SOME SPECIAL $W$-ALMOST PERIODIC SETS
ON THE INTEGERS

ERLING FØLNER

We begin by specializing, starting from an arbitrary topological group and passing to the discrete abelian group $\mathbb{Z}$ of the integers, some definitions and theorems concerning $W$-almost periodic ($W$-ap) functions on the group. For this we refer mainly to [1] and [2, I].

Definition. A bounded complex-valued function $f(n)$ on $\mathbb{Z}$ is called a $W$-almost periodic function on $\mathbb{Z}$ if to every $\varepsilon > 0$ there exists a trigonometric polynomial on $\mathbb{Z}$

$$s(n) = \sum_{p=1}^{P} c_p e^{i\gamma_p n}$$

where the $c_p$ are complex numbers and the $\gamma_p$ are real numbers, determined modulo $2\pi$, such that

$$\| f(n) - s(n) \|_W = \tilde{M} |f(n) - s(n)| < \varepsilon .$$

Here for a bounded real function $g(n)$ on $\mathbb{Z}$

$$\tilde{M}g(n) = \inf_A \sup_n \sum_{q=1}^{Q} \alpha_q g(n + n_q) ,$$

where

$$A = \{ \alpha_1, \ldots, \alpha_Q; n_1, \ldots, n_Q \}, \quad \alpha_q > 0, \quad \sum_{q=1}^{Q} = 1, \quad n_q \in \mathbb{Z} .$$

A set $B$ on $\mathbb{Z}$ is called $W$-almost periodic on $\mathbb{Z}$ if its characteristic function $B(n)$ is $W$-ap.

Analogously to (1) we put

$$Mg(n) = \sup_A \inf_n \sum_{q=1}^{Q} \alpha_q g(n + n_q) .$$

Received September 3, 1982; in revised form November 8, 1982.
If $\bar{M}g(n) = Mg(n)$ we denote this number by $Mg(n)$. For a complex-valued bounded function $f(n) = g(n) + ih(n)$ we put

$$Mf(n) = Mg(n) + iMh(n)$$

if $Mg(n)$ and $Mh(n)$ exist.

We omit the proof of the simple fact that if $Mf(n)$ exists, then

$$\frac{1}{N} \sum_{n=1}^{N} f(n) \to Mf(n) \quad \text{for } N \to \infty .$$

The sum and the product of two $W$-ap functions are $W$-ap, and the numerical value of a $W$-ap function is $W$-ap. For a $W$-ap function $f(n)$ the mean value $Mf(n)$ exists.

To a real $W$-ap function $f(n)$ we ascribe a Fourier series,

$$f(n) \sim \frac{1}{2}a_0 + \sum_{r=1}^{\infty} \{a_r \cos \gamma_r n + b_r \sin \gamma_r n\}$$

where

$$a_r = 2M \{f(n) \cos \gamma_r n\}, \quad b_r = 2M \{f(n) \sin \gamma_r n\}$$

and we consider at least the most denumerably many $\gamma_r$, determined modulo $2\pi$, for which $a_r$ or $b_r$ are different from zero.

The Fourier series $W$-converges to $f(n)$, that is

$$M\left|f(n) - \frac{1}{2}a_0 - \sum_{r=1}^{R} \{a_r \cos \gamma_r n + b_r \sin \gamma_r n\}\right| \to 0 \quad \text{for } R \to \infty .$$

**Main Theorem.** A bounded function $f(n)$ on $\mathbb{Z}$ is $W$-ap on $\mathbb{Z}$ if and only if to every $\varepsilon > 0$ there exists an

$$A = \{\alpha_1, \ldots, \alpha_Q; n_1, \ldots, n_Q\}, \quad \alpha_q > 0, \quad \sum_{q=1}^{Q} \alpha_q = 1, \quad n_q \in \mathbb{Z}$$

and finitely many integers $m_s$ such that to every integer $m$ there exists an $m_s$ with

$$\|f(n+m) - f(n+m_s)\|_{S_A} = \sup_{n} \sum_{q=1}^{Q} \alpha_q |f(n+n_q + m) - f(n+n_q + m_s)| < \varepsilon .$$

If $X_1, X_2, \ldots$ are at most denumerably many characters on $\mathbb{Z}$, then by $S(X_1, X_2, \ldots)$ we understand the at most denumerably many characters which are finite products of some, arbitrarily chosen, of the characters $X_1, \bar{X}_1, X_2, \bar{X}_2, \ldots$. Then we have
Theorem 2. Let \( f(n) \) be a real \( W \)-ap function on \( \mathbb{Z} \). For real \( \beta \) we consider on \( \mathbb{Z} \) the set

\[
A_\beta = [f(n) \leq \beta].
\]

