CONVEXITY OF MEASURES IN CERTAIN CONVEX CONES IN VECTOR SPACE $\sigma$-ALGEBRAS

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1. Introduction.

The Brunn–Minkowski theory on vector spaces deals with all types of connections between set functions and linear combination of sets. Below we will treat a special situation when the sets are restricted to a certain convex cone in the underlying $\sigma$-algebra.

Let $0 < \theta < 1$ and $-\infty \leq a \leq +\infty$ be fixed. For any $0 < s, t \leq +\infty$ define the mean

$$M^a_\theta(s, t) = (\theta s^a + (1 - \theta) t^a)^{1/a}, \quad a \in \mathbb{R} \setminus \{0\};$$

$$= \min (s, t), \quad a = -\infty;$$

$$= s^\theta t^{1-\theta}, \quad a = 0;$$

$$= \max (s, t), \quad a = +\infty.$$ 

Here $0^a = +\infty$, if $-\infty < a < 0$. Finally, for arbitrary $0 \leq s, t \leq +\infty$,

$$M^a_\theta(s, t) = 0, \quad \text{if} \ s = 0 \ \text{or} \ t = 0.$$ 

Throughout $E$ denotes a real, locally convex Hausdorff vector space and $C \ni 0$ stands for a fixed closed convex cone in $E$. Set

$$\langle C \rangle = \{ K - C; \ E \ni K \ \text{compact} \}.$$ 

Clearly,

$$s, t \geq 0, \quad A, B \in \langle C \rangle \Rightarrow sA + tB \in \langle C \rangle.$$ 

In addition, each set $A \in \langle C \rangle$ is $C$-invariant, that is, $A - C = A$. Given $-\infty \leq \alpha < +\infty$, we shall write $\mu \in \mathcal{M}_\alpha(E; C)$, if $\mu$ is a finite positive Radon measure on $E$ (abbreviated $\mu \in \mathcal{R}(E)$) and

$$\mu(\theta A + (1 - \theta)B) \geq M^a_\theta(\mu(A), \mu(B))$$

for all $A, B \in \langle C \rangle$ and every $0 < \theta < 1$. A measure satisfying these assumptions

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is said to be \(\alpha\)-concave in \(\langle C \rangle\). For brevity, \(\mathcal{M}_\alpha(E; \{0\})\) is written \(\mathcal{M}_\alpha(E)\) and an \(\alpha\)-concave measure in \(\langle \{0\} \rangle\) is simply called \(\alpha\)-concave. Below, by abuse of language, "\(\alpha\)-concave" is sometimes shortened to "\(\alpha\)-concave" ("\(\alpha\)-convex") if \(\alpha \geq 0\) (\(\alpha \leq 0\)).

The interest in \(1/n\)-concave measures originates from Brunn and Minkowski who show that the uniform distribution in an arbitrary convex body in \(\mathbb{R}^n\) is \(1/n\)-concave (restricted to convex bodies). The main features of 0-concave measures on \(\mathbb{R}^n\) are due to Davidović, Korenblum, and Hacet [11], Prekopa ([21], [22], [23]) and the author [3]. In ([4], [5], [6], [7]) we continue this program introducing \(\alpha\)-concave measures on possibly infinite-dimensional spaces. During the past few years this subject has been enriched on the foundational level, mainly by Brascamp and Lieb ([9], [10]), Dubuc ([12], [13]), and Hoffmann-Jørgensen [17].

