ONE-DIMENSIONAL K-TYPES
IN FINITE DIMENSIONAL REPRESENTATIONS OF
SEMISIMPLE LIE GROUPS:
A GENERALIZATION OF HELGASON'S THEOREM

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1. Introduction.

Let $G$ be a semisimple connected noncompact real Lie group, and let $K$ be a maximal compact subgroup. Let $(\pi, V)$ be a finite dimensional irreducible representation of $G$. A renowned theorem due to S. Helgason gives the condition in terms of the highest weight of $\pi$ under which $\pi$ is class one. This means that $V$ contains a nonzero vector fixed by $\pi(K)$, or in other words that $\pi$ contains the trivial $K$-type in its decomposition into irreducible representations of $K$. In this paper we generalize this theorem to give a complete description in terms of the highest weight of $\pi$ of all one-dimensional $K$-types contained in $\pi$.

Let $g = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $g$ of $G$, let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $j$ be a Cartan subalgebra of $g$ containing $\mathfrak{a}$. Then $j = \mathfrak{t}^+ \oplus \mathfrak{a}$ where $\mathfrak{t}^+ = j \cap \mathfrak{t}$. Assume that $G$ is contained in the complex simply connected Lie group with Lie algebra $g_{\mathbb{C}} = g + ig$. Since we are dealing only with finite dimensional representations of $G$ this assumption causes no loss of generality. Choose compatible orderings of $\mathfrak{a}$ and $j$, and let $\lambda \in j^\mathbb{C}$ be a complex linear form on $j$.

The precise content of Helgason’s theorem is as follows (cf. [2, III § 3]): If the restriction of $\lambda$ to $\mathfrak{t}^+$ is zero and if for all positive roots $\alpha$ of $\mathfrak{a}$ in $g$ the number $(\lambda, \alpha)/(\alpha, \alpha)$ is a nonnegative integer, then $\lambda$ is the highest weight of a finite dimensional class one representation of $G$, and all finite dimensional class one representations of $G$ occur in this way.

If $K$ is semisimple then the trivial representation is its only one-dimensional representation, and therefore no more can be said about one-dimensional $K$-types in $\pi$. However, if $K$ is not semisimple, or equivalently if $G/K$ is Hermitian symmetric, then there are other one-dimensional $K$-types than the trivial. It is for such groups $G$ our generalization of Helgason’s theorem applies.

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Let \( t_1 = [t, t] \) be the semisimple part of \( t \). The condition on the highest weight \( \lambda \) of \( \pi \) for one-dimensional \( K \)-types to occur is very similar to that of Helgason's theorem: the restriction of \( \lambda \) to \( t^+ \cap t_1 \) has to be zero, and furthermore \( \lambda \) has to satisfy a certain integrality condition (see Theorem 7.2). The method of our investigation is by reduction to the rank-one case in the manner of S. G. Gindikin and F. I. Karpelevič ([1]).

The paper is organized as follows: First the restricted root theorem of C. C. Moore is stated. In Section 3 we study the structure of \( K \) and determine the complete set of one-dimensional \( K \)-types. In the next section the centralizer of \( a \) in \( K \) and its intersection with the semisimple part of \( K \) are considered. In Section 5 the rank-one subgroups of \( G \) are studied, and in the succeeding section we look upon \( SU(n, 1) \) \((n \geq 1)\), which are the only rank-one groups in which \( K \) is not semisimple. Finally in Section 7 the main theorem is stated and proved.

The problem of generalizing Helgason's theorem in this direction emerged in [8]. Though we will not go into that here we point out, that Theorem 7.2 in combination with the results of [8] can be applied to the construction of interesting unitary representations, in particular of some exceptional groups.

2. Root structure.

Let \( G \) be a connected real simple noncompact Lie group, and let \( g \) be its Lie algebra. Assume that \( G \subset G_C \), where \( G_C \) is a simply connected complex Lie group with Lie algebra \( g_C \). Let \( g = t \oplus p \) be a Cartan decomposition, and let \( K \) be the corresponding maximal compact subgroup of \( G \). Let \( t_1 = [t, t] \) and assume that \( t_1 \neq t \), that is \( G/K \) is Hermitian symmetric (cf. [3, Ch. VIII]).

Let \( t \) be a Cartan subalgebra of \( t \), then \( t \) is also a Cartan subalgebra of \( g \). Let \( \Delta \subset t^* \) consist of the roots of \( t \) in \( g \), and let \( g_\gamma \subset g_C \) for \( \gamma \in \Delta \) denote the \( \gamma \) root space. Let \( \Delta_c \), respectively \( \Delta_n \) be the set of compact, respectively noncompact roots, i.e. those roots \( \gamma \) for which \( g_\gamma \subset t_C \), respectively \( g_\gamma \subset p_C \). Then \( \Delta = \Delta_c \cup \Delta_n \).

Let \( z \) be the center of \( t \), then \( \dim z = 1 \). As is well-known, we can choose an element \( Z_0 \in z \) such that \( \gamma(Z_0) = \pm i \) for all \( \gamma \in \Delta_n \). Choose an ordering of \( \Delta \) such that

\[
\Delta_n^+ = \{ \gamma \in \Delta \mid \gamma(Z_0) = i \},
\]

where \( \Delta_n^+ = \Delta^+ \cap \Delta_n \). Let \( \Delta_c^+ = \Delta^+ \cap \Delta_c \).

For \( \varphi \in t_C \) let \( H_\varphi \in t_C \) be defined by \( \varphi(H) = (H_\varphi, H) \) for all \( H \in t \), where \( (\cdot, \cdot) \) denotes the Killing form. Let \( \{ \gamma_1, \ldots, \gamma_r \} \subset \Delta_n \) be a maximal strongly orthogonal subset, such that \( \gamma_j \) is the highest element of \( \Delta_n \) strongly orthogonal to \( \{ \gamma_{j+1}, \ldots, \gamma_r \} \), for \( j = r, \ldots, 1 \) (cf. [3, p. 386]). Let
\[ t^- = \sum_{j=1}^{r} RiH_{\gamma_j} \]

and

\[ t^+ = \{ H \in t \mid \gamma_j(H) = 0, j=1, \ldots, r \} \]

then \( t = t^+ \oplus t^- \). Identify \( \gamma_j \) with its restriction to \( t^- \) (\( j = 1, \ldots, r \)).

