A NOTE ON $\tau$-CONSTANT FAMILIES
OF PLANE CURVES

STEIN ARILD STRØMME

Introduction.

If $\{X_y\}_{y \in Y}$ is an algebraic (or analytic) family of (germs of) isolated plane curve singularities with "constant $\tau$" (i.e. the dimension $\tau(X_y)$ of the base of the miniversal deformation of $X_y$ is independent of $y \in Y$), it seems to be well known (cf. the discussion in [6, p. 667]) that if the parameter space $Y$ is nonsingular, then the family is equisingular (in the sense of Zariski, [8]). The notion of equisingularity has been extended to families over Artinian base schemes by Wahl [7], and he proves that the equisingular deformation functor is smooth. On the other hand, the notion of $\tau$-constant families has an obvious infinitesimal analogue, called equicohomological families in this note. However, the implication equicohomological $\Rightarrow$ equisingular no longer holds; the purpose of this note is to provide a counterexample. The same example shows that the equicohomological stratum in the prorepresentable hull of the deformation functor is singular.

1. Equicohomological deformations.

(1.1). Let $R = k[[X, Y]]$ be the algebra of formal power series over an algebraically closed field $k$ of characteristic 0, let $f \in R$ be a reduced power series of order $r \geq 1$, that is $f \in (X, Y)^r - (X, Y)^{r+1}$, and put $B = R/(f)$. By definition, $B$ is an algebroid isolated plane curve singularity.

(1.2). Denote by $C$ the category of local $k$-algebras of finite length, and by $D : C \to \text{Sets}$ the deformation functor of $B$, and let $(H, \xi)$ be the prorepresentable hull of $D$, see [5]. It is described as follows: Let $g_1, \ldots, g_r \in R$ induce a $k$-basis of $R/(f, f_X, f_Y)$, and let $t_1, \ldots, t_r$ be variables. Then $H = k[[t_1, \ldots, t_r]]$, and if we put

$$f' = f + \sum_{i=1}^r t_i g_i \in R \otimes_k H = R_H,$$

then $\tilde{B} = R_H/(f')$ induces a semiuniversal family for $D$.  

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(1.3). Denote by $H^1(k, B, B)$ the algebra cohomology groups of André [1]; then the tangent space of $D$ (or of $H$) is canonically isomorphic to $H^1(k, B, B)$, which in turn is (non-canonically) isomorphic to $R/(f, f_x, f_y)$. Similarly, $H^1(H, \tilde{B}, \tilde{B})$ is isomorphic to $R_{H^1}(\tilde{f}, \tilde{f}_x, \tilde{f}_y)$.

(1.4). For any morphism $A' \to A$ in $C$ and any deformation $B_{A'} \in D(A')$, there is a natural base-change map

$$H^1(A', B_{A'}, B_{A'}) \otimes_{A'} A \to H^1(A, B_A, B_A),$$

where $B_A = B_A \otimes A$. Representing $B_{A'}$ as a quotient of $R_{A'}$ by $f' = R_{A'}$ and using similar isomorphisms as those in (1.3), we easily see that the base change map is an isomorphism.

(1.5). Definition. A deformation $B_A \in D(A)$ is equicohomological if $H^1(A, B_A, B_A)$ is a flat (hence free) $A$-module. In view of (1.4), these families form a subfunctor $EC$ of $D$.

(1.6). Let $\tau = \dim_k H^1(k, B, B)$, and denote by $J \subseteq H$ the $(\tau - 1)$th Fitting ideal of the finite $H$-module $H^1(H, \tilde{B}, \tilde{B})$. Put $\tilde{H} = H/J$ and $\tilde{B} = \tilde{B} \otimes_H \tilde{H}$. By [3, Lecture 8, case $n = 0$], for any morphism $H \to A$ with $A$ in $C$, it factors through $\tilde{H}$ if and only if $H^1(H, \tilde{B}, \tilde{B}) \otimes_H A$ is flat over $A$. In view of (1.4) above, this happens exactly when $\tilde{B} \otimes_H A$ is equicohomological. Hence $(\tilde{H}, \tilde{B})$ is a prorepresentable hull of $EC$.