The present paper is devoted to a study of \(\alpha\)-concave measures in convex cones of the type \(\langle C \rangle\) introduced above. One motivation for this is the following. Let \(X = (X_1, \ldots, X_n)\) be a random vector in \(\mathbb{R}^n\) with probability distribution function \(F_X(x_1, \ldots, x_n) = P[X_1 \leq x_1, \ldots, X_n \leq x_n]\). In a variety of different contexts it may be useful to know that \(F_X\) is \(\alpha\)-concave, that is, to know that the inequality \(F_X(\theta x + (1-\theta)y) \geq M^\alpha_x(F_X(x), F_X(y))\) is true for all \(x, y \in \mathbb{R}^n\) and each \(0 < \theta < 1\) (see e.g. Barlow and Proschan (reliability theory) [1], Berwald (convexity) [2], Hoffmann-Jørgensen, Shepp, and Dudley (absolute continuity of semi-norms) [18], Prekopa (stochastic programming) [24], and Rinott (statistics) [27]). Here two remarks are in order. Firstly, in almost all cases of interest, it is a non-trivial problem to decide whether \(F_X\) is \(\alpha\)-concave or not. Secondly, it seems to be an almost hopeless task to develop a convex analysis based on \(\alpha\)-concave distribution functions in \(\mathbb{R}^n(n > 1)\). In this context \(\alpha\)-concave measures in \(\langle \mathbb{R}^n_+ \rangle\) have some advantages as will be seen below.

A second reason for this paper is to deepen the Brunn–Minkowski approach to measures on linear spaces. Among other things, we prove zero-one laws and integrability of appropriate semi-norms.

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2. The basic results for the class \(\mathcal{M}_\alpha(\mathbb{R}^n)\).

Throughout, \(-\infty \leq \alpha < +\infty\), \(-\infty \leq \beta \leq +\infty\), and \(0 < \theta < 1\) are assumed to be fixed if not otherwise stated. Given \(\mu, \nu \in \mathcal{R}(E)\), we let

\[\mathcal{M}_\alpha^\theta(\mu, \nu; C) = \{\tau \in \mathcal{R}(E) \mid \tau(\theta A + (1-\theta)B) \geq M^\theta_x(\mu(A), \nu(B)), A, B \in \langle C \rangle\}\]

and set \(\mathcal{M}_\alpha^\theta(\mu, \nu; \{0\}) = \mathcal{M}^\theta(\mu, \nu)\). If \(V \subseteq E\) is universally Borel measurable and
convex, then the notation \( h \in \mathcal{F}_\beta(f, g \mid V) \) will mean that \( f, g, h : V \to [0, +\infty] \) are universally Borel measurable functions satisfying the inequality 
\[
h(\theta x + (1 - \theta)y) \leq M^\theta_\beta(f(x), g(y)) \text{ for all } x, y \in V.
\]
The members in the class 
\[
\mathcal{F}_\beta(V) = \{ f : V \to [0, +\infty] ; f \in \mathcal{F}_\beta^\theta(f, f \mid V), 0 < \theta < 1 \}
\]
are called \( \beta \)-concave functions in \( V \).

**Theorem 2.1.** ([3, Th. 3.1]) If \( f, g, h \in L^1_\alpha(m_n), -\infty \leq \alpha \leq 1/n, \) and 
\( h \in \mathcal{F}_\alpha(1-\alpha)(f, g \mid \mathbb{R}^n) \), then \( hm_n \in \mathcal{M}_\alpha(fm_n, gm_n) \).

Here \( m_n \) denotes Lebesgue measure in \( \mathbb{R}^n \) and \( \alpha/(1-\alpha) = -1/n, \alpha = -\infty, = +\infty, \alpha = 1/n \).

**Theorem 2.2.** ([3, Th. 3.2]) a) Let \( -\infty \leq \alpha \leq 1/n \) and suppose \( \mu \in \mathcal{M}_\alpha(\mathbb{R}^n) \). If the convex set \( \text{supp} \mu \) is \( n \)-dimensional, then \( \mu \) is absolutely continuous with respect to \( m_n \) and a suitable version of \( d\mu/dm_n \) is \( \alpha/(1-\alpha) \)-concave in \( \mathbb{R}^n \).

b) If \( \alpha > 1/n \) and \( \mu \in \mathcal{M}_\alpha(\mathbb{R}^n) \), then \( \text{dim \ sup} \mu < n \).

