TAKESAKI'S DUALITY FOR
A NON-DEGENERATE CO-ACTION

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Abstract.

Let \( \delta \) be a non-degenerate co-action of a locally compact group \( G \) on a C*-algebra \( A \). We can find an action \( \hat{\delta} \) on a \( \delta \)-crossed product \( A \times_\delta G \) and show that a crossed product \( (A \times_\delta G) \times_\delta G \) is isomorphic to \( A \otimes C(L^2(G)) \) where \( C(L^2(G)) \) is the algebra of all compact operators on \( L^2(G) \).

\( A \) and \( B \) are C*-algebras. We denote by \( M(A) \) the multiplier algebra of \( A \). If \( A \) is a concrete C*-algebra, we may define \( M(A) = \{ a \in A'' ; ab + ca \in A \text{ for } b,c \in A \} \). Following [1] we put

\[
\tilde{M}(A \otimes B) = \{ x \in M(A \otimes B) ; x(1 \otimes b) + (1 \otimes c)x \in A \otimes B \text{ for } b,c \in B \},
\]

where the symbol \( \otimes \) means the spatial tensor product.

Let \( G \) be a locally compact group. \( L^2(G) \) is the Hilbert space of square integrable functions on \( G \) with a left Haar measure \( ds \) on \( G \). The left and right regular representations of \( G \) on \( L^2(G) \) are defined by

\[
(\lambda(s)\xi)(t) = \xi(s^{-1}t)
\]

\[
(\rho(s)\xi)(t) = A^{-1}(s)\xi(ts)
\]

for \( s,t \in G \) and \( \xi \in L^2(G) \), where \( A \) is the modular function of \( G \). Let \( C_r^*(G) \) be the C*-algebra generated by \( \{ \lambda(f) ; f \in L^1(G) \} \) where

\[
\lambda(f) = \int_G f(s)\lambda(s) \, ds,
\]

which is called the reduced group C*-algebra of \( G \). We define a unitary operator \( W \) on \( L^2(G \times G) \) by

\[
(W\xi)(s,t) = \xi(s, st)
\]

for \( \xi \in L^2(G \times G) \) and we set \( \delta_G(x) = W^*(x \otimes 1)W = \text{Ad} \, W^*(x \otimes 1) \) for \( x \in C_r^*(G) \). Then we can show easily that \( \delta_G \) is an isomorphism of \( C_r^*(G) \) into \( \tilde{M}(C_r^*(G) \otimes C_r^*(G)) \). Since \( \delta_G(C_r^*(G))(1 \otimes C_r^*(G)) \) generates \( C_r^*(G) \otimes C_r^*(G) \), for

Received May 31, 1983.
each approximate identity \( \{e_i\} \) of \( C_r^*(G) \), \( \delta_G(e_i) \) converges to 1 in the strict topology of \( M(C_r^*(G) \otimes C_r^*(G)) \).

Let \( \theta \) be a homomorphism of \( A \) into \( M(B) \) satisfying that \( \theta(u_i) \) converges to 1 in the strict topology of \( M(B) \) for each approximate identity \( \{u_i\} \) of \( A \). Then \( \theta \) extends uniquely to a homomorphism (also denoted by \( \theta \)) of \( M(A) \) into \( M(B) \) ([8, Lemme 0.2.6]). The above \( \delta_G \) has a property
\[
(\delta_G \otimes 1)\delta_G = (1 \otimes \delta_G)\delta_G,
\]
where \( 1 \) is the identity map of \( C_r^*(G) \) (the above \( \delta_G \otimes 1 \) and \( 1 \otimes \delta_G \) are homomorphisms on \( M(C_r^*(G) \otimes C_r^*(G)) \)).

**Definition.** Let \( \delta \) be an isomorphism of \( A \) into \( \tilde{M}(A \otimes C_r^*(G)) \). The isomorphism \( \delta \) is called a co-action of \( G \) on \( A \) if for each approximate identity \( \{u_i\} \) of \( A \), \( \delta(u_i) \) converges to 1 in the strict topology of \( M(A \otimes C_r^*(G)) \) and
\[
(\delta \otimes 1)\delta = (1 \otimes \delta_G)\delta.
\]

We define a linear map \( \delta_u \) by \( \delta_u(a) = L_u \delta(a) \) for \( u \in B_r(G) \equiv C_r^*(G)^* \), \( a \in A \) where \( L_u \) is the left slice map of \( u \) (see [2]). Since \( \delta \) is a map into \( \tilde{M}(A \otimes C_r^*(G)) \), by [7, Theorem 2.1] \( \delta_u \) is a linear map of \( A \) into \( A \).