**Theorem 2.1.** (C. C. Moore). The set of nonzero restrictions of the elements of \( \Delta^+ \) to \( t^- \) is one of the following two sets:

- Case I: \( \{ \gamma_i, \frac{1}{2}(\gamma_j \pm \gamma_k) \mid 1 \leq i \leq r, 1 \leq k < j \leq r \} \)
- Case II: \( \{ \frac{1}{2}\gamma_i, \frac{1}{2}(\gamma_j \pm \gamma_k) \mid 1 \leq i \leq r, 1 \leq k < j \leq r \} \).

Furthermore the nonzero restrictions of compact roots have the form \( \frac{1}{2}\gamma_i \) or \( \frac{1}{2}(\gamma_j - \gamma_k) \), and the restrictions of noncompact roots have the form \( \frac{1}{2}\gamma_i \), \( \gamma_i \) or \( \frac{1}{2}(\gamma_j + \gamma_k) \).

The roots \( \gamma_1, \ldots, \gamma_r \) do all belong to the set of longest roots in \( \Delta \). In Case II, only one root length occurs in \( \Delta \).

Unless when \( t^+ = 0 \), \( \gamma_1, \ldots, \gamma_r \) are the only restricted roots of multiplicity one.

**Proof.** This can be verified case by case from the diagrams [3, pp. 532–34], or it can be proved by combinatorial arguments, cf. [6].

**Remark 2.2.** In [4] it is shown that Case I is necessary and sufficient for \( G/K \) to be a tube domain. Here is the classification of the possible algebras \( g \) (cf. [3]):

- Case I: \( \text{su} (n, n) \) (\( n \geq 2 \)), \( \text{so} (n, 2) \) (\( n \geq 5 \)), \( \text{so}^* (4n) \) (\( n \geq 3 \)), \( \text{sp} (n, R) \) (\( n \geq 1 \)) and \( \varepsilon_{7(-25)} \).
- Case II: \( \text{su} (p, q) \) (\( q > p \geq 1 \)), \( \text{so}^* (4n+2) \) (\( n \geq 2 \)) and \( \varepsilon_{6(-14)} \).

Among these, \( t^+ = 0 \) only happens for \( \text{sp} (n, R) \) (\( n \geq 1 \)).

For each \( \gamma \in \Delta_n \), choose \( X_\gamma \in g_\gamma \setminus \{0\} \) subject to \( \bar{X}_\gamma = X_{-\gamma} \) and \( \gamma([X_\gamma, X_{-\gamma}]) = 2 \), where the bar denotes conjugation with respect to the real form \( g \) of \( g_\mathbb{C} \). Let

\[ a = \sum_{j=1}^{r} R(X_{\gamma_j} + X_{-\gamma_j}) \]

then \( a \) is a maximal abelian subspace of \( p \). Let \( c \) be the automorphism of \( g_\mathbb{C} \) given by

\[ c = \text{Ad} \exp \frac{\pi}{4} \sum_{j=1}^{r} (X_{\gamma_j} - X_{-\gamma_j}) \].
Then $c$ maps it bijectively to $a$ and fixes $t^+$ (cf. [4]).

Let $j = t^+ + a$ and let $c_\ast : t^+_C \to j_C$ denote the adjoint of $c^{-1} : j_C \to t_C$. Then $c_\ast \Delta$ consists of the roots of the Cartan subalgebra $j$ of $g$. Let $\Sigma$ denote the set of nonzero restrictions of $c_\ast \Delta$ to $a$, and let $\Sigma^+$ consists of the nonzero restrictions of $c_\ast \Delta^+$ to $a$. Let $\alpha_j = c_\ast \gamma_j$, then exchanging the $\gamma$'s in Case I and II above with $\alpha$'s, we get the two possible forms of $\Sigma^+$.

3. The structure of $K$.

Let $K_1$ denote the analytic subgroup of $K$ with Lie algebra $t_1$.

**Lemma 3.1.** Let $\Phi$ denote the set of simple roots for $\Delta^+$, and let $s_\varphi \in \mathbb{R}$ for each $\varphi \in \Phi$. Then

$$\exp \left( \sum_{\varphi \in \Phi} s_\varphi \frac{2iH_\varphi}{(\varphi, \varphi)} \right) = e$$

if and only if $s_\varphi \in 2\pi \mathbb{Z}$ for all $\varphi \in \Phi$.

**Proof.** Let $U$ be the analytic subgroup of $G_C$ with Lie algebra $u = t + i u$. Then $U$ is compact and simply connected, and $t$ is a Cartan subalgebra of $u$. The lemma then follows from [9, Theorem 4.6.7].

**Corollary 3.2.** $K_1$ is simply connected.

**Proof.** It is easily seen that $\Phi \cap \Delta_c$ consists of the simple roots for $\Delta_c^+$. From Lemma 3.1

$$\exp \left( \sum_{\varphi \in \Phi \cap \Delta_c} s_\varphi \frac{2iH_\varphi}{(\varphi, \varphi)} \right) = e$$

if and only if $s_\varphi \in 2\pi \mathbb{Z}$ for all $\varphi \in \Phi \cap \Delta_c$, and so $K_1$ is simply connected, again by [9, Theorem 4.6.7].

Let $a$ denote the length of the longest roots in $\Delta$. The short roots, if there are any, then have length $a/\sqrt{2}$.

