A NOTE ON CONTINUITY OF
PSEUDODIFFERENTIAL OPERATORS
IN HARDY SPACES

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The following sharp result concerning $L_p$-bounds for pseudo differential operators was proven by C. Fefferman in [3]: If $T$ is a pseudo differential operator of class $L_{q, \delta}^{-m}$ where $0 \leq \delta < q \leq 1$ and $m \geq (1-q)n/2 - n/p$ then

\begin{equation}
T: L_p \to L_p, \quad 1 < p < \infty.
\end{equation}

Moreover, if $m \geq (1-q)n/2$ then

\begin{equation}
T: H_1 \to L_1.
\end{equation}

Here $L_p$ is the Lebesgue space in $\mathbb{R}^n$ and $H_1$ is the Hardy space in the sense of Fefferman and Stein (cf. [5]).

There arises the question if it is possible to extend this result to the case $0 < p < 1$ and further whether it is possible for $p = 1$ to have the same space of both sides. There is indeed a natural candidate for such a generalization, using the local or non-homogeneous Hardy spaces $h_p$ (cf. [6], [10] or [12] p. 124).

We have not been able to prove this but only the following weaker result.

**Theorem.** Let $m \in \mathbb{R}, 0 \leq \delta < q \leq 1$ and $T \in L_{q, \delta}^m$. Then for all $0 < p, q, r < \infty$, $s \in \mathbb{R}$ and $s_1 < s + m - (1-q)n|1/p - 1/2|$ it holds

\begin{equation}
T: F_{pq}^s \to F_{pr}^{s_1}.
\end{equation}

For the definition and properties of Triebel spaces $F_{pq}^s$ we refer to [9] or [12]. Note that $h_p = F_{p2}^0$ for $0 < p < \infty$.

**Remark 1.** For $s_1 > s + m - (1-q)n|1/p - 1/2|$ the claim is clearly false.

**Remark 2.** We recall that a symbol $r(x, \xi)$ is said to be in class $S_{q, \delta}^m, 0 \leq \delta \leq 1$, if

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\[ |D_x^\alpha \partial_x^\beta r(x, \xi)| \leq C_{\alpha \beta} (1 + |\xi|)^{m + |\beta| - \varrho |\alpha|} \]

holds for any multi-indexes \( \alpha \) and \( \beta \) and for each pair \( (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \). If \( r(x, \xi) \in S_{\omega, \delta}^m \) we say that the corresponding pseudodifferential operator \( r(x, D) \) belongs to the class \( L_{\omega, \delta}^m \).

**Proof of Theorem.** If suffices to show that

\[ T: F_{pq}^s \rightarrow F_{pq}^{s_1} \quad \text{for } s_1 < s + m - (1 - \varrho)(n/p + \text{const}) \]

In fact, if we combine this with Hörmander's \( L_2 \) estimate (cf. [7])

\[ T: F_{22}^s \rightarrow F_{22}^s, \quad T \in L_{\omega, \delta}^0 \]

the desired result follows by non-trivial interpolation (cf. [4] or [12 p. 73]). Thus for (1) it is sufficient to prove the following lemma.

**Lemma.** Let \( T \in L_{\omega, \delta}^{-\infty}, 0 \leq \delta < \varrho \leq 1 \) and \( m \geq (1 - \varrho)(n/\min (p, q) + n + 1) \). Then for all \( 0 < p, q < \infty \) and \( s \in \mathbb{R} \)

\[ T: F_{pq}^s \rightarrow F_{pq}^s . \]

**Proof.** For simplicity we suppose that \( s = 0 \). We write \( r(x, D) \) for \( T \). Let \( (\varphi_k) \) be the sequence of test functions as in the standard definition of \( F_{pq}^s \) (cf. [8]). What we should do is to estimate the norm

\[ \| (\varphi_j(D)r(x, D)f(x)) \|_{L_p^\infty}^{j=0} \|_{L_p^\infty} \]

by the norm \( \| (\varphi_j(D)f) \|_{L_p^\infty}^{j=0} \|_{L_p^\infty} \). Thus the main task will be to commute the operators \( \varphi_j(D) \) and \( r(x, D) \). We shall do this in the well-known manner by first invoking the Leibniz rule (cf. [11 p. 46]).

\[ \varphi_j(D)r(x, D) \sim \sum_{\beta \geq 0} \frac{1}{\beta!} r_{(\beta)}(x, D) \varphi_j^{(\beta)}(D) . \]

Here we have used the notation \( p_{(\beta)}^{(\varrho)}(x, \xi) = D_x^\beta (iD_\xi)^\varrho p(x, \xi) \).

