ON C*-ALGEBRA EXTENSIONS RELATIVE TO A FACTOR OF TYPE II$_\infty$

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Abstract.

The commutative semigroup of strong equivalence classes of unital extensions of the norm-closed two-sided ideal of an infinite semifinite countably decomposable factor by a separable unital C*-algebra which is the direct limit of a sequence of C*-algebras with continuous trace is a group. The identity of this group is the class of any trivial unital extension. In the case that the quotient C*-algebra is the direct limit of a sequence of finite-dimensional C*-algebras, and the factor is of type II, this group consists of a single element.

1. Introduction.

Let $M$ be an infinite semifinite countably decomposable factor and denote by $I$ the norm closure of the ideal of elements of finite rank. Let $A$ be a separable unital C*-algebra. In [5], Brown, Douglas, and Fillmore introduced the commutative semigroup $\text{Ext}_s(A,I)$ of strong equivalence classes of unital (essential) extensions of $I$ by $A$. This consists of the unital embeddings of $A$ into $M/I$, up to unitary equivalence by the image in $M/I$ of a unitary in $M$, with the sum of extensions $\tau_1$ and $\tau_2$ defined as the class of $u_1\tau_1 u_1^* + u_2\tau_2 u_2^*$, where $u_1$ and $u_2$ are the images in $M/I$ of isometries in $M$ whose range projections are orthogonal with sum 1. If a unital embedding of $A$ into $M/I$ is the image of a unital embedding of $A$ into $M$, we shall say that it is trivial.

Brown, Douglas, and Fillmore showed that $\text{Ext}_s(A,I)$ is a group, with identity the class of any trivial unital extension, if $M$ is of type I and $A$ is commutative. If $M$ is of type II, this was shown by Fillmore in [7].

In the case that $M$ is of type I, it was shown by Voiculescu in [12] that for arbitrary (separable) $A$ the class of a trivial unital extension is unique, and is an identity for the semigroup $\text{Ext}_s(A,I)$. It was shown by Choi and Effros in [6], using an idea of Arveson in [1], that if $A$ is nuclear then $\text{Ext}_s(A,I)$ is a group. For a unified exposition of these results, see [2].

While we cannot extend the result of Voiculescu in full generality to the case

Received May 29, 1980; in revised form October 5, 1982.
that $M$ is of type II, we can take a step beyond the result of Fillmore, by dealing with C*-algebras with continuous trace (not just commutative ones), and also with direct limits of sequences of such C*-algebras.

The result of Arveson, Choi, and Effros holds with the same proof in the case that $M$ is of type II. In the absence of the analogue of Voiculescu's theorem, the result must be stated as follows: if $A$ is nuclear then for any unital extension there is another unital extension such that the sum is trivial. This of course says that $\text{Ext}_s(A, I)$ modulo the trivial classes is a group.

We can compute the group $\text{Ext}_s(A, I)$ if $A$ is the direct limit of a sequence of finite-dimensional C*-algebras. In the case that $M$ is of type I, this was done by Pimsner and Popa in [10] and Pimsner in [11] (for a different generalization of this see [8]); the answer is $\text{Ext}(K_0 A / Z, Z)$, where $Z$ is embedded in $K_0 A$ as multiples of the class [1] of the unit of $A$. In the case that $M$ is of type II, the group is zero.

**2. Theorem.** Let $A$ be the C*-algebra direct limit of a unital sequence $A_1 \to A_2 \to \ldots$ of unital C*-algebras with continuous trace. Then there is a unique element of $\text{Ext}_s(A, I)$ which is the class of a trivial extension, and this element is an identity for the semigroup $\text{Ext}_s(A, I)$.