3. Some simple construction methods of \( \alpha \)-concave measures in convex cones.

To begin with, note that 
\[
\mathcal{M}_\alpha(E; C) \supseteq \mathcal{M}_\alpha(E; C), \quad \alpha_1 \leq \alpha_2 ,
\]
\[
\mathcal{M}_\alpha(E; C_1) \subseteq \mathcal{M}_\alpha(E; C_2), \quad C_1 \subseteq C_2 ,
\]
\[
\mathcal{M}_\alpha(E; E) = \mathcal{R}(E)
\]

and 
\[
\mathcal{M}_{-\infty}(E; H) = \mathcal{R}(E), \quad \text{H closed half space}.
\]

Also, by the Zorn lemma, any \( \mu \in \mathcal{M}_\alpha(E; C) \) belongs to at least one class \( \mathcal{M}_\alpha(E; C_\alpha(\mu)) \), where \( C_\alpha(\mu) \) is minimal.

The one-dimensional case \( E = \mathbb{R} \) is especially simple to treat since there only are four closed convex cones in \( \mathbb{R} \). Recall that a smooth positive \( \beta \)-concave function \( (\beta \in \mathbb{R}) \) \( f \) on a subinterval of \( \mathbb{R} \) is characterized by the differential inequality \( ff'' + (\beta - 1)f'^2 \leq 0 \). Often, this enables us to construct measures on \( \mathbb{R} \) which are \( \alpha \)-concave in the cones in question. However, there are lots of interesting exceptional cases and, in such a case, Theorem 2.1 may sometimes be helpful.

**Example 3.1.** We claim that each stable probability measure \( \mu \) on \( \mathbb{R} \) with
topological support $R_+^\infty$ is $0$-concave in $\langle R_+ \rangle$. In fact, due to a representation formula of Zolotarev [31] there exist $\delta > 0$ and $0 < \alpha < 1$ such that

$$\mu([-\infty, \delta x]) = \frac{1}{\pi} \int_0^\infty \exp (-v_\alpha(x, t)) \, dt, \quad x > 0,$$

where for all $x > 0$ and $0 < t < \pi$,

$$v_\alpha(x, t) = x^{\alpha/(\alpha-1)} \left( \frac{\sin \alpha t}{\sin t} \right)^{\alpha/(1-\alpha)} \frac{\sin (1-\alpha)t}{\sin t} .$$

Thus the claim above follows if we prove that $v_\alpha$ is convex. To see this we write

$$v_\alpha(x, t) = x^{\alpha/(\alpha-1)} \left[ \left( \frac{\sin \alpha t}{\sin t} \right)^{\alpha} \left( \frac{\sin (1-\alpha)t}{\sin t} \right)^{1-\alpha} \right]^{1-\alpha/(\alpha-1)}$$

and note that the function $(\xi, \eta) \mapsto \xi^\alpha \eta^{1-\alpha}$, $\xi, \eta > 0$, is convex for each $\alpha < 0$. Consequently, $v_\alpha$ is convex if the function

$$\left( \frac{\sin \alpha t}{\sin t} \right)^{\alpha} \left( \frac{\sin (1-\alpha)t}{\sin t} \right)^{1-\alpha}, \quad 0 < t < \pi,$$

is convex, which is obvious as

$$\frac{d^2}{dt^2} \ln \frac{\sin \alpha t}{\sin t} = \frac{\sin^2 \alpha t - \alpha^2 \sin^2 t}{\sin^2 \alpha t \sin^2 t} > 0, \quad 0 < t < \pi .$$

It is well-known from the early Brunn–Minkowski theory that each concave function, defined on a convex body in $\mathbb{R}^n$, induces a distribution measure which is $1/n$-concave in $\langle R_+ \rangle$. Before pushing this into a more general framework we introduce some new definitions.