**Lemma 3.3** Let $t_\gamma \in \mathbb{R}$ for each $\gamma \in \Delta_n^+$ and let

$$x = \exp \left( \sum_{\gamma \in \Delta_n^+} t_\gamma \frac{2iH_\gamma}{(\gamma, \gamma)} \right).$$

Then $x \in K_1$ if and only if
\[ \sum_{\gamma \in D^*_n} \frac{t_\gamma}{(\gamma, \gamma)} \in \frac{2\pi}{a^2} \mathbb{Z}. \]

**Proof.** Assume that \( x \in K_1 \). Since \( x \) centralizes \( t \cap t_1 \), which is a Cartan subalgebra of \( t_1 \), \( x = \exp Y \) for some \( Y \in t \cap t_1 \). Then

\[ \exp \left( \sum_{\gamma \in \Phi} t_\gamma \frac{2iH_{\gamma}}{(\gamma, \gamma)} - Y \right) = e \]

and it follows from Lemma 3.1 that

\[ \sum_{\gamma \in D^*_n} t_\gamma \frac{2iH_{\gamma}}{(\gamma, \gamma)} = \sum_{\varphi \in \Phi} s_{\psi} \frac{2iH_\varphi}{(\varphi, \varphi)} \]

for some \( s_\varphi \in 2\pi \mathbb{Z} \) (\( \varphi \in \Phi \)). Taking inner product with \( \frac{1}{2} Z_0 \) it follows that

\[ \sum_{\gamma \in D^*_n} t_\gamma \frac{2iH_{\gamma}}{(\gamma, \gamma)} = \frac{s_\psi}{(\psi, \psi)} \in \frac{2\pi}{a^2} \mathbb{Z} \]

where \( \psi \) is the unique simple noncompact root.

Assume conversely that

\[ t = \sum_{\gamma \in D^*_n} t_\gamma \frac{2iH_{\gamma}}{(\gamma, \gamma)} \]

Then

\[ \sum_{\gamma \in D^*_n} t_\gamma \frac{2iH_{\gamma}}{(\gamma_1, \gamma_1)} - ta^2 \frac{2iH_{\gamma_1}}{(\gamma_1, \gamma_1)} \in t_1 \]

since it is orthogonal to \( Z_0 \). However

\[ \exp \left( ta^2 \frac{2iH_{\gamma_1}}{(\gamma_1, \gamma_1)} \right) = e \]

by Lemma 3.1, and hence \( x \in K_1 \).

Let \( N = \sum_{\gamma \in D^*_n} a^2/(\gamma, \gamma) \), then \( N \) is the number of longest roots in \( D^*_n \) plus twice the number of short roots, if there are any. Define \( Z \in t \) by

\[ Z = \frac{1}{N} \sum_{\gamma \in D^*_n} \frac{2iH_{\gamma}}{(\gamma, \gamma)}. \]

**Proposition 3.4.** Let \( t \in \mathbb{R} \)

(i) \( Z \in z \setminus \{0\} \),

(ii) \( \exp tZ \in K_1 \) if and only if \( t \in 2\pi \mathbb{Z} \).
PROOF. (i) From (3.1) it follows that

$$ (Z, Z_0) = -\frac{2}{a^2}. $$

Therefore $Z \neq 0$. Let $\varphi \in A^+_c$ and $\gamma \in A^+_n$. If $(\varphi, \gamma) = 0$ then $\gamma$ contributes nothing to $\varphi(Z)$. If $(\varphi, \gamma) \neq 0$ then also the reflected root $\sigma_{\varphi, \gamma}$ is positive and noncompact since $\sigma_{\varphi, \gamma}(Z_0) = i$. From $(\varphi, \gamma + \sigma_{\varphi, \gamma}) = 0$ it then follows that $\varphi(Z) = 0$ for all $\varphi \in A^+_c$, and hence $Z \in z$.

(ii) follows immediately from Lemma 3.3.

Let $l \in \mathbb{Z}$, and define $\chi_l: K \to \mathbb{C}$ by $\chi_l(k) = 1$ for $k \in K_1$ and $\chi_l(\exp iZ) = e^{ilt}$ for $t \in \mathbb{R}$. From Proposition 3.4 we get that $\chi_l$ is a well defined one dimensional representation of $K$, and that all one dimensional representations of $K$ have this form.

REMARK 3.5. In Case I, we can give a simpler formula for $Z$ as follows

$$ Z = \frac{1}{r} \sum_{j=1}^r \frac{2iH_{jj}}{(\gamma_j, \gamma_j)}. $$

In fact, let $Z'$ denote the right hand side of (3.3). Then $Z' \in z$ by Theorem 2.1, and $(Z, Z_0) = (Z', Z_0)$ from (3.2), so (3.3) follows. In particular we have $z \subset t^-$ in Case I (cf. also [4, Proposition 3.12]).

4. The structure of $M$.

Let $M$ denote the centralizer of $a$ in $K$, and let $m$ be its Lie algebra. Then $t^+$ is a Cartan subalgebra of $m$. For any Lie group $F$, let $F_0$ denote its identity component. It is well-known (cf. [2, p. 75]) that

$$ (4.1) \quad M = M_0 \cdot (\exp \mathfrak{a} \cap K) $$

and also that if $H_\alpha \in \mathfrak{a}$ for $\alpha \in \mathfrak{a}^*$ is defined by $(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{a}$, then (cf. [2, p. 77])

$$ (4.2) \quad \{ H \in \mathfrak{a} \mid \exp iH \in K \} = \left\{ \sum_{\alpha \in \Sigma} s_\alpha \frac{H_\alpha}{(\alpha, \alpha)} \mid s_\alpha \in 2\pi \mathbb{Z} \text{ for all } \alpha \in \Sigma^+ \right\}. $$

From the description of $\Sigma^+$ by Theorem 2.1 it follows that (4.2) can be restated as follows:

$$ (4.3) \quad \{ H \in \mathfrak{a} \mid \exp iH \in K \} = \left\{ \sum_{j=1}^r s_j \frac{H_{jj}}{(\alpha_j, \alpha_j)} \mid s_j \in 2\pi \mathbb{Z} \text{ for } j = 1, \ldots, r \right\}. $$
Lemma 4.1.

\[ \exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)} = \exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)} \]

for \( j = 1, \ldots, r. \)

Proof. There is a homomorphism \( \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}_C \) for which

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix} \rightarrow \frac{2H_{\gamma_j}}{(\alpha_j, \alpha_j)} \quad \text{and} \quad
\begin{pmatrix}
i & 0 \\
0 & -i \\
\end{pmatrix} \rightarrow \frac{2iH_{\gamma_j}}{(\gamma_j, \gamma_j)}.
\]

The lemma then follows, since in \( SL(2, \mathbb{C}) \)

\[
\exp \pi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \exp \pi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Lemma 4.2. In Case II, \( M \) is connected.