**Proof.** We must show that if $\sigma \in \text{Ext}_s(A, I)$, and if $\tau \in \text{Ext}_s(A, I)$ is the class of a trivial extension, then

$$\sigma + \tau = \sigma .$$

It is sufficient to prove that if $\sigma$ and $\tau$ are as above, then there exists $\sigma' \in \text{Ext}_s(A, I)$ such that

$$\sigma = \sigma' + \tau .$$

Indeed, then, as in [12], we may take in place of $\tau$ a trivial extension $\tau'$ for which

$$\tau' + \tau = \tau' ,$$

and then

$$\sigma + \tau = (\sigma' + \tau') + \tau = \sigma' + (\tau' + \tau) = \sigma' + \tau' = \sigma .$$

($\tau'$ may be taken, as in [12], to be the class of the extension defined by the orthogonal sum of infinitely many copies of a unital splitting of $\tau$, i.e., the sum $u_1 \pi u_1^* + u_2 \pi u_2^* + \ldots$ where $\pi$ is a unital splitting of $\tau$ and $u_1, u_2, \ldots$ are isometries in $M$ with $u_1 u_1^* + u_2 u_2^* + \ldots = 1$.)

In other words, it is sufficient to prove that if $\sigma$ and $\tau$ are unital embeddings of $A$ in $M/I$, with $\tau$ trivial, then there exists a proper isometry $v \in M/I$ such that $vv^*$ commutes with $\sigma(A)$, and
We shall begin by establishing two basic facts about extensions of $I$ by $A_1$ (instead of $A$).

First, in the language of [3] (Definition 6.6), every point of $\hat{A}_1$ is pull-out-able. In the case that $M$ is of type II, we define this as follows. If $\lambda \in \hat{A}_1$, that is, $\lambda$ is the unitary equivalence class of a morphism of $A_1$ onto $M_n\mathbb{C}$ for some $n = 1, 2, \ldots$, embed $M_n\mathbb{C}$ unitarily in $M/I$ and consider the composed morphism $A_1 \to M/I$. Note that, as $M$ is of type II, any two unital embeddings of $M_n\mathbb{C}$ in $M/I$ are strongly equivalent. The strong equivalence class of the morphism $A_1 \to M/I$ therefore depends only on $\lambda$; denote this class by $\lambda_M$. We shall say that $\lambda$ is pull-out-able (with respect to $M$) if for any $\sigma \in \text{Ext}_s(A_1, I)$, $\sigma + \lambda_M = \sigma$.

The proof that any $\lambda \in \hat{A}_1$ is pull-out-able in the case that $M$ is of type II is similar to the proof in the case that $M$ is of type I, given in the first half of the paragraph following 6.7 of [3]. This proof is designed for weak equivalence rather than strong, but is easily modified for strong (and, furthermore, by the remark preceding 5.3 of [3], and in view of 6.8d of [3], does not even need to be so modified). We note that the case $A_1$ is commutative, to which the general case is reduced, is now to be deduced from 2.9 of [7], rather than [5].

The second basic fact about $A_1$ is as follows. For any unital embedding $\pi: A_1 \to M$, any finite projection $f$ in $M$, any finite subset $S$ of $A_1$, and any $\varepsilon > 0$, there exist a finite projection $g$ in $M$ with $g \geq f$, a finite-dimensional sub-$C^*$-algebra $C$ of $gMg$ containing $g$, and a unital morphism $\varrho: \pi(A_1) \to C$ such that, for each $a \in \pi(S)$,

$$\|ga - ag\| < \varepsilon,$$

$$\|gag - \varrho(a)\| < \varepsilon.$$
dimensional commutative sub-C*-algebra $C_2$ of $gMg$ commuting with $gD_1$, and a surjective morphism $\varrho_2: D_2 \to C_2$ such that, for each $d \in S_2$,

$$||gd - dg|| < \varepsilon_1,$$

$$||gdg - \varrho_2(d)|| < \varepsilon_1.$$  

Since $h$ is approximately contained in a finite-dimensional sub-C*-algebra of $D_2$, it follows that $h$ is approximately contained in a finite-dimensional sub-C*-algebra of $D$ containing $D_1$. Denote by $g$ the smallest projection in $M$ containing $f$ and commuting with this finite-dimensional algebra. Then $g$ is finite, and $g$ approximately commutes with $h$. In particular, since the elements of $S_2$ are approximately polynomials in $h$, $g$ approximately commutes with $S_2$. Furthermore, since $ghg + (1 - g)h(1 - g)$ is close to $h$, each point of the spectrum of $ghg$ is close to some point in the spectrum of $h$. Therefore, $ghg$ is close to a selfadjoint element $h'$ of $(ghg)'$ such that the spectrum of $h'$ is finite and is contained in the spectrum of $h$. This defines a surjective morphism $\varrho_2$ from $D_2$ onto a finite-dimensional subalgebra $C_2$ of $(ghg)'$, a subalgebra commuting with $gD_1$ since $ghg$ does, such that $gdg$ is close to $\varrho_2(d)$ for all $d \in S_2$. Thus, if the commutative finite-dimensional subalgebra of $D_2$ chosen above approximates $h$ sufficiently well, we have $||gd - dg|| < \varepsilon_1$, $||gdg - \varrho_2(d)|| < \varepsilon_1$ for all $d \in S_2$. In particular, if a central projection $e$ of $D_1$ belongs to $S_2$, and $\varepsilon_1 \leq 1$, then $g$ and $e$ both commute with $\varrho_2(e)$, and $||ge - \varrho_2(e)|| < 1$, so $\varrho_2(e) = ge$.