**Lemma 1.** Let \( \delta \) be a co-action of \( G \) on \( A \). For \( x = \delta_u(a) \), \( a \in A \) and \( u \in B_r(G) \cap K(G) \), we have
\[
\int_G \delta_{\phi \lambda(s)^*}(x) \otimes \lambda(s) z \, ds = \delta(x)(1 \otimes \lambda(\phi)z)
\]
for \( \phi \in B_r(G) \cap K(G) \) and \( z \in C_r^*(G) \), where \( \phi(s) = \phi(s^{-1}) \) and \( \langle z, \phi \lambda(s)^* \rangle = \langle \lambda(s)z, \phi \rangle \) (\( K(G) \) is the family of continuous functions on \( G \) with compact supports).

**Proof.** Both functions \( s \in G \to \lambda(s)z \in C_r^*(G) \) and \( s \in G \to \phi \lambda(s)^* \in C_r^*(G)^* \) are norm-continuous. The integrand:
\[
s \in G \to \delta_{\phi \lambda(s)^*}(x) \otimes \lambda(s)z
\]
is continuous in the norm topology of \( A \otimes C_r^*(G) \), whose support is contained in a compact set \( (\text{supp} \, u) \cdot (\text{supp} \, \varphi)^{-1} \). Hence \( \int_G \delta_{\phi \lambda(s)^*}(x) \otimes \lambda(s)z \, ds \) is contained in \( A \otimes C_r^*(G) \). For \( \omega \in A^* \), \( \psi \in B_r(G) \cap K(G) \) and \( z = \lambda(f) \), \( f \in K(G) \), we have
\[
\left\langle \int_G \delta_{\phi \lambda(s)^*}(x) \otimes \lambda(s)z \, ds, \, \omega \otimes \psi \right\rangle
\]
\[
= \int_G \left\langle \delta(x), \omega \otimes \phi \lambda(s)^* \right\rangle \left\langle \lambda(s)z, \psi \right\rangle ds,
\]
since the function

\[ s \in G \to \langle \lambda(s)z, \psi \rangle = \langle \lambda(s)\lambda f, \psi \rangle \]

is continuous whose support is contained in a compact set \((\text{supp } \psi) \cdot (\text{supp } f)^{-1}\),

\[
\langle \delta(x), \omega \otimes \int_G \langle \lambda(s)z, \psi \rangle \varphi \lambda(s)^* ds \rangle.
\]

For \( g \in L^1(G) \), we get

\[
\begin{align*}
\langle \lambda(g), \int_G \langle \lambda(s)z, \psi \rangle \varphi \lambda(s)^* ds \rangle & \\
= & \int_G \langle \lambda(s)z, \psi \rangle \langle \lambda(s)^* \lambda(g), \varphi \rangle ds & \\
= & \int_{G \times G} \langle \lambda(s)z, \psi \rangle \varphi \lambda(t) \langle \lambda(s^{-1} t), \varphi \rangle dt ds & \\
= & \int_{G \times G} \langle \lambda(\varphi h) \lambda(h)z, \psi \rangle \varphi \lambda(h^{-1}), \varphi \rangle dt dh & (s^{-1} t = h^{-1}) & \\
= & \int_G \langle \lambda(g) \lambda(h)z, \psi \rangle \langle \lambda(h^{-1}), \varphi \rangle dh & \\
= & \int_G \varphi(h) \langle \lambda(g) \lambda(h)z, \psi \rangle dh = \langle \lambda(g) \lambda(\varphi)z, \psi \rangle & \\
= & \langle \lambda(g), \lambda(\varphi)z \psi \rangle.
\end{align*}
\]

Therefore we have \( \int_G \langle \lambda(s)z, \psi \rangle \varphi \lambda(s)^* ds = \lambda(\varphi)z \psi \). Hence

\[
(2) = \langle \delta(x), \omega \otimes \lambda(\varphi)z \psi \rangle = \langle \delta(x)(1 \otimes \lambda(\varphi)z), \omega \otimes \psi \rangle.
\]

Since \( B(G) \cap K(G) \) is dense in the Fourier algebra \( A(G) \) (see [2]), we obtain

\[
(3) \int_G \delta_{\varphi \lambda(s)^* \varphi(s)} (x) \otimes \lambda(s)z \ ds = \delta(x)(1 \otimes \lambda(\varphi)z)
\]

for \( z = \lambda(f), f \in K(G) \). Both sides in (3) are continuous with respect to \( z \). Then we have the equation (3) for all \( z \in C_r^*(G) \).