Proof. By (4.3) and Lemma 4.1, it suffices to prove that for \( j = 1, \ldots, r \):

\[ \exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)} \in M_0. \]

Let \( \varphi \in \Delta^+ \) be a root whose restriction to \( t^- \) is \( \frac{1}{2}\gamma_j \). Then obviously \( H_{\gamma_j} - 2H_{\varphi} \in t^+ \). From Lemma 3.1 we have

\[ \exp \left( 2\pi \frac{2iH_{\varphi}}{(\varphi, \varphi)} \right) = e \]

and therefore

\[ \exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)} = \exp \frac{2\pi i}{a^2} (H_{\gamma_j} - 2H_{\varphi}) \in M_0. \]

Let \( W \) be the Weyl group of \( \Delta \), and let \( \theta = \sigma_{\gamma_1} \cdots \sigma_{\gamma_r} \in W \). Then \( \theta(S + T) = S - T \) for \( S \in t^+ \) and \( T \in t^- \). Let \( \Xi \subset \Delta^+ \) denote the set of roots whose restriction to \( t^- \) is \( \frac{1}{2}\gamma_j \) for some \( j \), and let \( \Xi_n = \Xi \cap \Delta_n \) and \( \Xi_c = \Xi \cap \Delta_c \). Note that \( \theta(\Xi_n) = -\Xi_n \), since if \( \xi \in \Xi_n \) with \( \xi|_{t^-} = \frac{1}{2}\gamma_j \), then \( \theta \xi = \xi - \gamma_j \in -\Xi_n \).

Let \( R \) denote the number of elements of \( \Xi_n \) (or \( \Xi_c \)). Define \( X \in t \) by \( X(0) = 0 \) in Case I and in Case II:

(4.4) \[ X = \frac{1}{R} \left[ \sum_{\xi \in \Xi_n} \frac{2iH_{\xi}}{(\xi, \xi)} - \sum_{\xi \in \Xi_c} \frac{2iH_{\xi}}{(\xi, \xi)} \right]. \]
Lemma 4.3. In Case II, the following holds:

(i) \( Z + \theta Z = \frac{R}{N} X \).

(ii) \( X \in t^+, X \perp t^+ \cap t_1 \).

(iii) \( X \neq 0 \).

(iv) \( Z - X \in t_1 \).

(v) \( \text{For } t \in \mathbb{R}: \exp t X \in K_1 \text{ if and only if } t \in 2\pi \mathbb{Z} \).

Proof. (i) is clear from (4.4) since \( \theta(\Xi_n) = -\Xi_c \).

(ii) follows from (i). From (4.4) it follows that

(4.5) \( (X, Z_0) = -\frac{2}{a^2} \)

and from this (iii) is obvious.

(iv) follows from (4.5) and (3.2).

(v) is obvious from (iv) and Proposition 3.4 (ii).

Note that it follows that \( z \notin t^- \) in Case II (cf. also [4, Section 4]).

Lemma 4.4. In Case II, \( M \cap K_1 \) is connected.

Proof. Since \( M \) is connected and \( m = (m \cap t_1) \oplus \mathbb{R} X \) by the preceding lemmas, it suffices to prove that \( \exp \mathbb{R} X \cap K_1 \subset (M \cap K_1)_0 \). By Lemma 4.3 (v) it then suffices to prove that \( \exp 2\pi X \in (M \cap K_1)_0 \).

Choose \( \xi \in \Xi_n \) then \( 0 \xi \in A_c \). Therefore \( (H_{\xi} + \theta H_{\xi}, Z_0) = i \), and from (4.5) it follows that

\( X - \frac{2i}{a^2} (H_{\xi} + \theta H_{\xi}) \in t^+ \cap t_1 \).

However, by Lemma 3.1

\[ \exp \left[ \frac{2i}{a^2} (H_{\xi} + \theta H_{\xi}) \right] = e \]

and hence \( \exp 2\pi X \in \exp t^+ \cap t_1 \).

We can now state an analogue of (4.1) and (4.3).
PROPOSITION 4.5. (i) \( M \cap K_1 = (M \cap K_1)_0 \exp i a \cap K_1. \)

(ii) \[ \{ H \in a \left| \exp iH \in K_1 \} \]
\[ = \left\{ \sum_{j=1}^{r} s_j \frac{H_{s_j}}{(s_j, s_j)} : s_j \in 2\pi \mathbb{Z} \text{ for } j = 1, \ldots, r \text{ and } \sum_{j=1}^{r} s_j \in 4\pi \mathbb{Z} \right\} \].

PROOF. (i) In Case I, \( M_0 \subset K_1 \) and hence \( (M \cap K_1)_0 = M_0. \) Therefore (i) follows from (4.1). In Case II (i) is obvious from Lemma 4.4.

(ii) follows from (4.3), Lemma 4.1, and Lemma 3.3.

