sigma \leqq c_0/16c_2 \quad \text{and} \quad \sum_{\sigma\in\Gamma^*,\tau\in\Gamma'} \eta/\varepsilon_\sigma \leqq c_0/16c_2$$

hold and taking c_3 such that

$$\sum_{\sigma\in\Gamma^*,\tau\in\Gamma'} c(\eta)/\varepsilon_\sigma = c_3/c_2$$

holds we obtain

$$c_2 \sum\nolimits_1 \leqq (c_0/8)\|\varphi\|_{\Gamma^*}^2 + c_3\|\varphi\|_0^2 \,. \tag{15}$$

The estimate for $\sum_2$ is obtained by changing the role of σ and τ. Before treating $\sum_3$ we observe that

$$\sum\nolimits_3 \leqq \sum_{\sigma,\tau\in\Gamma'} (\|D^\tau\varphi\|_0^2 + \|D^\sigma\varphi\|_0^2)\,.$$

Now, since σ and τ do not belong to Γ^*, there exists α_σ and α_τ in Γ^* such that $\sigma < \alpha_\sigma$ and $\tau < \alpha_\tau$. Applying Lemma 1, we get for $\varepsilon > 0$

$$\sum\nolimits_3 \leqq 2\|\varphi\|_{\Gamma^*}^2 \sum_{\sigma,\tau\in\Gamma'} \varepsilon + \|\varphi\|_0^2 \sum_{\sigma,\tau\in\Gamma'} c_\sigma(\varepsilon) + c_\tau(\varepsilon)\,.$$

Taking ε such that $2\sum_{\sigma,\tau\in\Gamma'}\varepsilon \leqq c_0/8c_2$ we obtain

$$c_2 \sum\nolimits_3 \leqq (c_0/8)\|\varphi\|_{\Gamma^*}^2 + c_5\|\varphi\|_0^2 \,. \tag{16}$$

Hence we find

$$\sum \leqq (3c_0/8)\|\varphi\|_{\Gamma^*}^2 + (c_3+c_4+c_5)\|\varphi\|_0^2 \tag{17}$$

and by (13) with $c_0' = c_0/8$

$$\operatorname{Re} B(\varphi,\varphi) \geqq c_0'\|\varphi\|_{\Gamma^*}^2 - c_1\|\varphi\|_0^2 \,.$$

A generalized homogeneous Dirichlet problem.

We now pose

PROBLEM 1. Let G be a bounded open set in R^n, $L(.,D)$ a differential operator of form (1), and $f \in L^2(G)$ a given function. Find all elements $u \in H_0^\Gamma(G)$ such that

$$B(u,\varphi) = (f,\varphi)_0 \tag{18}$$

holds for all $\varphi \in C_0^\infty(G)$, where B is defined by (2).

The solution of Problem 1 is given by

THEOREM 4. *If the differential operator $L(.,D)$ fulfils the assumptions of Theorem 3 and if the imbedding of $H_0^{\Gamma^*}(G)$ into $L^2(G)$ is compact (see Theorem 2), then the Fredholm alternative holds for Problem 1.*

The proof of Theorem 4 follows with Theorem 3 and the obvious continuity of B on $H_0^{\Gamma^*}(G) \times H_0^{\Gamma^*}(G)$, as the Fredholm alternative follows for strongly uniformly elliptic differential operators, see (Theorem 14.6) [3].

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REFERENCES

1. K. Doppel and N. Jacob, *A Hilbert space approach for a differential operator in the theory of multiparameter stochastic processes*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. 8 (1983), 375–389.
2. E. B. Dynkin, *Harmonic functions associated with several Markov-Processes*, Adv. in Appl. Math. 2 (1981), 260–283.
3. A. Friedman, *Partial differential equations*, Holt, Rinehart and Winston, Inc., New York, 1969.
4. N. Jacob, *Bemerkungen über verallgemeinerte homogene lineare Dirichlet-Probleme*, Thesis, Freie Universität Berlin, 1980.
5. J. S. Louhivaara and C. G. Simader, *Über koerzitive lineare partielle Differentialoperatoren: Fredholmsche verallgemeinerte Dirichletprobleme und deren Klasseneinteilung*, in: *Complex analysis and its applications.* (A collection of papers dedicated to I. N. Vekua on his 70th birthday.) Eds. N. N. Bogoljubov, M. A. Lavrent'er, and A. V. Bicadze. Izdat. "Nauka", Moscow, 1978, pp. 342–345.

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