It follows immediately from what was shown in the preceding paragraph that for any finite projection $f$ in $M$, any finite subset $S$ of $D$, and any $\varepsilon > 0$, there exist a finite projection $g$ in $M$ with $g \geq f$, a finite-dimensional sub-C*-algebra $C$ of $gMg$ containing $g$, and a surjective morphism $\varrho: D \to C$ such that $||ga - ag|| < \varepsilon$, $||gag - \varrho(a)|| < \varepsilon$ for all $a \in S$. Just choose $g$ as above, with $\varepsilon_1$ sufficiently small, and with $S_2$ such that $S$ is contained in the algebra generated by $S_2$ and $D_1$, and such that each central projection of $D_1$ belongs to $S_2$, and take for $C$ the subalgebra generated by $C_2$ defined above and by $gD_1$, and for $\varrho$ the unique extension of $\varrho_2$ from $D_2$ to $D$ which on $D_1$ coincides with multiplication by $g$. (Note that if $\varepsilon_1 < 1$ then $\varrho_2(e) = ge$ for all $e \in D_2 \cap D_1$.)

This establishes the two basic facts about $A_1$ (or $A_2$, or $A_3$, . . .) that we shall need. Now let $\sigma$ and $\tau$ be unital embeddings of $A$ in $M/I$, with $\tau$ trivial, and let $\pi$ be a unital splitting of $\tau$. Denote the preimage of $\sigma(A)$ in $M$ by $B$, choose a dense sequence $(a_k)$ in $A$, and choose a sequence $(b_k)$ in $B$ with $b_k + I = \sigma(a_k)$.

By the second basic fact established above, applied to $A_n, A_n, \ldots$ and $S_1 \subseteq A_n, S_2 \subseteq A_n, \ldots$ where $S_k$ is a finite set containing elements strictly within distance $2^{-k}$ of $a_1, \ldots, a_k$, $k = 1, 2, \ldots$, there exist an increasing sequence

$$g_1 \leq g_2 \leq \ldots$$
of finite projections in $M$, with supremum 1, finite-dimensional sub-$C^*$-algebras $C_1 \subseteq g_1 M g_1, C_2 \subseteq g_2 M g_2, \ldots$, with $g_1 \in C_1, g_2 \in C_2, \ldots$, and surjective morphisms $\varrho_1: A_{n_1} \to C_1, \varrho_2: A_{n_2} \to C_2, \ldots$ such that, for all $a \in \pi(S_k)$, 

$$\|g_k a - a g_k\| < 2^{-k},$$

$$\|g_k a g_k - \varrho_k \pi^{-1}(a)\| < k^{-1}.$$  

We remark that we do not need that the sequence $A_1, A_2, \ldots$ is increasing; the only assumption we need on the separable unital $C^*$-algebra $A$ is that any finite subset can be approximated by elements of some unital sub-$C^*$-algebra with continuous trace. Thus, the class of separable unital $C^*$-algebras to which $A$ may belong is closed under taking direct limits. 