Later on, we need the following lemma:

LEMMA 4.6. (i) If \( \gamma \in A^+ \setminus \Xi, \) then \( \gamma(X) = 0. \)

(ii) If \( \gamma \in \Xi_m, \) then \( \gamma(X) = -a^2(X, X)i/4, \)

(iii) If \( \gamma \in \Xi_c, \) then \( \gamma(X) = a^2(X, X)i/4. \)

PROOF. Assume Case II. Let \( b \in \mathbb{R} \) be given by \( Z = bZ_0. \)

(i) Let \( \gamma \in A^+ \setminus \Xi. \) If \( \gamma \) is compact, then \( \theta \gamma \) is also compact by Theorem 2.1, and hence \( (\gamma + \theta \gamma)(Z) = 0. \) If \( \gamma \) is noncompact, then \( \theta \gamma \) is also noncompact but negative, and hence \( (\gamma + \theta \gamma)(Z) = ib - ib = 0. \) Then \( \gamma(X) = 0 \) by Lemma 4.3 (i).

(ii)–(iii) If \( \gamma \in \Xi_m, \) then \( \theta \gamma \in -\Xi_c \) and hence \( \gamma(Z + \theta Z) = ib. \) Therefore \( \gamma(X) = ibN/R, \) and since \( \theta X = X, \) we then have \( -\theta \gamma(X) = -ibN/R. \) But then by (4.4)
\[
(X, X) = \frac{1}{R} \left[ \sum_{\xi \in \Xi_a} \frac{2i \xi(X)}{(\xi, \xi)} - \sum_{\xi \in \Xi_c} \frac{2i \xi(X)}{(\xi, \xi)} \right] = -\frac{4bN}{Ra^2}
\]
so \( bN/R = -a^2(X, X)/4. \)

5. The rank-one reduction.

Let \( \alpha \in \Sigma^+ \setminus 2\Sigma^+ \) and let \( g^\alpha \) be the subalgebra of \( g \) generated by the root spaces \( g_{\alpha} \) and \( g_{-\alpha}. \) Let \( G^\alpha \) be the corresponding analytic subgroup of \( G. \) Then \( G^\alpha \) is a simple Lie group of real rank one, and \( g^\alpha = \mathfrak{l}^\alpha \oplus p^\alpha \) is a Cartan decomposition, where \( \mathfrak{l}^\alpha = \mathfrak{l} \cap g^\alpha \) and \( p^\alpha = p \cap g^\alpha. \) Therefore \( K^\alpha = K \cap G^\alpha \) is a maximal compact subgroup of \( G^\alpha. \) (For these well-known facts, see [3, pp. 407–409].) Let \( m(\alpha) \) denote the multiplicity of \( \alpha. \)

LEMMA 5.1. In Case I, \( G^\alpha/K^\alpha \) is Hermitian symmetric if and only if \( m(\alpha) = 1, \)
and then \( g^\alpha \cong su(1, 1). \) In Case II, \( G^\alpha/K^\alpha \) is Hermitian symmetric if and only if \( \alpha = \frac{1}{2} \alpha_j \) for some \( j \in \{1, \ldots, r\}, \) and then \( g^\alpha \cong su(n, 1), \) where \( n = 1 + \frac{1}{2}m(\alpha). \)

PROOF. According to classification the only rank one Hermitian symmetric
spaces are $SU(n, 1)/S(U(n) \times U(1)) \ (n \in \mathbb{N})$. If $G^x/K^x$ is Hermitian symmetric, it follows therefore that $g^x \cong su(n, 1)$ for some $n \in \mathbb{N}$. In Case I, $2x$ is not a root, and hence $g^x \cong su(1, 1)$. Obviously this happens if and only if $m(x) = 1$. In Case II, if $x$ is one of the roots $\frac{1}{2}(x_i \pm x_j)$, then $m(x) > 1$ and $2x \notin \Sigma$, so $g^x$ cannot be isomorphic to $su(n, 1)$ for any $n \in \mathbb{N}$. On the other hand, if $x = \frac{1}{2}x_j$, then $m(2x) = 1$, and therefore $g^x \cong su(n, 1)$ with $n + \frac{1}{2}m(x)$ by the classification of real rank one algebras.

Let $l \in \mathbb{Z}$. We will determine the restriction of $\chi_l$ to $K^x$, and assume therefore that $K^x$ is not semisimple.

In Case II, $g^x \cong su(1, 1)$ and in this identification
\[
\frac{2ic^{-1}H_x}{(x, x)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]
By Remark 3.5
\[
\frac{1}{(Z, Z)} \left( \frac{2ic^{-1}H_x^i}{(x, x)}, Z \right) = \sum_{j=1}^{r} \frac{(x, x_j)}{(x, x)}
\]
and therefore
\[
\chi_l \left( \exp \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) = \begin{cases} 1 & \text{if } x = \frac{1}{2}(x_i - x_j) \\ e^{ilt} & \text{if } x = x_i \\ e^{i2lt} & \text{if } x = \frac{1}{2}(x_i + x_j) \ (i \neq j) \end{cases}.
\]

In Case II we have $x = \frac{1}{2}x_j$ and $g^x \cong su(n, 1)$. Let $\Delta_{n}^+(x)$ denote the set of noncompact positive roots of $\mathfrak{t} \cap g^x$ in $g^x$. Note that the cardinality of $\Delta_{n}^+(x)$ is $n$, and put
\[
Z(x) = \frac{1}{n} \sum_{\gamma \in \Delta_{n}^+(x)} \frac{2iH_{\gamma \gamma}}{(\gamma, \gamma)}.
\]
Then, in the identification with $su(n, 1)$, $Z(x)$ is the diagonal matrix with $i/n$ in the first $n$ entries and $-i$ in the last entry. From (5.2) and (3.2) it follows that $Z - Z(x) \in \mathfrak{t}_1$. Therefore
\[
\chi_l(\exp t Z(x)) = e^{ilt}.
\]

6. A lemma concerning $SU(n, 1)$.

Let $G = KAN$ be the Iwasawa decomposition of $G$ corresponding to $\Sigma^+$, and define maps $\kappa: G \to K$ and $H: G \to a$ by
\[
x \in \kappa(x) \exp H(x)N
\]
for \( x \in G \). Let \( g = \frac{1}{2} \sum x \in \Sigma^* m(x) x \) and let \( \hat{N} \) be the group opposite to \( N \), i.e.

\[
\hat{N} = \exp \left( \sum_{x \in \Sigma^*} g_x \right).
\]

Let \( d\hat{n} \) denote some Haar measure on \( \hat{N} \). Let \( n \in N \) and \( k \in ]0, \infty[ \). Let \( \mathbb{Z}_+ = N \cup \{0\} \).