Under the conditions on $E$ and $C$ stated in the Introduction, the ordered pair $(E; C)$ is called a semi-ordered, locally convex Hausdorff space over $R$. For all $x, y \in (E; C)$, the shorthand notation $x \prec y$ means that $y - x \in C$. Suppose $(F; D)$ is another semi-ordered, locally convex Hausdorff space over $R$ and let $u$ be a mapping of a convex subset $V$ of $(E; C)$ into $(F; D)$. Then $u$ is said to be increasing if $[x, y \in V, x \prec y \Rightarrow u(x) < u(y)]$ and convex if

$$[x, y \in V, 0 < \theta < 1 \Rightarrow u(\theta x + (1-\theta)y)] < \theta u(x) + (1-\theta)u(y) .$$

**Theorem 3.1.** Let $\tau \in M_0^0(\mu, \nu; C)$, let $u: (E; C) \to (F; D)$ be Lusin $\mu$, $\nu$, and $\tau$-measurable, and suppose there exists a $C$-invariant convex support $V$ of the measure $\mu + \nu$. If $u|_V$ is increasing and convex, then $u(\tau) \in M_0^0(u(\mu), u(\nu); D)$.

**Proof.** Let $A, B \subseteq F$ be $D$-invariant. It is readily seen that
\[ u^{-1}(\theta A + (1 - \theta)B) \supseteq \theta(u^{-1}(A) \cap V) + (1 - \theta)(u^{-1}(B) \cap V) \]

where the sets \( u^{-1}(A) \cap V \) and \( u^{-1}(B) \cap V \) are \( C \)-invariant. Finally, using that

\[ m_{\text{inner measure}}(u^{-1}(\cdot)) = (u(m)_{\text{inner measure}} \quad m = \mu, \nu, \tau, \]

(see e.g. Schwartz [28, p. 25]) we are done.

**Example 3.2.** Let \( E \neq \{0\} \) be a Banach space and suppose \( \mu \in \mathcal{M}_\alpha(E) \) \( (\alpha > -\infty) \) has topological support \( E \). Then each sphere in \( E \) is a \( \mu \)-null set. In the Gaussian case the same result is due to Gross [16]. The interest of such a message has been further emphasized by Topsøe [30], who studies uniform weak convergence of measures in restricted Banach spaces.

A combination of Theorems 2.1 and 3.1 yields

**Corollary 3.1.** Suppose \( \mu = f m_n \in \mathcal{R}(\mathbb{R}^n) \) and let \( u : \mathbb{R}^n \to (\mathbb{R}^n; C) \) be a \( C^2 \) mapping. Moreover, assume there exists an open convex set \( V \subseteq \mathbb{R}^n \) such that \( u(V) \) supports \( \mu \) and such that \( u|_V \) is injective and convex. Then \( \mu \in \mathcal{M}_\alpha(\mathbb{R}^n; C) \) \( (-\infty \leq \alpha \leq 1/n) \), if \( (f \circ u)|Ju| \) is \( \alpha/(1-\alpha) \)-concave in \( V \), where \( Ju \) denotes the Jacobian of \( u \).

**Example 3.3.** Let \( X_1, \ldots, X_n, Y \) be stochastically independent \( N(0; 1) \)-distributed random variables and set \( Z = (X_1^2/Y^2, \ldots, X_n^2/Y^2) \). The density function \( f_Z \) of \( Z \) vanishes off \( \mathbb{R}^n_+ \) and

\[ f_Z(z) = \text{const.} z_1^{\frac{1}{2}} \cdots z_n^{\frac{1}{2}}(1 + z_1 + \ldots + z_n)^{-\frac{n+1}{2}}, \quad z > 0. \]

Introducing \( u(\xi) = (\xi_1^2, \ldots, \xi_n^2), \) \( \xi \in \mathbb{R}^n \), and applying Corollary 3.1 we now conclude that \( P_Z \in \mathcal{M}_{-1}(\mathbb{R}^n; \mathbb{R}^n_+) \). From the proof it also follows that \( P_{(X_i/Y_1, \ldots, X_n/Y)} \in \mathcal{M}_{-1}(\mathbb{R}^n) \).