**Lemma 2.** Let \( \delta \) be as above. The closure \( I(A) \) of \( \{ \delta_{\varphi}(a); a \in A, \varphi \in A(G) \} \) is a C*-subalgebra of \( A \). Moreover for \( x \in I(A) \) and \( z \in C_r^*(G) \) the element \( \delta(x)(1 \otimes z) \) is contained in \( I(A) \otimes C_r^*(G) \).
PROOF. Since $K(G) \cap A(G)$ is norm-dense in $A(G)$ and $\| \phi \| \leq \| \delta \|$ for $\phi \in B_\tau(G)$, $I(A)$ is the closure of $\{ \delta_\phi(a); a \in A, \phi \in K(G) \cap A(G) \}$. Since $A(G)$ is a regular ring (see [2]), we can find, for $\phi_1, \phi_2 \in K(G) \cap A(G)$, $\phi_3$ in $K(G) \cap A(G)$ with $\phi_3 \equiv 1$ on a neighbourhood of $(\text{supp } \phi_1) \cdot (\text{supp } \phi_2)$ and $\text{supp } \phi_1 \cup \text{supp } \phi_2$. Then we obtain

$$
\delta_{\phi_3}(\delta_{\phi_1}(x) \delta_{\phi_2}(y)) = \delta_{\phi_1}(x) \delta_{\phi_2}(y)
$$

$$
\delta_{\phi_3}(\delta_{\phi_1}(x) + \delta_{\phi_2}(y)) = \delta_{\phi_1}(x) + \delta_{\phi_2}(y)
$$

for all $z, y \in A$. Therefore $I(A)$ is a C*-subalgebra of $A$. When we choose an approximate identity $\{ \phi_i \}$ of $L^1(G)$ in the set $K(G) \cap B_\tau(G) = K(G) \cap A(G)$, by the equation (1) we have

$$
\lim_i \int_G \delta_{\phi_1}(x) \otimes \lambda(s) ds = \lim_i \delta(x)(1 \otimes \lambda(\phi_i))z = \delta(x)(1 \otimes z),
$$

that is $\delta(x)(1 \otimes z)$ is contained in $I(A) \otimes C_\tau^*(G)$ for $x = \delta_u(y)$, some $y \in A$ and $u \in K(G) \cap A(G)$. Therefore $\delta(x)(1 \otimes z)$ is contained in $I(A) \otimes C_\tau^*(G)$ for $x \in I(A)$.

**Lemma 3.** Let $\delta$ be as above. The closed subspace $[\delta(I(A))(1 \otimes C_\tau^*(G))]$ generated by $\delta(I(A))(1 \otimes C_\tau^*(G))$ contains $I(A) \otimes C_\tau^*(G)$.

PROOF. Take $x = \delta_u(y)$, $(y \in A, u \in A(G) \cap K(G))$, and by (1) we have, for $\phi \in A(G) \cap K(G)$,

$$
\delta(x)(1 \otimes \lambda(\phi)) = \int_G \delta_{\phi\lambda(s)}(x) \otimes \lambda(s) ds \quad \text{(in the strict topology)}.
$$

For $v \in A(G) \cap K(G)$, we obtain

$$
i \otimes L_v[(i \otimes \delta_G)(\delta(x)(1 \otimes \lambda(\phi)))]
= i \otimes L_v \left( \int_G \delta_{\phi\lambda(s)}(x) \otimes \lambda(s) \otimes \lambda(s) ds \right)
= \int_G v(s) \delta_{\phi\lambda(s)}(x) \otimes \lambda(s) ds.
$$

On the other hand, we get, for $\omega_1 \in A^*$, $\omega_2 \in C_\tau^*(G)^*$

$$
\langle i \otimes L_v[(i \otimes \delta_G)(\delta(x)(1 \otimes \lambda(\phi)))], \omega_1 \otimes \omega_2 \rangle
= \langle ((i \otimes \delta_G)\delta(x))(1 \otimes \delta_G(\lambda(\phi))), \omega_1 \otimes \omega_2 \otimes v \rangle
= \langle (\delta \otimes i)\delta(x), \omega_1 \otimes \left( \int_G \phi(s)\lambda(s) \otimes \lambda(s) ds \right)(\omega_2 \otimes v) \rangle.
$$
\[ \int_G \tilde{\phi}(s) \langle (\delta \otimes t)\delta(x), \omega_1 \otimes \lambda(s)\omega_2 \otimes \lambda(s)v \rangle \, ds = \int_G \tilde{\phi}(s) \langle \delta_{\lambda(s)v}(x)(1 \otimes \lambda(s)), \omega_1 \otimes \omega_2 \rangle \, ds = \left\langle \int_G \tilde{\phi}(s)\delta_{\lambda(s)v}(x)(1 \otimes \lambda(s)) \, ds, \omega_1 \otimes \omega_2 \right\rangle. \]