**Lemma 6.1.** Assume \( G = SU(n, 1) \), and let \( \beta \in \Sigma^+ \) be the root for which \( 2\beta \notin \Sigma \). Put \( v = \frac{1}{2} k \beta \). Then the integral

\[
\int_{\hat{N}} e^{-v \cdot g, H(\hat{n})} \chi_l(\kappa(\hat{n})) d\hat{n}
\]

converges absolutely for all \( l \in \mathbb{Z} \), and it is nonzero if and only if \( |l| \notin k + n + 2\mathbb{Z}_+ \).

**Proof.** We have

\[
K = \left\{ \begin{pmatrix} U & 0 \\ 0 & \det U^{-1} \end{pmatrix} \left| U \in U(n) \right. \right\}
\]

and

\[
\chi_l \left( \begin{pmatrix} U & 0 \\ 0 & \det U^{-1} \end{pmatrix} \right) = (\det U)^l.
\]

We put \( H = E_{1, n+1} + E_{n+1, 1} \), where \( E_{ij} \) denotes the \( n+1 \) square matrix with 1 on the \( i,j \)'th entry and all other entries 0. With \( a = RH \) we get \( \beta(H) = 2 \) and \( \varrho(H) = n \).

The root spaces are given as follows

\[
\varrho_{\beta} = \text{Ri}(E_{1,1} - E_{1,n+1} + E_{n+1,1} - E_{n+1,n+1})
\]

\[
\varrho_{\frac{1}{2}\beta} = \left\{ \sum_{j=1}^{n} (z_j E_{1,j+1} - \bar{z}_j E_{j+1,1} + \bar{z}_j E_{j+1,n+1} + z_j E_{n+1,j+1}) \bigg| z_1, \ldots, z_{n-1} \in \mathbb{C} \right\}
\]

\[
\varrho_{-\frac{1}{2}\beta} = \left\{ \sum_{j=1}^{n} (z_j E_{1,j+1} - \bar{z}_j E_{j+1,1} - \bar{z}_j E_{j+1,n+1} - z_j E_{n+1,j+1}) \bigg| z_1, \ldots, z_{n-1} \in \mathbb{C} \right\}
\]

\[
\varrho_{-\beta} = \text{Ri}(E_{1,1} + E_{1,n+1} - E_{n+1,1} - E_{n+1,n+1}).
\]

From the first two of these equations it is easily seen that \( H(x) \) and \( \chi_l(\kappa(x)) \) for \( x \in SU(n, 1) \) can be computed as follows: Let \( \eta(x) \) denote the sum of the first and the last element in the last row of \( x \), then

\[
H(x) = \log|\eta(x)|H
\]
\[ \chi_i(\kappa(x)) = \left( \frac{\eta(x)}{|\eta(x)|} \right)^{-1}. \]

From the expressions for \( q_{-1/\theta} \) and \( q_{-\theta} \), it then follows that the integral (6.1) except for a constant (nonzero) factor equals

\[ (6.2) \quad \int_{\mathbb{C}^{n-1}} \int_{\mathbb{R}} \left[ (1 + |z|^2)^2 + s^2 \right]^{-\frac{k+n}{2}} \left( \frac{1 + |z|^2 - is}{|1 + |z|^2 - is|} \right)^{-1} dsdz. \]

If \( n > 1 \) we use polar coordinates in \( \mathbb{C}^{n-1} \) and get

\[ \int_0^\infty \int_{\mathbb{R}} \left[ (1 + r^2)^2 + s^2 \right]^{-\frac{k+n}{2}} \left( \frac{1 + r^2 - is}{|1 + r^2 - is|} \right)^{-1} dsr^{2n-3} dr. \]

Let \( c_n = \int_0^\infty (1 + r^2)^{-k-n+1}r^{2n-3} dr \) if \( n > 1 \) and \( c_1 = 1 \) (note that this integral converges because \( k > 0 \)). Substitution of \( s = (1 + r^2)t^2 \) if \( n > 1 \) and \( s = t^2 \) if \( n = 1 \) gives the following integral instead of (6.2):

\[ c_n \int_{-\pi/2}^{\pi/2} (\cos t)^{k+n-2}e^{it} dt. \]

This integral can in fact be computed in terms of the gamma function since \( k > n + 1 \), and the result is

\[ c_n 2^{k+n-2} \Gamma\left(\frac{1}{2}(k+n+1)\right)\Gamma\left(\frac{1}{2}(k+n-l)\right) \]

(cf. [7, p. 158 (5)-(7)]). The lemma now follows since the denominator has poles precisely when \( |l| \in k+n+2\mathbb{Z} \).

7. The main theorem.

Let \( \lambda \in \mathcal{X} \), let \( m_0 = \lambda(iX) \) and

\[ m_j = \frac{2(\lambda, x_j)}{(x_j, x_j)} \quad (j = 1, \ldots, r). \]

Note that in Case I, \( m_0 = 0 \).

**Proposition 7.1.** If \( \lambda|_{\Gamma \cap \Gamma_1} = 0 \), then \( \lambda \) is dominant integral (with respect to \( c_{\mathcal{X}}\Delta^+ \)) if and only if the following three conditions hold:

(i) \( m_0, m_1, \ldots, m_r \) are integers satisfying \( |m_0| \leq m_1 \leq \ldots \leq m_r \).