Next we will discuss a quite different construction method which only makes sense for \( C \neq \{0\} \).

Let \( C \neq \{0\} \) be a closed convex cone in \( E \) and suppose \( \alpha \geq 1 \) is fixed. We now choose a non-empty Borel set \( C_0 \subseteq C \setminus \{0\} \) such that \( (x\mathbb{R}_+) \cap C_0 = \{x\}, \) \( x \in C_0 \), and a bounded Borel function \( f : E \to \mathbb{R}_+ \) possessing the following properties;

(i) the measure \( \nu_x : A \mapsto \int_0^\infty f(rx)1_A(rx) \, dr \) is \( \alpha \)-concave in \( \langle C \rangle \) for each \( x \in C_0 \),

(ii) \( 0 \in \text{supp} \nu_x, \quad x \in C_0 \).

Let \( \tau \in \mathcal{R}(E) \) be supported on \( C_0 \). We claim that the Radon measure
\[ \mu = \int v_x(\cdot) \, d\tau(x) \]

is an \( \alpha \)-concave measure in \( \langle C \rangle \). To see this, assume \( A, B \in \langle C \rangle \) are both of positive \( \mu \)-measure and note that

\[ v_x(\theta A + (1 - \theta)B) \geq (\theta v_x^\alpha(A) + (1 - \theta)v_x^\alpha(B))^{1/\alpha}, \quad x \in C_0, \]

because \( 0 \in A \cap B \). Finally, using the Minkowski inequality it follows that \( \mu \in \mathcal{M}_\alpha(E; C) \).

The above construction shows the necessity in the following

**Theorem 3.2.** Let \( \alpha > 0 \). Each \( \mu \in \mathcal{M}_\alpha(E; C) \) is concentrated on a finite-dimensional subspace of \( E \) if and only if \( C \) is finite-dimensional.

**Proof.** Suppose \( C \) is finite-dimensional and represent \( E \) as a topological direct sum of \( C - C \) and a complementary subspace \( F \) of \( E \). Let \( u: (E; C) \to (F; \{0\}) \) be the canonical map and note that \( u \) is increasing and convex. Thus, for any \( \mu \in \mathcal{M}_\alpha(E; C) \), \( u(\mu) \in \mathcal{M}_\alpha(F) \) and Theorem 2.2 implies that \( u(\mu) \) is concentrated on a finite-dimensional subspace of \( F \). Consequently, \( \mu \) is concentrated on a finite-dimensional subspace of \( E \).


In the sequel, \( E' \) denotes the topological dual of \( E \) and \( C^+ = \{ \xi \in E'; \xi|_C \geq 0 \} \). If \( \tau \in \mathcal{M}_\alpha^0(\mu, \nu; C) \) and \( \xi_1, \ldots, \xi_n \in C^+ \), then, by Theorem 3.1, \( u(\tau) \in \mathcal{M}_\alpha^0(u(\mu), u(\nu); \mathbb{R}^n) \), where \( u = (\xi_1, \ldots, \xi_n) \). To begin with in this section we shall prove the following converse result.

**Theorem 4.1.** Assume that the cone \( C^\ast \subseteq C^+ \) strictly separates \( C \) and points belonging to the complement of \( C \). If \( \mu, \nu, \tau \in \mathcal{B}(E) \) and \( u(\tau) \in \mathcal{M}_\alpha^0(u(\mu), u(\nu); \mathbb{R}^n) \) for all \( u = (\xi_1, \ldots, \xi_n) \) such that \( \xi_1, \ldots, \xi_n \in C^\ast \), \( n \in \mathbb{N}_+ \), then \( \tau \in \mathcal{M}_\alpha^0(\mu, \nu; C) \).