Therefore
\[ (4) \]
\[ \int G \tilde{\phi}(s)\delta_{\lambda(s)v}(x)(1 \otimes \lambda(s)) \, ds = \int G v(s)\delta_{\varphi\lambda(s)v}(x) \otimes \lambda(s) \, ds. \]

For \( z \in C_r^*(G) \) we obtain,
\[ \int G \tilde{\phi}(s)\delta_{\lambda(s)v}(x)(1 \otimes \lambda(s)z) \, ds = \int G v(s)\delta_{\varphi\lambda(s)v}(x) \otimes \lambda(s)z \, ds. \]

Since the integrands in the above equation are norm-continuous,
\[ \lim_{v} \int G \tilde{\phi}(s)\delta_{\lambda(s)v}(x)(1 \otimes \lambda(s)z) \, ds = \delta_{\varphi}(x) \otimes z \]

in the norm topology, when the measure \( v(s)ds \) tends to a Dirac measure at the identity of \( G \). Then \([\delta(I(A))(1 \otimes C_r^*(G))] \) contains \( I(A) \otimes C_r^*(G) \).

**Remark.** The restriction \( \delta|_{I(A)} \) of \( \delta \) to \( I(A) \) is a co-action of \( G \) on \( I(A) \).

**Lemma 4.** Let \( \delta \) be as above. The closed linear span \([\delta(A)(1 \otimes C_r^*(G))] \) is coincided with \( I(A) \otimes C_r^*(G) \).

**Proof.** Without the condition \( x = \delta_y(y) \) in the proof of the former equality in (4), we have, for \( v, \varphi \in A(G) \cap K(G), \, x \in A, \)
\[ (5) \]
\[ \int G \tilde{\phi}(s)\delta_{\lambda(s)v}(x)(1 \otimes \lambda(s)\varphi) \, ds = \int G v(s)\delta_{\varphi\lambda(s)v}(x)(1 \otimes \lambda(s)) \, ds. \]

Since \( A \otimes C_r^*(G) \) contains \( \delta(x)(1 \otimes \lambda(\tilde{\phi})) \), the norm closure of \( \{ (1 \otimes (\delta_G)_v)(\delta(x)(1 \otimes \lambda(\tilde{\phi}))) ; v \in A(G) \cap K(G) \} \) contains \( \delta(x)(1 \otimes \lambda(\tilde{\phi})) \). The norm closure of
\[ \left\{ \int G \tilde{\phi}(s)\delta_{\lambda(s)v}(x)(1 \otimes \lambda(s)z) \, ds ; \, v \in A(G) \cap K(G) \right\} \]
contains $\delta(x)(1 \otimes \lambda(\phi)z),$ $(z \in C^*_\omega(G)).$ Since the element $\delta_{\lambda(s)v}(x)$ is in $I(A),$ Lemma 2 implies that $\int_G \Phi(s)\delta(\lambda(s)v(x))(1 \otimes \lambda(s)z)ds$ is in $I(A) \otimes C^*_\omega(G),$ that is

$$\delta(x)(1 \otimes \lambda(\phi)z) \in I(A) \otimes C^*_\omega(G).$$

By taking $\varphi$ as an approximate identity of $L^1(G),$ we get $\delta(x)(1 \otimes z) \in I(A) \otimes C^*_\omega(G)$ for $x \in A$ and $z \in C^*_\omega(G).$ By Lemma 3, we have $[\delta(A)(1 \otimes C^*_\omega(G))] = I(A) \otimes C^*_\omega(G).$

**Theorem 5.** Let $\delta$ be as above. The following statements are equivalent.

(i) $A = I(A),$

(ii) $[\delta(A)(1 \otimes C^*_\omega(G))] = A \otimes C^*_\omega(G),$

(iii) $[\delta(A)(1 \otimes C(L^2(G))] = A \otimes C(L^2(G)),$

(iv) $\delta$ is non-degenerate in Landstad's sense, i.e. for each non-zero linear functional $\omega$ in $A^*,$ we can find $u \in B_r(G)$ with $(\omega \otimes u)\delta \neq 0.$