(ii) In Case I, if \( t^+ \neq 0 \), then \( (-1)^{m_1} = \ldots = (-1)^{m_r} \).

(iii) In Case II, \( (-1)^{m_0} = (-1)^{m_1} = \ldots = (-1)^{m_r} \).
Proof: If \( t^+ = 0 \) the statement is obvious. Assume \( t^+ \neq 0 \) and Case I, then \( t^+ \subset t_1 \). Let \( \beta \) be a root of \( j \) in \( \mathfrak{g}_C \) which is not supported on \( \alpha \) and has restriction \( \frac{1}{2}(\alpha_i \pm \alpha_j) \). Then necessarily \( \beta \) is a long root, and hence

\[
2(\lambda, \beta) = \frac{1}{2}(m_i \pm m_j).
\]

The statement then follows from (7.1) and Theorem 2.1.

Assume next Case II. From Lemma 4.6, if \( \beta \) is a root of \( j \) in \( \mathfrak{g}_C \) and if \( \beta|_{\alpha} = \frac{1}{2}(\alpha_i \pm \alpha_j) \), then (7.1) holds again. On the other hand if \( \beta|_{\alpha} = \frac{1}{2}\alpha_i \) then

\[
\frac{2(\lambda, \beta)}{(\beta, \beta)} = \frac{2(\lambda|_\alpha, \beta|_\alpha)}{(\beta, \beta)} + \frac{2\lambda(X)\beta(X)}{(X, X)(\beta, \beta)} = \begin{cases} 
\frac{1}{2}(m_i + m_0) & \text{if } \beta \in E \, \Xi_n, \\
\frac{1}{2}(m_i - m_0) & \text{if } \beta \in E_c \, \Xi_c.
\end{cases}
\]

With that the proposition follows.

Assume now that \( \pi \) is a finite dimensional irreducible representation of \( G \) having \( \lambda \) as its highest weight. It is well known (cf. [9, Lemma 8.5.3]) that the space of \( N \)-fixed vectors for \( \pi \) is invariant under \( M \), and that this representation \( \delta \) of \( M \) is irreducible. Note that \( \lambda|_{t^+} \) is a highest weight of \( \delta \).

**Theorem 7.2.** The following three conditions are equivalent.

(i) \( \lambda|_{t^+ \cap t_1} = 0 \) and \( (-1)^{m_1} = \ldots = (-1)^{m_r} \).

(ii) \( \delta|_{M \cap K_1} \) is trivial.

(iii) \( \pi \) has nonzero \( K_1 \)-fixed vectors.

If these conditions hold, then \( \pi \) contains precisely the following one dimensional \( K_1 \)-types, each contained once:

**In Case I:** \( \chi_l \) for \( l = -m_1, -m_1 + 2, \ldots, m_1 - 2, m_1 \).

**In Case II:** \( \chi_{m_0} \).

Proof. First the equivalence of (i) and (ii) is proved. Obviously \( \delta \) is trivial on \( (M \cap K_1)_0 \) if and only if \( \lambda|_{t^+ \cap t_1} = 0 \). We have

\[
\delta\left(\exp \frac{2\pi i H_j}{(\alpha_j, \alpha_j)}\right) = \exp \left(\frac{2\pi i (\lambda, \alpha_j)}{(\alpha_j, \alpha_j)}\right) = (-1)^{m_i},
\]

and from Proposition 4.5 it follows therefore that (i) and (ii) are equivalent.

Let \( \nu = g + \lambda|_\alpha \in \mathfrak{a}^* \) and let \( V \) be the representation space of \( \delta \). Consider the principal series representation \( I_{\delta, \nu} \) of \( G \), the definition of which we recall:

Let \( C_G^p \) denote the space of \( V \)-valued \( C_\infty \)-functions \( f \) on \( G \) satisfying \( f(\text{gman}) \).
\[ = a^{-\tau - \epsilon \delta(m^{-1})} f(g) \quad \text{for all } g \in G, \; m \in M, \; a \in A, \; \text{and } n \in N. \] 

For the opposite minimal parabolic subgroup \( \bar{P} = MA \hat{N} \) we define similarly \( l_{\delta, \nu}^P \) as the representation of \( G \) on
\[ C_{\delta, \nu}^P \ = \ \{ f \in C^\infty(G, V) \mid f(g \epsilon \alpha) = a^{-\nu + \epsilon \delta(m^{-1})} f(g), \; \forall g \in G, \; m \in M, \; a \in A, \; \alpha \in \hat{N} \} . \]

It is well known that \( \pi \) is equivalent to a subrepresentation of \( l_{\delta, \nu}^P \) (cf. [9, Lemma 8.5.7]), and also that this subrepresentation can be realized as follows:

For each \( f \in C_{\delta, \nu}^P \) and \( x \in G \) the integral
\[ Af(x) = \int_N f(x \alpha \bar{n}) d\bar{n} \]
is absolutely convergent and defines a \( G \)-homomorphism \( A : C_{\delta, \nu}^P \rightarrow C_{\delta, \nu}^P \) (cf. [9, Section 8.10]), whose image is equivalent to \( \pi \) (cf. [5, p. 75]).

Let \( l \in \mathbb{Z} \). By Frobenius reciprocity \( \chi_l \) occurs in the \( K \)-decomposition of \( l_{\delta, \nu}^P \) if and only if \( \delta = \chi_l | M \). If \( \chi_l \) occurs in \( \pi \) it also occurs in \( C_{\delta, \nu}^P \) and therefore (iii) implies (ii), and also \( \chi_l \) has multiplicity at most one in \( \pi \).