**Proof.** Let \( A, B \subseteq E \) be compact. We shall prove the following inequality

\[ \tau(\theta A + (1 - \theta)B - C) \geq M_\alpha^0(\mu(A - C), \nu(A - C)) \]

To this end, first note that

\[ \theta A + (1 - \theta)B - C = \\bigcap \{ \theta A + (1 - \theta)B - [\xi_1 \geq -1, \ldots, \xi_n \geq -1]: \xi_1, \ldots, \xi_n \in C^\ast, n \in \mathbb{N}_+ \} \]

as \( \theta A + (1 - \theta)B \) is compact. Now let \( \varepsilon > 0 \) be fixed and choose

\[ C_0 = [\xi_1 \geq -1, \ldots, \xi_n \geq -1] \quad (\xi_1, \ldots, \xi_n \in C^\ast) \]
satisfying the estimate
\[ \tau(\theta A + (1 - \theta)B - C) + \varepsilon \geq \tau(\theta A + (1 - \theta)B - C_0). \]
Moreover, by compactness, we may pick \( a_1, \ldots, a_p \in A, b_1, \ldots, b_q \in B \) such that \( A \subseteq \{a_1, \ldots, a_p\} - C_0 \) and \( B \subseteq \{b_1, \ldots, b_q\} - C_0 \). Then
\[ \tau(\theta A + (1 - \theta)B - C_0) \geq \tau(\theta\{a_1, \ldots, a_p\} + (1 - \theta)\{b_1, \ldots, b_q\} - C_0) \]
where the last member does not exceed
\[ M_\varepsilon^\theta(\mu(\{a_1, \ldots, a_p\} - C_0), \nu(\{b_1, \ldots, b_q\} - C_0)) \geq M_\varepsilon^\theta(\mu(A - C), \nu(B - C)). \]
Summing up, we have
\[ \tau(\theta A + (1 - \theta)B - C) + \varepsilon \geq M_\varepsilon^\theta(\mu(A - C), \nu(B - C)) \]
and (4.1) follows at once.

Theorem 4.1 raises the question how to characterize the classes \( \mathcal{M}_x(\mathbb{R}^n; \mathbb{R}_+^n) \) in a simple way, which, however, seems to be very complicated for each \( n > 1 \). It should be remarked that an \( \alpha \)-concave measure in \( \langle \mathbb{R}^2_+ \rangle \) is not generally, a convex image of an \( \alpha \)-concave measure on \( \mathbb{R}^2 \) even if \( \alpha \leq \frac{1}{2} \).

**Example 4.1.** Let \( I_1, I_2, I_3 \subseteq \{x| = 1, x \in \mathbb{R}^2_+\} \) be mutually disjoint closed arcs of positive lengths. Set \( S_i = \text{convex hull} \{0 \cup I_i\}, i = 1, 2, 3, \) and introduce the measure \( \mu(dx) = 1_{S_1 \cup S_2 \cup S_3}(x)dx/|x| \). Of course, \( \mu \ll m_2 \) and from the previous section we know that \( \mu \) is 1-concave in \( \langle \mathbb{R}^2_+ \rangle \). However, there do not exist a \( v \in \mathcal{M}_{-\infty}(\mathbb{R}^n) \) and a convex function \( u: \text{supp} \nu \rightarrow \text{supp} \mu \) such that \( u(\nu) = \mu \). In fact, assuming the converse, necessarily, \( k = \dim \text{supp} \nu > 0 \) and \( \dim u^{-1}(\{0\}) \leq k - 1 \). Consequently, there exists a continuous curve in \( (\text{supp} \mu) \setminus \{0\} \) connecting two of the three connected components of \( \text{int} \text{supp} \mu \), which is absurd.

We must leave the above question unanswered here and shall next discuss some applications of Theorem 4.1.

Below, if a net \( (\mu_i) \) in \( \mathcal{R}(E) \) converges weakly to \( \mu \in \mathcal{R}(E) \), this fact is expressed \( \mu_i \Rightarrow \mu \).