By the first basic fact established above, for each $k = 1, 2, \ldots$ there exist an infinite projection $p_k$ in $M$ commuting modulo $I$ with the preimage $B_{n_k}$ in $M$ of $\sigma(A_{n_k}) \subseteq M/I$, a finite-dimensional sub-$C^*$-algebra $\bar{C}_k$ of $p_k M p_k$ containing $p_k$, with $\bar{C}_k \cap I = 0$, and an isomorphism $\theta_k: C_k \to \bar{C}_k$ such that for all $b \in B_{n_k}$,

$$p_k b p_k - \theta_k \varrho_k \sigma^{-1}(b + I) \in I.$$ 

It follows that there exists an orthogonal sequence $(f_1, f_2, \ldots)$ of finite projections in $M$ (with $f_k \subseteq p_k$) such that $1 - \sum_1^\infty f_k$ is infinite in $M$, $f_k$ commutes with $\bar{C}_k$, the map $f_k \theta_k: C_k \to f_k \bar{C}_k$ is determined by a partial isometry $u_k$ in $M$, i.e.,

$$u_k^* u_k = g_k, \quad u_k u_k^* = f_k,$$

$$u_k^* \theta_k(c) u_k = c, \quad c \in C_k,$$

and for each $b$ belonging to a fixed finite subset of $B$ which maps onto $\sigma(S_k) \subseteq \sigma(A_{n_k})$ and contains elements strictly within distance $2^{-k}$ of $b_1, \ldots, b_k$ (recall that $b_i$ maps onto $\sigma(a_i)$),

$$f_k b f_j = f_j b f_k = 0, \quad 1 \leq j < k,$$

$$\|f_k b f_k - f_k \theta_k \varrho_k \sigma^{-1}(b + I) f_k\| < k^{-1}.$$ 

We now have the following chain of inequalities, whenever $1 \leq i \leq k$, where $b \in B_{n_k}$ is chosen as above close to $b_i$:

$$\|u_k^* b_i u_k - u_k^* b u_k\| < 2^{-k},$$

$$\|u_k^* b u_k - u_k^* \theta_k \varrho_k \sigma^{-1}(b + I) u_k\| < k^{-1},$$

$$u_k^* \theta_k \varrho_k \sigma^{-1}(b + I) u_k = \varrho_k \sigma^{-1}(b + I),$$

$$\|\varrho_k \sigma^{-1}(b + I) - g_k \pi(\sigma^{-1}(b + I)) g_k\| < k^{-1},$$

$$\|g_k a - a g_k\| < 2^{-k},$$

$$\|g_k a g_k - \varrho_k \pi^{-1}(a)\| < k^{-1}.$$
\[ \| g_k \pi(\sigma^{-1}(b_i + I)) g_k - g_k \pi(\sigma^{-1}(b_i + I)) g_k \| < 2^{-k}, \]

\[ \sigma^{-1}(b_i + I) = a_i. \]

Hence by the triangle inequality, for \( 1 \leq i \leq k, \)

\[ \| u_k^* b_i u_k - g_k \pi(a_i) g_k \| < 4k^{-1}. \]

We also have, for \( 1 \leq i \leq k, \)

\[ \| g_k \pi(a_i) - \pi(a_i) g_k \| < 3 \cdot 2^{-k} < 2^{-k+2}, \]

\[ \| f_k b_i f_j \| < 2^{-k}, \quad \| f_j b_i f_k \| < 2^{-k}, \quad 1 \leq j < k. \]

Finally, set \( g_k - g_{k-1} = e_k, \quad k = 1, 2, \ldots, \) where \( g_0 = 0. \) Then \( \sum_{k=1}^{\infty} e_k = 1 \) in \( M, \) and if \( 1 \leq i < k, \) setting \( u_k e_k = v_k \) we have

\[ \| v_k^* b_i v_k - e_k \pi(a_i) e_k \| < 4k^{-1}, \]

\[ \| e_k \pi(a_i) - \pi(a_i) e_k \| < 2^{-k+3} + 2^{-k+2} < 2^{-k+4}, \]

\[ \| v_k^* b_i v_k \| < 2^{-k}, \quad \| v_j^* b_i v_k \| < 2^{-k}, \quad 1 \leq j < k. \]