Next we choose another sequence of test functions \( (\psi_j) \) with \( \psi_j(D) \varphi_j(D) = \varphi_j(D) \) valid for all \( j \). Moreover we suppose that \( \psi_j(\xi) \) is supported in a set where \( |\xi| \sim 2^j \). To estimate \( r_{(\beta)}(x, D) \varphi^{(\beta)}(D)f(x) \) we write it in the integral form

\[ r_{(\beta)}(x, D) \varphi^{(\beta)}(D)f(x) = \int K_\beta^j(x, y) f_j(y) dy \]

where \( f_j = \psi_j(D)f \) and

\[ K_\beta^j(x, y) = \int e^{-i(x-y)\xi} r_{(\beta)}(x, \xi) \varphi_j^{(\beta)}(\xi) d\xi . \]
For the kernel \( K^j_\beta(x, y) \) we can get the following estimate

\[
|K^j_\beta(x, y)| \leq C_\lambda \frac{2^{jn}}{(1 + 2|x-y|)^\lambda}, \quad \text{for } \lambda \leq [m]/(1 - q).
\]

Namely, by partial integration one obtains for each \( \alpha, |x| \leq [m]/(1 - q) \)

\[
|(x - y)^\alpha K^j_\beta(x, y)| \leq C_{2\beta} \sum_{\gamma \leq \alpha} 2^{jn} (1 + 2^j)^{-m+(1-\alpha)} 2^{-j|\alpha|} \leq C_{2\beta} 2^{jn} 2^{-j|\alpha|}.
\]

By using \( \lambda > (n/\min (p, q)) + n \) it follows from (2) and (3) that

\[
|r_{(\beta)}(x, D)\varphi_j^{(\beta)}(D)f(x)| \leq C f^*_j(\mu, x)
\]

with \( \mu > n/\min (p, q) \). Here \( f^*_j(\mu, x) \) is the Fefferman-Stein maximal function defined by

\[
f^*_j(\mu, x) = \sup_{y \in \mathbb{R}^n} \frac{|\varphi_k(D)f(y)|}{(1 + 2^k|x-y|)^\mu}.
\]

Hence, if we write

\[
\varphi_j(D)r(x, D)f(x) := \sum_{|\beta| < N} \frac{1}{\beta!} r_{(\beta)}(x, D)\varphi_j^{(\beta)}(D)f(x) + R^N_j f(x) := g^0_j(x) + g^1_j(x)
\]

we obtain from the Fefferman-Stein-Peetre inequality (cf. [5], [9] or [12, p. 47]) that

\[
\| (g^0_j(x))_{j=0}^\infty \|_{L_p(\mu)} \leq C_N \| f \|_{F^p_{\omega}}.
\]

It remains to give a similar estimate for the remainder \( R^N_j f(x) \). In order to do that we write \( R^N_j f \) in the form

\[
R^N_j f(x) = \int e^{ix(t + \xi)} \hat{f}(\xi)p^N_j(\eta, \xi) d\eta d\xi
\]

where

\[
p^N_j(\eta, \xi) = \hat{r}(\eta, \xi) \left( \varphi_j(\eta + \xi) - \sum_{|\beta| < N} \frac{1}{\beta!} \varphi_j^{(\beta)}(\xi)\eta^\beta \right)
\]

and \( \hat{r}(\eta, \xi) \) is the Fourier transform of \( r(x, \xi) \) with respect to \( x \). By using Lagrange's remainder term in Taylor's formula and by taking \( N \) large enough one can prove that

\[
\| (g^1_j)_{j=0}^\infty \|_{L_p(\mu)} \leq C \| f \|_{F^p_{\omega}}.
\]

For the details cf. [8].
Remarks. The use of interpolation yields the corresponding result for Besov spaces.

Finally, we ask whether Fefferman's theorem remains true if $0 \leq \delta < q \leq 1$ is replaced by $0 \leq \delta = q < 1$ or more generally whether the following result holds: Supposing $0 < p < \infty$, $0 \leq q < 1$ and $T \in L_{q,e}^{(1-e)\frac{1}{p-1/2}}$ we have

$$T: h_p \rightarrow h_p.$$  

For $p = 2$ this is true according to theorem of Calderón and Vaillancourt (cf. [1]). For $1 < p < \infty$ and $q = 0$ it is proved by Coifman and Meyer in [2 p. 140].

REFERENCES


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