**Proof.** The equivalence of (i) and (ii) follows from Lemma 4. Since $C^*_\omega(G) \cdot C_0(G)$ generates $C(L^2(G)),$ we have (ii) $\Rightarrow$ (iii). We shall prove (iii) $\Rightarrow$ (i). We define a rank one operator $\xi \otimes \eta^c$ with $(\xi \otimes \eta)(\zeta) = \langle \xi, \eta \rangle \zeta$ for $\xi, \eta$ and $\zeta \in L^2(G).$ For $\xi_i, \eta_i$ $(i = 1, 2),$ $\xi, \eta \in L^2(G)$ and elements $p, q$ of the universal Hilbert space for $A,$ we have

$$\langle \{1 \otimes (\xi_1 \otimes \eta_1^c)\} \delta(a)\{1 \otimes (\xi_2 \otimes \eta_2^c)\}p \otimes \xi, q \otimes \eta \rangle$$

$$= \langle \delta(a)(p \otimes \langle \xi_1, \eta_1 \rangle \xi_2), q \otimes \langle \eta_1, \xi_1 \rangle \eta_1 \rangle$$

$$= \langle \delta(a)(p \otimes \xi_2), q \otimes \langle \eta_1, \xi_1 \rangle \langle \xi_1, \eta_2 \rangle \eta_1 \rangle$$

$$= \langle \delta_{\omega_{\xi_2, \eta_1}}(ap)q \langle \langle \xi_1, \eta_2 \rangle \xi_1, \eta \rangle$$

where

$$\omega_{\xi_2, \eta_1}(z) = \langle z \xi_2, \eta_1 \rangle$$

$$= \langle \delta_{\omega_{\xi_2, \eta_1}}(a) \otimes (\xi_1 \otimes \eta_1^c)(p \otimes \xi), q \otimes \eta \rangle.$$

Then

$$[1 \otimes (\xi_1 \otimes \eta_1^c)]\delta(a)[1 \otimes (\xi_2 \otimes \eta_2^c)] = \delta_{\omega_{\xi_2, \eta_1}}(a) \otimes (\xi_1 \otimes \eta_1^c).$$

Since the family of finite rank operators on $L^2(G)$ generates $C(L^2(G)),$ $\delta(A)(1 \otimes C(L^2(G))$ is contained in $I(A) \otimes C(L^2(G)).$ Then by (iii), we have $A \otimes C(L^2(G)) = I(A) \otimes C(L^2(G)),$ which implies $A = I(A).$ If (ii) holds, for non zero functionals $\omega$ in $A^*$ and $u$ in $B_r(G),$ we can find $a$ in $A$ with $(\omega \otimes au)\delta \neq 0.$ Suppose that $I(A)$ is a proper $C^*$-subalgebra of $A.$ We can find a non zero linear functional $\omega$ in $A^*$ with $\omega(I(A)) = 0.$ Then it follows from Lemma 4 and
[7, Theorem 2.1] that \((\omega \otimes u)\delta = 0\) for all \(u \in B_r(G)\), which is a contradiction with non-degeneracy of \(\delta\).

It is found in [4, Lemma 3.8] that a co-action \(\delta\) of a discrete or amenable group \(G\) is automatically non-degenerate. Also a canonical co-action on a reduced crossed product for a \(C^*\)-dynamical system is automatically non-degenerate. The author has been unable to prove the automatic non-degeneracy of \(\delta\) for arbitrary locally compact group. For convenience of readers, we prove the automatic non-degeneracy for a discrete or amenable group. We prove the condition (i) in Theorem 5 in a slight different way.

**Proposition 6 ([4]).** Let \(G\) be a discrete or amenable group. A co-action \(\delta\) of \(G\) on \(A\) is automatically non-degenerate.

**Proof.** By [7, Theorem 2.1], for \(u \in B_r(G)\), we find \(a\) in \(A\) and \(v \in B_r(G)\) with \(u = av\). Then we have
\[
\delta_u(g) = \delta_{uv}(x) = L_v(\delta(x)(1 \otimes a))
\]
in \(I(A)\) by Lemma 4. We have
\[
\left\{ \begin{array}{l}
\delta(\delta_u(x)) = \delta L_u(\delta(x)) = (1 \otimes L_u)(\delta \otimes 1)(\delta(x)) \\
= (\delta \otimes L_u)(1 \otimes \delta)(\delta(x))) = (1 \otimes (\delta \otimes L_u)(\delta(x)))
\end{array} \right.
\] (5)

Suppose that \(G\) is amenable, we take the identity \(u_1(s) \equiv 1\) in \(B_r(G) = B(G)\). Since \(1 \otimes (\delta \otimes L_u)(\delta(x)) = \delta(x)\) for \(x \in A\), which implies \(x = \delta_u(x) \in I(A)\) by the injectivity of \(\delta\). Suppose that \(G\) is discrete. Then \(\delta(x)\) is contained in \(A \otimes C^*_r(G)\). Therefore it is easy to prove that the closure of \(\{\delta \otimes L_u(\delta(x)) ; u \in B_r(G)\}\) contains \(\delta(x)\). By (5), \(\delta(I(A))\) contains \(\delta(x)\) for \(x \in A\), that is \(x \in I(A)\). In both cases we have \(A = I(A)\).