Moreover, \( \sum_{k=1}^{\infty} v_k \) is an isometry in \( M, \) with cokernel \( 1 - \sum_{k=1}^{\infty} v_k v_k^* \) containing \( 1 - \sum_{k=1}^{\infty} f_k \) which is infinite. Denote the image of \( \sum_{k=1}^{\infty} v_k \) in \( M/I \) by \( v. \) Then \( v \) is a proper isometry in \( M/I, \) and for each \( i = 1, 2, \ldots, \)

\[ v^* \sigma(a_i) v = v^* (b_i + I) v = \left( \sum_{k=1}^{\infty} v_k^* b_i v_k \right) + I = \left( \sum_{k=1}^{\infty} e_k \pi(a_i) e_k \right) + I = \left( \sum_{k=1}^{\infty} e_k \pi(a_i) \right) + I = \pi(a_i) + I = \tau(a_i). \]

By continuity,

\[ v^* \sigma(a) v = \tau(a), \quad a \in A. \]

This implies in particular that \( vv^* \) commutes with \( \sigma(A) \) (see the proof of Corollary 1, page 338 of [2]), and so the proof of the Theorem is complete.

3. Theorem. Let \( A \) be a separable unital nuclear \( C^* \)-algebra. Then for any \( \tau \in \text{Ext}_s(A, I) \) there exists \( \tau' \in \text{Ext}_s(A, I) \) such that \( \tau + \tau' \) is trivial.
Proof. This follows from Corollary 3.11 of [6] (see also the Corollary of Theorem 7 of [2]) applied with \( B = M \) and \( J = I \), by using the idea of [1], as in the case that \( M \) is of type I.

4. Corollary. Let \( A \) be as in 2. Then \( \text{Ext}_r(A, I) \) is a group.

Proof. This follows from 2 and 3.

5. Remark. The methods used above, slightly modified, yield analogues of 2 and 4 for extensions of \( I \) by a nonunital separable \( C^* \)-algebra \( A \) which is the direct limit of \( C^* \)-algebras with continuous trace.

In this case, by 3.4 of [5], strong equivalence coincides with weak equivalence, and hence by 3.15 of [5], we may suppose that \( A \) is stable. Then \( A \) is the direct limit of stable \( C^* \)-algebras with continuous trace. Since a stable \( C^* \)-algebra with continuous trace and with totally disconnected spectrum is trivial, and is hence a direct limit of unital \( C^* \)-algebras with continuous trace, the proof of 2 is applicable with only minor modifications.

6. Theorem. Let \( A \) be the \( C^* \)-algebra direct limit of a unital sequence \( A_1 \rightarrow A_2 \rightarrow \ldots \) of finite-dimensional \( C^* \)-algebras, and suppose that \( M \) is of type II. Then every unital extension of \( I \) by \( A \) is trivial.

Proof. It would be enough to show that if \( 1 \in A_1 \subseteq A_2 \subseteq M/I \), and \( 1 \in B_1 \subseteq M \) maps isomorphically onto \( A_1 \) by the quotient map \( M \rightarrow M/I \), then there exists \( B_2 \subseteq M \) with \( B_1 \subseteq B_2 \), mapping isomorphically onto \( A_2 \). (With \( 1 \in A_1 \subseteq A_2 \subseteq \ldots \subseteq A \subseteq M/I \), one could just choose successively \( B_1 \subseteq B_2 \subseteq \ldots \subseteq M \) mapping isomorphically onto \( A_1 \subseteq A_2 \subseteq \ldots \)) If the finite-dimensional \( C^* \)-algebra \( A \), is not simple, however, this is not true.

The proof seems to require a less direct approach, using \( K \)-theory. The argument consists of five steps. Let \( I \rightarrow B \rightarrow A \) be an essential unital extension of \( I \) by \( A \).

Step 1. The induced sequence \( K_0 I \rightarrow K_0 B \rightarrow K_0 A \) defines an abelian group extension of the group \( K_0 I (= \mathbb{R}) \) by the group \( K_0 A \).

This can easily be shown directly, using that any projection in \( M/I \) is the image of a projection in \( M \) (Theorem 3.2 of [13]), so that any projection in \( A \) is the image of a projection in \( B \). What is needed — namely, that the sequence of abelian groups

\[ 0 \rightarrow K_0 I \rightarrow K_0 B \rightarrow K_0 A \rightarrow 0 \]
is exact — can also be seen, as Brown has pointed out in the different situation in which $I$ is an approximately finite-dimensional C*-algebra ([4]), by observing that in the six-term exact sequence

$$
K_0I \rightarrow K_0B \rightarrow K_0A \\
\uparrow \quad \quad \downarrow \\
K_1A \leftarrow K_1B \leftarrow K_1I
$$

of Bott periodicity, the groups $K_1A$ and $K_1I$ are zero.