**Corollary 4.1.** The map \( (\mu, \nu) \rightarrow \mathcal{M}_x^\theta(\mu, \nu; C) \) is weakly closed, that is, if \( \tau_i \in \mathcal{M}_x^\theta(\mu_i, \nu_i; C) \) and \( \mu_i \Rightarrow \mu, \nu_i \Rightarrow \nu, \tau_i \Rightarrow \tau \), then \( \tau \in \mathcal{M}_x^\theta(\mu, \nu; C) \).

**Proof.** By Theorem 4.1 we may assume that \( (E; C) = (\mathbb{R}^n; \mathbb{R}_+^n) \) and the result follows at once (compare [4, Th. 2.2]).

**Theorem 4.2.** If \( \mu, \nu \in \mathcal{M}_x(E) \), then \( \mu \land \nu \in \mathcal{M}_x(E) \).
Theorem 4.2 does not extend to arbitrary $\alpha$-concave measures in convex cones. Note, however, that $\mu \wedge \nu \in \mathcal{M}_x \wedge _1(R; R_+)$ if $\mu, \nu \in \mathcal{M}_x(R; R_+)$, which follows by differentiation.

PROOF. The finite-dimensional case is a consequence of Theorems 2.1 and 2.2. In the general case we argue as follows.

Let $u: E \to R^n$ be an arbitrary linear continuous mapping. It shall be proved that $u(\mu \wedge \nu)$ is $\alpha$-concave. To this end, suppose $A, B$ are compact subsets of $R^n$. Moreover, let $G$ be a Borel set in $R^p$ and choose an arbitrarily linear continuous map $f: E \to R^p$. Then, setting $H = R^p \setminus G$, we have

$$
\mu(u^{-1}(\theta A + (1-\theta)B) \cap f^{-1}(G)) + \nu(u^{-1}(\theta A + (1-\theta)B) \cap f^{-1}(H))
= \mu(u, f)((\theta (A \times R^p) + (1-\theta)(B \times R^p)) \cap (R^n \times G)) +
+ \nu(u, f)((\theta (A \times R^p) + (1-\theta)(B \times R^p) \cap (R^n \times H))
$$

where the last expression does not exceed

$$
(\mu(u, f) \wedge \nu(u, f))(\theta (A \times R^p) + (1-\theta)(B \times R^p)) \geq M^\theta_2((\mu(u, f) \wedge \nu(u, f))(A \times R^p), (\mu(u, f) \wedge \nu(u, f))(B \times R^p))
$$

Finally, using the inequality $\mu(u, f) \wedge \nu(u, f) \geq (\mu \wedge \nu)(u, f)$, Theorem 4.2 follows at once.

5. Multiplication by densities.

For all $\alpha, \beta \in R$ satisfying $\alpha + \beta \geq 0$, we introduce half the harmonic mean

$$
\kappa(\alpha, \beta) = \begin{cases} 
(\alpha^{-1} + \beta^{-1})^{-1}, & \alpha + \beta > 0, \alpha \neq 0, \beta \neq 0, \\
-\infty, & \alpha + \beta = 0, (\alpha, \beta) \neq (0,0), \\
0, & \alpha = \beta = 0.
\end{cases}
$$

THEOREM 5.1. Suppose $\tau \in \mathcal{M}_x^\theta(\mu, \nu; C) (\alpha \in R)$ and let $h \in \mathcal{F}_\beta^\theta(f, g, |E) (\beta \in R)$, where $\alpha + \beta \geq 0$. If $f, g, h: (E, C) \to R$ are bounded and decreasing, then $ht \in \mathcal{M}_x^\theta(\kappa(\alpha, \beta)f \mu, g \nu; C)$.

Here and throughout $R$ is assumed to be endowed with its usual cone ordering if not otherwise stated.