Let \(C^*(G)\) be the enveloping \(C^*\)-algebra of \(L^1(G)\), and \(U\) be the universal representation of \(G\). We can define an isomorphism \(\overline{\delta}_G\) of \(C^*(G)\) into \(\overline{\delta}_G(C^*(G) \otimes C^*(G))\) such that
\[
\overline{\delta}_G(U(f)) = \overline{\delta}_G\left( \int_G f(s)U(s)\,ds \right)
\]
\[
= \int_G f(s)U(s) \otimes U(s)\,ds
\]
for \(f \in L^1(G)\). Moreover \((1 \otimes \overline{\delta}_G)\overline{\delta}_G = (\overline{\delta}_G \otimes 1)\overline{\delta}_G\) and
\[
[\overline{\delta}_G(C^*(G))(1 \otimes C^*(G))] = C^*(G) \otimes C^*(G)
\]
(see [3, Theorem 3.9]).
Let $\delta$ be an injective homomorphism of $A$ into $\tilde{M}(A \otimes C^*(G))$ and $\delta(e_n)$ converges 1 in the strict topology of $A \otimes C^*(G)$ for each approximate identity $\{e_n\}$ of $A$ and $(\delta \otimes 1) \delta = (1 \otimes \delta_G) \delta$. Let $\pi$ be a canonical homomorphism of $C^*(G)$ onto $C^*_r(G)$. Note that $\delta$ automatically satisfies the statements in Theorem 5 by the same proof as in the case of an amenable group $G$ (Proposition 6). Set

$$
\delta^1(x) = (\iota \otimes \pi) \delta(x) \quad \text{for } x \in A.
$$

Since $\delta^1$ is not in general injective, set

$$
I = \ker \delta^1 \quad \text{and} \quad \delta'(\theta(x)) = (\theta \otimes 1) \delta^1(x) \quad \text{for } x \in A,
$$

where $\theta$ is a canonical homomorphism of $A$ onto $A/I$.

**Proposition 7.** The map $\delta'$ is a non-degenerate co-action of $G$ on $A/I$.

**Proof.** For $f \in L^1(G)$ and $x \in A$, we have

$$
\delta'(\theta(x))(1 \otimes \lambda(f)) = (\theta \otimes 1) \delta^1(x)(1 \otimes \lambda(f))
$$

$$
= [(\theta \otimes 1)(i \otimes \pi) \delta(x)](1 \otimes \lambda(f)) = \theta \otimes \pi(\delta(x)(1 \otimes U(f))),
$$

because of $\pi(U(f)) = \lambda(f)$. Then $\delta'(\theta(x))(1 \otimes z)$ is contained in $A/I \otimes C^*_r(G)$ for $x \in A$ and $z \in C^*_r(G)$. Suppose $\delta'(\theta(x)) = 0$ ($x \in A$). Then for $\omega \in C^*_r(G)^*$, we have

$$
0 = L_\omega(\delta'(\theta(x))) = L_\omega((\theta \otimes 1) \delta^1(x))
$$

$$
= \theta(L_\omega \delta^1(x)) = \theta(\delta^1_{\omega}(x)).
$$

Therefore $\delta^1_{\omega}(x)$ is contained in $I$. Since

$$(1 \otimes L_\omega)(\delta^1 \otimes 1) \delta^1(x) = \delta^1(\delta^1_{\omega}(x)) = 0 \quad \text{for } \omega \in C^*_r(G)^*,
$$

we have $(\delta^1 \otimes 1) \delta^1(x) = 0$. Since

$$
(\delta^1 \otimes 1) \delta^1 = [(i \otimes \pi \otimes 1)(\delta \otimes 1)][(i \otimes \pi) \delta]
$$

$$
= (i \otimes \pi \otimes \pi)(\delta \otimes 1) \delta = (i \otimes \pi \otimes \pi)(i \otimes \delta_G) \delta
$$

$$
= (i \otimes \delta_G)(i \otimes \pi) \delta = (i \otimes \delta_G) \delta^1
$$

because of $(\pi \otimes \pi) \delta_G = \delta_G \circ \pi$, then we obtain $(i \otimes \delta_G) \delta^1(x) = 0$. Since $(i \otimes \delta_G)$ is an isomorphism of $\tilde{M}(A \otimes C^*_r(G))$ (see [1, Proposition 2.4]), we get $\delta^1(x) = 0$, that is $\delta'$ is an isomorphism of $A/I$. We have, on $A$,

$$
(\delta' \otimes 1) \delta'(\theta(x)) = [((\theta \otimes 1) \delta^1) \otimes 1][(\theta \otimes 1) \delta^1(x)]
$$