**Step 2.** The abelian group extension $K_0I \rightarrow K_0B \rightarrow K_0A$ defined above splits.

To see this note that the subgroup $K_0I$ (= R) of $K_0B$ is divisible. It is then by Theorem 2 of [9] a direct summand of $K_0B$.

**Step 3.** There exists a splitting map $K_0A \rightarrow K_0B$ for the abelian group extension $K_0I \rightarrow K_0B \rightarrow K_0A$ which takes [1] into [1] (i.e., takes the class of the unit of $A$ into the class of the unit of $B$).

To see this, first choose by Step 2 some splitting map $K_0A \rightarrow K_0B$. To get a different splitting we must add a map from $K_0A$ to $K_0I$, and it is sufficient for us to show that there is such a map which is nonzero on $[1] \in K_0A$. (Since $K_0I$ = R it follows that there is such a map which is arbitrary on this element.) The existence of such a map $K_0A \rightarrow K_0I$ follows from the facts that $K_0I$ (= R) is divisible and that $K_0A$ is torsion free. (A maximal additive extension of a nonzero map from $Z \subseteq K_0A$ into $K_0I$ must then be defined on all of $K_0A$.)

**Step 4.** Any splitting map $K_0A \rightarrow K_0B$ for the abelian group extension $K_0I \rightarrow K_0B \rightarrow K_0A$ is positive with respect to the natural preorder in $K_0A$ and $K_0B$.

It is sufficient to show that if $h$ is an element of $K_0B$ such that the image of $h$ in $K_0A$ is nonzero and positive, then $h$ is positive in $K_0B$. After tensoring with a suitable full matrix algebra of finite order, what we must show is that if $e$ and $f$ are orthogonal projections in $B$ with $f \in I$ and $e \notin I$, then there exists a projection $f' \in I$ equivalent to $f$ with $f' \leq e$. That this is true for $B$ follows, as $B \subseteq M$, from the fact that it is true for $M$.

**Step 5.** Let $\varphi: K_0A \rightarrow K_0B$ be a splitting of $K_0I \rightarrow K_0B \rightarrow K_0A$ which is unital, i.e., takes [1] into [1]. Then there is a unital splitting $A \rightarrow B$ of the extension of C*-algebras $I \rightarrow B \rightarrow A$ which induces $\varphi$.

Denote by
the composition of $\phi$ with the induced map $K_0 A_n \to K_0 A$ (which may not be injective). It will suffice to show that if $n = 1, 2, \ldots$ and if

$$\psi_{n-1} : A_{n-1} \to B$$

is a unital morphism for which the given map $B \to A$ is a left inverse, where $A_0 = C \subseteq A$, and such that the induced map $K_0 A_{n-1} \to K_0 B$ is $\varphi_{n-1}$, where $\varphi_0[1] = [1]$, then there is an extension of $\psi_{n-1}$ to a morphism

$$\psi_n : A_n \to B$$

for which the map $B \to A$ is a left inverse, and such that the induced map $K_0 A_n \to K_0 B$ is $\varphi_n$. The closure of the common extension of $\psi_1, \psi_2, \ldots$ is then a unital splitting $A \to B$, inducing $\varphi : K_0 A \to K_0 B$.

It is enough to consider the case $n = 2$. Choose a maximal orthogonal set $S_1$ of minimal projections in $A_1$, and a maximal orthogonal set $S_2$ of minimal projections in $A_2$, such that every element of $S_2$ lies inside some element of $S_1$. Fix $p \in S_1$. Choose a morphism $\psi$ from $p A_2 p$ to $\psi_1(p) M \psi_1(p)$ for which the map $M \to M/I$ is a left inverse; necessarily, $\psi(p A_2 p) \subseteq B$. For each $q \in S_2$ with $q \leq p$, the element $\varphi_2[q] - [\psi q]$ of $K_0 B$ belongs to the kernel of $K_0 B \to K_0 A$ and hence, by Step 1, to the image of $K_0 I \to K_0 B$. Since $K_0 I = R$, each such $\varphi_2[q] - [\psi q]$ is either positive or negative. If $\varphi_2[q] - [\psi q]$ is negative, then, since $\psi q \in M \setminus I$, $\psi q$ may be increased by a projection in $I$ so that $\varphi_2[q] - [\psi q] = 0$. Furthermore, this may be done simultaneously for all the projections in $S_2$ equivalent in $A_2$ to $q$, in such a way that $\psi$ is still a morphism. We may therefore suppose that $\varphi_2[q] - [\psi q]$ is positive for each $q \in S_2$ with $q \leq p$. Since