Assume (ii). We will first determine the set of \( l \in \mathbb{Z} \) such that \( \delta = \chi_l | M \). In Case I, \( m \subset t_1 \) and hence \( \delta | M_o = \chi_l | M_o = 1 \) for all \( l \in \mathbb{Z} \). By (4.3) and Lemma 4.1, \( \delta = \chi_l | M \) if and only if
\[ \delta \left( \exp \frac{2\pi i H_{\gamma_l}}{(\gamma_l, \gamma_l)} \right) = \chi_l \left( \exp \frac{2\pi i H_{\gamma_l}}{(\gamma_l, \gamma_l)} \right) . \]

By (5.1) the right hand side of (7.3) is \((-1)^l\). Comparing with (7.2) we see that \( \delta = \chi_l | M \) if and only if \( l \) has the same parity as \( m_1, \ldots, m_r \). In Case II, \( M \) is connected, and since both \( \delta \) and \( \chi_l \) are trivial on \( M \cap K_1 \) it follows from Lemma 4.3 that \( \delta = \chi_l | M \) if and only if
\[ \delta(\exp t X) = \chi_l(\exp t X) \quad \text{for all } t \in \mathbb{R} . \]

However \( \delta(\exp t X) = e^{im \theta} \) and \( \chi_l(\exp t X) = e^{it \theta} \) by Lemma 4.3 (iv). Therefore \( \delta = \chi_l | M \) if and only if \( l = m_0 \).

Assume now that \( l \) is such that \( \delta = \chi_l | M \). As mentioned \( \chi_l \) then occurs in \( l_{\delta, \nu}^P \).

In fact, if we define for \( g \in G \)
\[ f_l(g) = e^{-(\nu + \epsilon \delta(H_l(g)))} \chi_l(g)^{-1} , \]
then \( f_l \in C_{\delta, \nu}^P \) and \( f_l(k^{-1}g) = \chi_l(k) f_l(g) \) for \( k \in K, \; g \in G \), so \( f_l \) generates the \( K \)-type \( \chi_l \) in \( l_{\delta, \nu}^P \). Therefore \( \pi \) contains \( \chi_l \) if and only if \( Af_l \neq 0 \).

From Iwasawa decomposition \( G = KA \hat{N} \) it follows that \( Af_l \neq 0 \) if and only if \( Af_l(v) \neq 0 \), i.e. if and only if
\[(7.4) \quad \int_{\bar{N}} e^{-(v+\varphi(H(l)) \chi_l(\bar{\bar{n}}))^{-1} d\bar{n} \neq 0}.
\]

Note that if \(l = 0\), (7.4) is obvious. This implies that \(\pi\) contains the trivial \(K\)-type \(\chi_0\) if and only if \(\delta\) is trivial (this is the main step in the proof of Helgason’s theorem, cf. [2, III Corollary 3.8]).

By the method of Gindikin and Karpelević (see [9, Proof of Theorem 8.10.16]) the problem of proving (7.4) is reduced to the real rank-one case. Thus (7.4) holds if and only if

\[(7.5) \quad \int_{N'} e^{-(v+\varphi(H(l)) \chi_l(\bar{n})}^{-1} d\bar{n} \neq 0 \]

for all \(\bar{\bar{\alpha}} \in \Sigma^+ \setminus 2\Sigma^+\), where \(\bar{\bar{N}} = G^\alpha \cap N\).

When \(K^\alpha\) is semisimple (7.5) is clear, so we may assume that \(g^\alpha = so(n,1)\) (cf. Lemma 5.1).

In Case I, we have \(g^\alpha = su(n,1)\) and \(\chi_l|_{K^\alpha}\) is determined by (5.1). If \(\alpha = \frac{1}{2}(\alpha_i - \alpha_j)\), then \(\chi_l|_{K^\alpha}\) is trivial and (7.5) is obvious. If \(\alpha = \alpha_j\), then by Lemma 6.1 we get that (7.5) holds precisely when

\[(7.6) \quad \|l\| \not\equiv \frac{2(v,\alpha)}{\langle \alpha, \alpha \rangle} + 1 + 2\mathbb{Z}_+ = m_i + 2\mathbb{N}.
\]

(It is easily seen that the conclusion of Lemma 6.1 also holds for any group covered by \(SU(n,1)\), as long as \(\chi_l\) is well defined on this group.) Finally, if \(\alpha = \frac{1}{2}(\alpha_i + \alpha_j)\) we get that (7.5) holds when

\[(7.7) \quad \|2l\| \not\equiv \frac{2(v,\alpha)}{\langle \alpha, \alpha \rangle} + 1 + 2\mathbb{Z}_+ = m_i + m_j + 2\mathbb{N}.
\]

By Proposition 7.1 (i), we see that (7.6) and (7.7) holds precisely when \(\|l\| \leq m_1\), and thus the theorem follows in Case I.

In Case II we have \(\alpha = \frac{1}{2} \alpha_j\) and \(\chi_l|_{K^\alpha}\) is determined by (5.3). From Lemma 6.1 we get that (7.5) holds when

\[\|l\| \leq \frac{2(\beta_j,\alpha_j)}{\langle \alpha_j, \alpha_j \rangle} = m_j.
\]

However by our assumption that \(\delta = \chi|_M\) we have \(l = m_0\), and therefore (7.5) holds by Proposition 7.1 (i).

**Remark 7.3.** From the proof of Lemma 6.1 it follows that the integral

\[c(v, l) = \int_{\bar{N}} e^{-(v+\varphi(H(l)) \chi_l(\bar{n})^{-1} d\bar{n}}
\]
for $G = \text{SU}(n, 1)$ takes the following value

$$
(7.8) \quad \frac{2^{n-k} \Gamma(n) \Gamma(k)}{\Gamma(\frac{1}{2}(k+n+l)) \Gamma(\frac{1}{2}(k+n-l))}.
$$

Here $\tilde{d}^i$ is so normalized that $c(g, 0)$ equals one. From the proof of Theorem 7.2 it then follows that $c(v, l)$ for arbitrary $G$ can be given an explicit formula as product of expressions like (7.8) and the usual factors in the product formula for the $c$-function (cf. [1]).

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