$$
= \{(\theta \otimes 1) \delta^1 \theta\} \otimes 1 \delta^1(x) = \{(\theta \otimes 1) \delta^1 \otimes 1\} \delta^1(x)
$$

$$
= \{(\theta \otimes 1) \delta^1 \otimes 1\} \delta^1(x)
$$

$$
= \{(\theta \otimes 1) \delta^1 \otimes 1\} \delta^1(x)
$$
\begin{align*}
= (\theta \otimes 1 \otimes 1)(\delta^1 \otimes 1)\delta^1(x) &= (\theta \otimes 1 \otimes 1)(1 \otimes \delta_G)\delta^1(x) \\
= (1 \otimes \delta_G)(\theta \otimes 1)\delta^1(x) &= (1 \otimes \delta_G)\delta'(\theta(x)) .
\end{align*}

Since \( \delta(e_n) \) converges to 1 in the strict topology of \( M(A \otimes C^*(G)) \) for each approximate identity \( \{ e_n \} \) of \( A \), it follows from [8, Lemme 0.2.6] that \( \delta' \) has the same property for \( A/I \). Then we have proved that \( \delta' \) is a co-action of \( G \) on \( A/I \).

Also \( \delta'_u(\theta(x)) = \theta(\delta_u(x)) \) for \( x \in A \) and \( u \in A(G) \) and by the same proof as in the case of an amenable group \( G \) (Proposition 6), \( A \) is generated by \( \{ \delta_u(x); \ u \in A(G), x \in A \} \). Therefore \( \{ \delta'_u(x); u \in A(G), x \in A/I \} \) generated \( A/I \), that is \( \delta' \) is non-degenerate.

The isomorphism \( \delta \) of \( A \) into \( \tilde{M}(A \otimes C^*(G)) \) (respectively \( \tilde{M}(A \otimes C^*(G)) \)) satisfying \( (\delta \otimes 1)\delta = (1 \otimes \overline{\delta})\delta \) (respectively \( (\delta \otimes 1)\delta = (1 \otimes \delta)\delta \)) is related with crossed product (respectively reduced crossed product).

Before we prove Takesaki's duality for a co-action, we need some notations and definitions. And we note that the discussion which we make below is the same which Landstad [5], Nakagami and Takesaki [6] and Van Heeswijk [9] do.

Let \( \delta \) be a co-action of \( G \) on \( A \) and let \( C_0(G) \) be the family of continuous functions on \( G \) vanishing at infinity. The crossed product \( A \times_{\delta} G \) by \( \delta \) is the C*-algebra generated by \( \delta(A)(1 \otimes C_0(G)) \) in the full operator algebra \( B(L^2(G, H)) \) (\( H \) is the universal Hilbert space for \( A \) and \( C_0(G) \) acts as multiplication on \( L^2(G) \)). Let \( V \) be a unitary operator on \( L^2(G \times G, H) \) satisfying
\[
(V \xi)(s, t) = A(t)^{1/2} \xi(st^{-1}, t)
\]
for \( \xi \in L^2(G \times G, H) \) and \( A \) is the modular function of \( G \). Set a dual action \( \hat{\delta} \) of \( G \),
\[
\hat{\delta}(x) = V(x \otimes 1)V^*
\]
for \( x \in A \times_{\delta} G \). Then \( \hat{\delta}(\delta(x)) = \delta(x) \otimes 1 \) (\( x \in A \)) and \( \hat{\delta}(1 \otimes f) = 1 \otimes \alpha_G(f) \) (\( f \in C_0(G) \)), where
\[
\alpha_G(f)(s, t) = f(st^{-1}) .
\]
Therefore \( \hat{\delta} \) is an isomorphism of \( A \times_{\delta} G \) into \( \tilde{M}(A \times_{\delta} G \otimes C_0(G)) \) such that \( \hat{\delta}(e_n) \) converges to 1 in the strict topology of \( M(A \otimes C_0(G)) \) for each approximate identity \( \{ e_n \} \) of \( A \times_{\delta} G \) and \( (\delta \otimes 1)\hat{\delta} = (1 \otimes \alpha_G)\delta \). The crossed product \( (A \times_{\delta} G) \times_{\delta} G \) by the action \( \delta \) is the C*-algebra generated by \( \delta(A \times_{\delta} G)(1 \otimes 1 \otimes C^*(G)) \). Set a co-action \( \hat{\delta} \) of \( G \) on \( (A \times_{\delta} G) \times_{\delta} G \),
\[
\hat{\delta}(x) = (1 \otimes 1 \otimes W^*)(x \otimes 1)(1 \otimes 1 \otimes W)
\]
for \( x \in (A \times_{\delta} G) \times_{\delta} G \). Then \( \hat{\delta} \) is easily proved to be a non-degenerate co-action of \( G \).
THEOREM 8. Let \( \delta \) be a non-degenerate co-action of \( G \) on \( A \). The \( \mathbb{C}^* \)-algebra \( (A \times_\delta G) \times_\delta G \) is isomorphic to \( A \otimes C(L^2(G)) \), moreover its isomorphism transfers \( \delta \) to \( \delta^\sim \), where