$$[\psi_1 p] = \varphi_1[p] = \varphi_2[p] = \sum_{q \in S_2, q \leq p} \varphi_2[q],$$

we have the decomposition

$$\left[\psi_1 p - \sum_{q \in S_2, q \leq p} \psi q\right] = \sum_{q \in S_2, q \leq p} (\varphi_2[q] - [\psi q]),$$

in the positive part of the image of $K_0 I \to K_0 B$, and since any positive element of this image is the image of the class in $K_0 I$ of a projection in $I$, it follows that $\psi q$ may be increased by a projection in $I$, for each $q \in S_2$ with $q \leq p$, with class $\varphi_2[q] - [\psi q]$ in the image of $K_0 I \to K_0 B$, in such a way that

$$\sum_{q \in S_2, q \leq p} \psi q = \psi_1 p.$$

Moreover, since the change is by projections of the same dimension in $I$ for
different $q \in S_2$ which are equivalent to $q$ in $A_2$, also the partial isometries between different $\psi q$'s may be extended so that $\psi$ is still a morphism. We now have

$$\varphi_2[q] - [\psi q] = 0, \quad q \in S_2, q \preceq p.\]$$

Carry out the construction of $\psi$ as above for each $p \in S_1$. Then we have a morphism

$$\psi: \sum_{p \in S_1} pA_2 p \rightarrow B$$

such that for each $p \in S_1$,

$$\sum_{q \in S_2, q \preceq p} \psi q = \psi_1 p,$$

and for each $q \in S_2$,

$$\varphi_2[q] - [\psi q] = 0.$$

It is now straightforward to construct a common extension of $\psi$ and of $\phi_1$ to a morphism $\phi_2$ of all of $A_2$ into $B$, necessarily inducing $\varphi_2: K_0 A_2 \rightarrow K_0 B$. We make only the following remarks. If $p_1$ and $p_2$ are distinct elements of $S_1$ which are equivalent in $A_1$ then a common extension of $\psi$ and of $\phi_1$ to $(p_1 + p_2)A_2(p_1 + p_2)$ is unique. If $p_1$ and $p_2$ are elements of $S_1$ which are not equivalent in $A_1$, and if $q_1$ and $q_2$ are elements of $S_2$ with $q_1 \preceq p_1$ and $q_2 \preceq p_2$, such that $q_1$ and $q_2$ are equivalent in $A_2$, then

$$[\psi q_1] = \varphi_2[q_1] = \varphi_2[q_2] = [\psi q_2].$$

Hence if $v$ is a partial isometry in $A_2$ with $v^* v = q_1$ and $v v^* = q_2$, and $u$ is any partial isometry in the preimage of $v$ in $\psi(q_1)M\psi(q_2)$, then

$$[\psi q_1 - u^* u] = [\psi q_1] - [u^* u] = [\psi q_2] - [u u^*] = [\psi q_2 - u u^*],$$

and so $u$ may be extended by adding a partial isometry in $I$ in such a way that $u^* u = \psi q_1$, $u u^* = \psi q_2$.

7. Corollary. Let $A$ be the C*-algebra direct limit of a unital sequence $A_1 \rightarrow A_2 \rightarrow \ldots$ of finite-dimensional C*-algebras, and suppose that $M$ is of type II. Then Ext$_s(A, I)$ has only one element.

Proof. This follows from 2 and 6.

Acknowledgements. The second author would like to thank the members of the Mathematics Institute at the University of Copenhagen, especially E.
Christensen, G. A. Elliott, H. P. Jakobsen, E. Kehlet, D. Olesen, and G. K. Pedersen, for their warm hospitality.

We are indebted to A. Kishimoto for a critical reading of the manuscript.

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