(1.7). Remark. In [2] it is proved that $(\tilde{H}, \tilde{B})$ actually prorepresents $EC$, and that $EC$ is the maximal prorepresentable subfunctor of $D$ (in the terminology of that paper, $EC$ is the prorepresentable substratum of $D$).

(1.8). By Hilbert's syzygy theorem, the $R$-module $H^1(k, B, B)$ has a free resolution of the form

$$0 \to R^2 \xrightarrow{\phi} R^3 \xrightarrow{\psi} R,$$

where $\psi$ is the row vector $(f, f_x, f_y)$. Furthermore, since coker $(\psi)$ has finite support, the image of $\psi$ is the ideal generated by the maximal minors of $\varphi$. The $3 \times 2$ matrix $\varphi = (\varphi_{i,j})$ will play a key role in the description of the tangent space $T_{EC}$ of $EC$. By definition, this is $EC(k[\varepsilon])$, where $k[\varepsilon]$ is the ring of dual numbers $(\varepsilon^2 = 0)$. $T_{EC}$ is a subspace of $T_D = H^1(k, B, B)$ and can be described as follows:

(1.9). Proposition. For any $g \in R$, let $B_g = R[\varepsilon]/(f + \varepsilon g)$. Then the following are equivalent:
(i) $B_\varepsilon$ is equicohomological, i.e. in $T_{EC}$.
(ii) $g\varphi_{1j} + g_x\varphi_{2j} + g_y\varphi_{3j} \in (f, f_X, f_Y) \subseteq R \quad (j = 1, 2)$.

**Proof.** Put $\psi_1 = (g, g_X, g_Y)$ (the vector), then $B_\varepsilon \in T_{EC}$ iff

$$\text{coker } (\psi + \varepsilon\psi_1) : R[\varepsilon]^3 \to R[\varepsilon]$$

is flat over $k[\varepsilon]$. By a well-known theorem [4] this happens iff $\varphi$ can be lifted to a matrix $\varphi + \varepsilon\varphi_1$ such that $(\psi + \varepsilon\psi_1)(\varphi + \varepsilon\varphi_1) = 0$, that is $\psi\varphi_1 + \psi_1\varphi = 0$. The existence of such a $\varphi_1$ is clearly equivalent to (ii).

2. An example.

(2.1). Put $f = (X^4 - Y^4)^2 - X^{10}$. This example has been studied by Wahl [7, 6.8]. Let $R/(f, f_X, f_Y) \to T_D$ be the isomorphism induced by the correspondence $g \to B_\varepsilon$ as in (1.9); then $T_D$ will be identified with $R/(f, f_X, f_Y)$ via this isomorphism. Wahl shows that the tangent space $T_{\overline{ES}}$ of the equisingular deformation functor $\overline{ES}$ is the ideal generated by $(X, Y)^{10}$ and $X^2Y^2(X^4 - Y^4)$ in $R/(f, f_X, f_Y)$.

(2.2). **Proposition.**

(i) A $k$-basis for $T_D$ is given by $\{X^iY^j \mid (i, j) \in B\}$, where

$$B = \{0, \ldots, 5\} \times \{0, \ldots, 6\} \cup \{(6, 0), (6, 1), (6, 2), (6, 3), (7, 3), (8, 3)\}.$$

(ii) A $k$-basis for $T_{EC}$ is given by the following:

(a) $X^iY^j, \quad (i, j) \in B, \ i + j \geq 9$,

(b) $X^4 - Y^4 + 4X^6 - 4X^2Y^4

$$X^5 - XY^4

48X^4Y - 48Y^5 - 5X^6Y

X^5Y - XY^5

32X^4Y^2 - 32Y^6 - 5X^6Y^2

X^5Y^2 - XY^6

X^6Y - X^2Y^5

X^6Y^2 - X^2Y^6

X^3Y^5.$$

In particular, $\tau = 48$ and $\dim T_{EC} = 18$. 
(2.3). Corollary. There exist equicohomological deformations that are not equisingular (not even equimultiple along any section, see [7] for definitions).