With exception of the at most denumerably many \( \beta \) where

\[
\psi(\beta) = \sim M A_\beta
\]

makes a jump, \( A_\beta \) is a \( W \)-ap set and

\[
M\{[f(n) = \beta]\} = 0.
\]

If \( X_1, X_2, \ldots \) are the Fourier characters of \( f(n) \), the Fourier characters of \( A_\beta \) are in \( S(X_1, X_2, \ldots) \).

As the reader will see, this can easily be deduced by the method put forward in [4, p. 78], confer [2, I, p. 34]. For the more general result, see [2, I, p. 60].

Now let \( \alpha \) be an arbitrary fixed irrational number. Then for every integer \( h \neq 0 \)

\[
e^{2\pi i h n}
\]

is a character on \( \mathbb{Z} \) which is not the main character 1. Hence from (3)

\[
\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h n} \rightarrow M e^{2\pi i h n} = 0 \quad \text{for } N \rightarrow \infty.
\]

From this Weyl showed, see [3] for general results concerning the asymptotic distribution of real numbers and in particular equi-distribution, that if \( g(x) \) is real and periodic on \( \mathbb{R} \) with period 1 and in a period interval is continuous, except for finitely many points of jump, if any, then

(5) \[
\frac{1}{N} \sum_{n=1}^{N} g(\alpha n) \rightarrow \int_{0}^{1} g(x) dx \quad \text{for } N \rightarrow \infty.
\]

Let \( 0 < \lambda < 1 \) and let first

\[
g(x) = \begin{cases} 
1 & \text{for } 0 \leq x \leq \lambda \\
0 & \text{elsewhere in } 0 \leq x < 1
\end{cases}
\]

and periodic with period 1.

Our aim is to prove that the set \( C_\lambda \) on \( \mathbb{Z} \) written

\[
C_\lambda = [g(\alpha n) = 1]
\]

i.e. the set of \( n \in \mathbb{Z} \) for which
\[ 0 \leq \alpha n \leq \lambda \pmod{1} \]

is \(W\)-ap and to find its Fourier series. Note that

\[ C_\lambda(n) = g(\alpha n). \]

We have

\[
C_\lambda = \left[ \frac{-\lambda}{2} \leq \alpha n - \frac{\lambda}{2} \leq \frac{\lambda}{2} \pmod{1} \right]
\]

\[
= \left[ \cos 2\pi \left( \frac{\alpha n - \lambda}{2} \right) \geq \cos 2\pi \frac{\lambda}{2} \right]
\]

\[
= [\cos (2\pi \alpha n - \pi \lambda) \geq \cos \pi \lambda].
\]

Let \(0 < \lambda_1 < \lambda < \lambda_2 < 1\) and put

\[ C_{\lambda, \lambda_1} = [\cos (2\pi \alpha n - \pi \lambda) \geq \cos \pi \lambda_1] \]

\[
= \left[ \frac{-\lambda_1}{2} + \frac{\lambda_2}{2} \leq \alpha n \leq \frac{\lambda_1}{2} + \frac{\lambda_2}{2} \pmod{1} \right],
\]

\[ C_{\lambda, \lambda_2} = [\cos (2\pi \alpha n - \pi \lambda) \geq \cos \pi \lambda_2] \]

\[
= \left[ \frac{-\lambda_2}{2} + \frac{\lambda_2}{2} \leq \alpha n \leq \frac{\lambda_2}{2} + \frac{\lambda_2}{2} \pmod{1} \right].
\]

In the last expression for \(C_\lambda\) above, \(\cos (2\pi \alpha n - \pi \lambda)\) is a trigonometric polynomial on \(\mathbb{Z}\) containing only the two characters \(e^{i2\pi \alpha n}\) and \(e^{-i2\pi \alpha n}\). Using Theorem 2 to

\[ f(n) = -\cos (2\pi \alpha n - \pi \lambda) \]

we see that \(C_{\lambda, \lambda_1}\) and \(C_{\lambda, \lambda_2}\) are \(W\)-ap with Fourier characters amongst

\[ e^{i2\pi r n}, \quad r = 0, \pm 1, \pm 2, \ldots \]

when we assume that \(\lambda_1\) and \(\lambda_2\) avoid certain at most denumerably many values. Thus we may also let \(\lambda_2 - \lambda_1 \to 0\).