\[
\delta^\sim(x) = (1 \otimes W)[(1 \otimes \sigma)(\delta \otimes 1)(x)]1 \otimes W^*
\]

and \( \sigma \) is a flip map of \( C^*_r(G) \otimes C(L^2(G)) \) onto \( C(L^2(G)) \otimes C^*_r(G) \).

PROOF. Let \( D \) be the \( \mathbb{C}^* \)-algebra generated by

\[
S(1 \otimes W)(1 \otimes W^*)(\delta(A) \otimes 1)(1 \otimes W)(1 \otimes 1 \otimes C(L^2(G)))(1 \otimes W^*)S^*,
\]

where \( S \) is a unitary operator defined by

\[
(S\xi)(s,t) = A(t)^{-\frac{1}{2}}\xi(s, t^{-1}) \quad (\xi \in L^2(G \times G, \mathcal{M})).
\]

Then

\[
(1 \otimes W^*)(\delta(A) \otimes 1)(1 \otimes W)(1 \otimes 1 \otimes C(L^2(G)))
\]

\[
= (1 \otimes \delta g)\delta(A)(1 \otimes 1 \otimes C(L^2(G)))
\]

\[
= (\delta \otimes 1)\delta(A)(1 \otimes 1 \otimes C(L^2(G)))
\]

\[
= (\delta \otimes 1)(\delta(A)(1 \otimes C(L^2(G))).
\]

Since \( \delta \) is non-degenerate, by Theorem 5 (iii), \( \delta(A)(1 \otimes C(L^2(G))) \) generates \( A \otimes C(L^2(G)) \). Then \( D \) is isomorphic to \( A \otimes C(L^2(G)) \). Therefore we have only to prove that \( D \) coincides \( (A \times_\delta G) \times_\delta G \). We prove easily the following facts:

\[
\begin{align*}
S(\delta(a) \otimes 1)S^* &= \delta(a) \otimes 1 & (a \in A) \\
S(1 \otimes W)(1 \otimes 1 \otimes \nu(g))(1 \otimes W^*)S^* &= 1 \otimes 1 \otimes \lambda(g) & (g \in L^1(G))
\end{align*}
\]

(6)

where \( \nu \) is the right regular representation of \( G \)

\[
S(1 \otimes W)(1 \otimes 1 \otimes f)(1 \otimes W^*)S^* = 1 \otimes \alpha_C(f) & (f \in C_0(G))
\]

\[
C(L^2(G)) \text{ is generated by } \{ f \cdot \nu(g); f \in C_0(G), g \in L^1(G) \}.
\]

By extending \( \delta \) and \( \delta^\sim \) to their multipliers, we have

\[
\begin{align*}
\tilde{\delta}(\delta(a)) &= \delta(a) \otimes 1 & (a \in A) \\
\tilde{\delta}(1 \otimes f) &= 1 \otimes f \otimes 1 & (f \in C_0(G)) \\
\tilde{\delta}(\nu(g)) &= \int_G g(s)\nu(s) \otimes \nu(s) \, ds & (g \in L^1(G))
\end{align*}
\]

and
\[ (8) \begin{align*}
\hat{\delta}(\delta(a) \otimes 1) &= \delta(a) \otimes 1 \otimes 1 & (a \in A) \\
\hat{\delta}(1 \otimes \varepsilon_G(f)) &= 1 \otimes \varepsilon_G(f) \otimes 1 & (f \in C_0(G)) \\
\hat{\delta}(1 \otimes 1 \otimes \lambda(g)) &= \int_G g(s)(1 \otimes 1 \otimes \lambda(s) \otimes \lambda(s)) \, ds & (g \in L^1(G)).
\end{align*} \]

By (6), \( D \) is isomorphic to \((A \times_\delta G) \times_\delta G\). By (7), (8) its isomorphism transfers \( \hat{\delta} \) to \( \hat{\delta} \).

When \( G \) is a discrete or amenable group, Takesaki's duality by co-action of \( G \) holds true without non-degeneracy of \( \delta \). If \( G \) is compact, Landstad has already solved it in [5, Theorem 3].

REFERENCES

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