(2.4). Proposition. EC is obstructed, that is \( \tilde{H} \) is singular.

Proof of (2.2). (i) is basically an exercise: First find a monomial base of \( R/(f_X, f_Y) \), then express the residue class of \( f \) and its multiples in this basis. (In fact \( \mu = \dim R/(f_X, f_Y) = 57 \), and the annihilator of \( f \) modulo \( (f_X, f_Y) \) is the ideal \( (X^3, Y^3) \).) The multiplication table in \( T_D \) should be generated by the relations \( X^{10} = X^6 Y^4 = (X, Y)^{12} = 0 \), \( Y^7 = X^4 Y^3 \), \( X^7 = X^3 Y^4 + \frac{3}{4} X^5 Y^4 \).

(ii) From these computations we may construct the following matrix \( \varphi \):

\[
\varphi = \begin{bmatrix}
40Y^3 & 8X^3 - 10X^5 \\
-4XY^3 & \frac{1}{2}Y^4 - X^4 + X^6 \\
X^4 - 5Y^4 & -\frac{3}{4}X^3 Y + \frac{5}{4}X^5 Y
\end{bmatrix}
\]

and one checks easily that its minors are \( f, f_X, \) and \( f_Y \). With all this, we are in a position to apply the test of (1.9), reducing everything to a system of \( k \)-linear equations.

(2.5). Remark. To make a check on these computations, put for example \( g = X^5 - XY^4 \). Then

\[
\varphi_{11}g + \varphi_{21}g_X + \varphi_{31}g_Y = -2Xf_Y
\]

\[
\varphi_{12}g + \varphi_{22}g_X + \varphi_{32}g_Y = (11 - 20X^2)f + (2X^3 - X)f_X + (\frac{3}{8}X^2 Y - \frac{7}{8} Y)f_Y
\]

hence \( f + \varepsilon g \) and its partial derivatives are the maximal minors of \( \varphi + \varepsilon \varphi_1 \), where

\[
\varphi_1 = \begin{bmatrix}
0 & 20X^2 - 11 \\
0 & X - 2X^3 \\
2X & \frac{3}{2}Y - \frac{5}{2}X^2 Y
\end{bmatrix}.
\]

This can of course be verified directly.

Proof of (2.4). From the explicit description of the semiuniversal family of (1.2) it is clear that it is algebraizable: it is, in fact, defined over \( \mathcal{H}_1 = k[t_1, \ldots, t_n] \). Passing to an affine open neighbourhood Spec \( \mathcal{H} \subseteq \text{Spec} \mathcal{H}_1 \) of the origin, \( H^1(\mathcal{H}, B_{\mathcal{H}}, B_{\mathcal{H}}) \) is a finite \( \mathcal{H} \)-module, and we may form a quotient \( \tilde{\mathcal{H}} \) of \( \mathcal{H} \) in the same way as in (1.6), the equicohomological stratum. Then \( \tilde{H} \) is the completion of \( \tilde{\mathcal{H}} \) at the origin, and it suffices to show that \( \tilde{\mathcal{H}} \) is singular. Assuming the contrary, we may extend the deformation \( f + \varepsilon g \), where \( g = X^5 \)
\(-XY^5\) to an equicohomological family defined over a nonsingular curve. The
general fiber in this family would be an isolated singularity no worse than that
defined by \(g = 0\), an ordinary 5-ple point. Hence the invariant \(\tau\) of the general
fiber is at most 16, whereas in any equicohomological family, \(\tau\) is constant,
since the formation of \(H^1\) commutes with base change. Since \(\tau = 48\) for the
special fiber, we have the desired contradiction.

(2.6). Remark. In this example, it happens that \(T_{ES} \subseteq T_{EC}\). In the general case,
this is not so, as can be seen from the example \(X^5 + Y^5 + eX^3Y^3\), which is
equisingular but not equicohomological.

Note added in proof. G. Pfister recently discovered that the family
\(X^5 + X^2Y^2 + Y^4 + tX^4\) is \(\tau\)-constant but not equisingular.

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