From (5), taking

\[ g(x) = \begin{cases} 
1 & \text{for } \frac{-\lambda_1}{2} + \frac{\lambda}{2} \leq x \leq \frac{\lambda_1}{2} + \frac{\lambda}{2} \\
0 & \text{elsewhere in } 0 \leq x < 1
\end{cases} \]

and periodic with period 1 we get, using also (3)

\[ MC_{\lambda, \lambda_1} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g(\alpha n) = \int_{0}^{1} g(x) dx = \lambda_1. \]
Analogously, taking \( g(x) = 1 \) for
\[
-\frac{\lambda_2}{2} + \frac{\lambda}{2} \leq x \leq \frac{\lambda_2}{2} + \frac{\lambda}{2},
\]
and repeated with period 1, and \( g(x) = 0 \) elsewhere, we get
\[
MC_{\lambda, \lambda_2} = \lambda_2.
\]
We have
\[
C_{\lambda, \lambda_1} \subset C_\lambda \subset C_{\lambda, \lambda_2}
\]
so that
\[
0 \leq \bar{M}(C_\lambda(n) - C_{\lambda, \lambda_1}(n)) \leq \bar{M}(C_{\lambda, \lambda_2}(n) - C_{\lambda, \lambda_1}(n)) = M(C_{\lambda, \lambda_2}(n) - C_{\lambda, \lambda_1}(n)) = \lambda_2 - \lambda_1 \to 0.
\]
Thus \( C_\lambda(n) \) can be \( W \)-approximated by the \( W \)-ap functions \( C_{\lambda, \lambda_1}(n) \) and hence is \( W \)-ap itself. The Fourier series of \( C_{\lambda, \lambda_1}(n) \) will converge formally towards the Fourier series of \( C_\lambda(n) \) so that the Fourier characters of \( C_\lambda(n) \) are amongst
\[
e^{i2\pi r n}, \quad r = 0, \pm 1, \pm 2, \ldots.
\]
Thus from (4)
\[
C_\lambda(n) \sim \frac{1}{2}a_0 + \sum_{r=1}^{\infty} \{a_r \cos 2\pi r n + b_r \sin 2\pi r n\}
\]
and using (5) with
\[
g(x) = \begin{cases} 
\cos 2\pi rx & \text{for } 0 \leq x \leq \lambda \\
0 & \text{elsewhere in } 0 \leq x < 1
\end{cases}
\]
and periodic with period 1 we get, using also (3), that
\[
a_r = 2M\{C_\lambda(n) \cos 2\pi r n\} = 2 \int_0^\lambda \cos 2\pi r x \, dx.
\]
Hence
\[
\frac{1}{2}a_0 = \lambda
\]
and for \( r > 0, \)
\[
a_r = \frac{\sin 2\pi r \lambda}{\pi r}.
\]
Analogously, for \( r > 0, \)
\[ b_r = 2M \{ C_\lambda(n) \sin 2\pi r x n \} = 2 \int_0^\lambda \sin 2\pi r x \, dx = \frac{1 - \cos 2\pi r \lambda}{\pi r}. \]

It is evident that

\[ \frac{1}{2} a_0 + \sum_{r=1}^\infty \{ a_r \cos 2\pi r x + b_r \sin 2\pi r x \} \]

is the Fourier series of our first \( g(x) \). Hence it converges to 1 for \( 0 < x < \lambda \) (mod 1), to \( \frac{1}{2} \) for

\[ x \equiv \begin{cases} 
0 \\
\lambda 
\end{cases} \pmod{1} \]

and to 0 elsewhere.

Therefore, the Fourier series of

\[ C_\lambda(n) = g(\alpha n) \]

converges to \( C_\lambda(n) \) for all \( n \), except for

\[ \alpha n \equiv \begin{cases} 
0 \\
\lambda 
\end{cases} \pmod{1}, \]

i.e. \( n = 0 \) and at most one further value of \( n \), because \( \alpha \) is irrational. For \( n = 0 \) and the eventual other \( n \) it converges to \( \frac{1}{2} \), and not 1.

The preceding result can be generalized from \( \mathbb{Z} \) to \( \mathbb{Z} \times \ldots \times \mathbb{Z} \). We state it for \( \mathbb{Z} \times \mathbb{Z} \). If \( 1, \alpha_1, \beta_1 \) or \( 1, \alpha_2, \beta_2 \) are linearly independent with respect to rational coefficients and \( 0 < \lambda_1 < 1, 0 < \lambda_2 < 1 \) and \( a_1 \) and \( a_2 \) are real numbers, then the set \( C \) on \( \mathbb{Z} \times \mathbb{Z} \), consisting of the \( (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z} \) for which both

\[ a_1 \leq \alpha_1 n_1 + \alpha_2 n_2 \leq \lambda_1 + a_1 \pmod{1} \]

and

\[ a_2 \leq \beta_1 n_1 + \beta_2 n_2 \leq \lambda_2 + a_2 \pmod{1} \]

is \( W \)-ap on the discrete abelian group \( \mathbb{Z} \times \mathbb{Z} \). On \( \mathbb{R} \times \mathbb{R} \) we put

\[ g(x_1, x_2) \]

equal to 1 when both

\[ a_1 \leq x_1 \leq \lambda_1 + a_1 \pmod{1} \quad \text{and} \quad a_2 \leq x_2 \leq \lambda_2 + a_2 \pmod{1}, \]

and equal to 0 elsewhere. Then the characteristic function of \( C \),

\[ C(n_1, n_2) = g(\alpha_1 n_1 + \alpha_2 n_2, \beta_1 n_1 + \beta_2 n_2) \]

has, being \( W \)-ap on \( \mathbb{Z} \times \mathbb{Z} \), a Fourier series on \( \mathbb{Z} \times \mathbb{Z} \), and this series is obtained from the Fourier series on \( \mathbb{R} \times \mathbb{R} \) of
by replacing \((x_1, x_2)\) by \((\alpha_1 n_1 + \alpha_2 n_2, \beta_1 n_1 + \beta_2 n_2)\).

From the preceding we can obtain a sharpening of Kronecker's approximation theorem.

Let \(0 < \lambda_1 < 1, \ldots, 0 < \lambda_p < 1\) and let \(a_1, \ldots, a_p\) be real numbers. Let further \(1, \alpha_1, \ldots, \alpha_p\) be linearly independent with respect to rational coefficients. Then Kronecker's approximation theorem may be stated as follows. The set \(C\) of integers \(n\) which satisfy all the inequalities

\[
a_1 \leq \alpha_1 n \leq a_1 + \lambda_1 \pmod{1}
\]

\[
\ldots
\]

\[
a_p \leq \alpha_p n \leq a_p + \lambda_p \pmod{1}
\]

is not empty. **We shall prove moreover that \(C\) is \(W\)-almost periodic and has a positive mean value**

\[
MC = \lambda_1 \ldots \lambda_p.
\]

Let \(C_1, \ldots, C_p\) be the sets of integers which satisfy the first, \(\ldots\), the \(p\)th of the above inequalities. From the preceding it follows that they are \(W\)-almost periodic and that

\[
MC_1 = \lambda_1, \ldots, MC_p = \lambda_p.
\]

It also follows that the Fourier characters of \(C_1, \ldots, C_p\) are among respectively

\[
e^{i2\pi r_1 \alpha_1 n}, \quad r_1 = 0, \pm 1, \pm 2, \ldots
\]

\[
\ldots
\]

\[
e^{i2\pi r_p \alpha_p n}, \quad r_p = 0, \pm 1, \pm 2, \ldots.
\]

We have

\[
C = C_1 \cap \ldots \cap C_p
\]

so that the characteristic functions satisfy

\[
C(n) = C_1(n) \ldots C_p(n),
\]

and hence \(C(n)\) is \(W\)-almost periodic, and its Fourier series taken on complex form is obtained by formal multiplication of the Fourier series of \(C_1(n), \ldots, C_p(n)\). We seek the constant term of the Fourier series of \(C(n)\), and hence seek the integers \(r_1, \ldots, r_p\) for which

\[
e^{i2\pi r_1 \alpha_1 n} \ldots e^{i2\pi r_p \alpha_p n} = 1,
\]

i.e.
\[ r_1 \alpha_1 + \ldots + r_p \alpha_p = \text{integer} \]

and hence \( r_1 = \ldots = r_p = 0 \), since \( 1, \alpha_1, \ldots, \alpha_p \) are linearly independent. Thus the only contribution to the constant term of the Fourier series of \( C(n) \) is

\[ MC = MC_1 \ldots MC_p = \lambda_1 \ldots \lambda_p . \]

REFERENCES