The proof of Theorem 5.1 is based on the next

LEMMA 5.1. Let $\alpha \in R$, $\beta \in R \setminus \{0\}$, and suppose $H \in \mathcal{F}_x^\theta(F, G| R_+)$. 

a) If $\alpha > 0 > \beta$ and $\alpha + \beta \geq 0$, then
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(5.1) \[ \int_0^\infty x^{1/\beta - 1} H(x) \, dx \geq M^\theta_{\alpha, \beta} \left( \int_0^\infty x^{1/\beta - 1} F(x) \, dx, \int_0^\infty x^{1/\beta - 1} G(x) \, dx \right). \]

b) If \( \alpha + \beta \geq 0 \) and \( F, G, H \) decrease, then (5.1) is true.

PROOF. Recall that the function \( \xi^a \eta^{1-a}, \xi, \eta > 0 \), is concave (convex) if \( 0 < a < 1 \) (\( a < 0 \) or \( a > 1 \)).

a) Since \( x^{1/\beta - 1} H(x) \in \mathcal{F}_{(\alpha^{-1} + \beta^{-1} - 1)^{-1}}(x^{1/\beta - 1} F(x), x^{1/\beta - 1} G(x) | R_+) \) the inequality (5.1) follows from Theorem 2.1.

b) STEP 1. \( 0 < \alpha \leq 1, \beta > 0 \).

**Proof of Step 1.** Set \( I_x = I^x, I = F, G, H \). Without loss of generality we may assume that \( I(a(l)) = 0 \) for a suitable \( a(l) > 0 \) and that the function \( I |_L | [0, a(l)] \) is strictly decreasing and \( \mathcal{C}^1 \). Then, by partial integration,

\[
\int_0^\infty x^{1/\beta - 1} I(x) \, dx = -\frac{\beta}{\alpha} \int_0^{a(l)} x^{1/\beta} I_x^{1/\alpha - 1} (x) I'_x (x) \, dx
\]

and if \( i_x \) denotes the inverse of the function \( I_x |_L | [0, a(l)] \), we have

\[
\int_0^\infty x^{1/\beta - 1} I(x) \, dx = \frac{\beta}{\alpha} \int_0^{i_x(0)} i_x^{1/\beta} (x) x^{1/\alpha - 1} \, dx.
\]

Moreover,

\[
h_x(\theta x + (1 - \theta)y) \geq \theta f_x(x) + (1 - \theta) g_x(y), \quad 0 \leq x \leq f_x(0), \quad 0 \leq y \leq g_x(0).
\]

Thus, defining \( i_x = 0, x > i_x(0) \), it follows that

\[
h_x^{1/\beta} (x) x^{1/\alpha - 1} \in \mathcal{F}_{(\alpha^{-1} + \beta^{-1} - 1)^{-1}}(f_x^{1/\beta} (x) x^{1/\alpha - 1}, g_x^{1/\beta} (x) x^{1/\alpha - 1} | R_+)
\]

and (5.1) is an immediate consequence of Theorem 2.1.

STEP 2. \( 1 < \alpha < +\infty, \beta > 0 \).

**Proof of Step 2.** Set \( I(\cdot, \xi) = \xi I, \xi > 0, I = F, G, H \), and note that for all fixed \( \xi, \eta > 0 \),

\[
H(\cdot, \theta \xi + (1 - \theta) \eta) \in \mathcal{F}_{\frac{\alpha}{\alpha + 1} + \beta^{-1}}(F(\cdot, \xi), G(\cdot, \eta) | R_+).
\]

Now using the previous step, we have

\[
(\theta \xi + (1 - \theta) \eta) \int_0^\infty x^{1/\beta - 1} H(x) \, dx \geq M^\theta_{(\alpha/(\alpha + 1)), \beta^{-1}} \left( \xi \int_0^\infty x^{1/\beta - 1} F(x) \, dx, \eta \int_0^\infty x^{1/\beta - 1} G(x) \, dx \right).
\]
If \( F = 0 \) or \( G = 0 \) a.s. \([m_1]\) there is nothing to prove. If not, we set
\[
\xi = \left( \int_0^\infty x^{1/\beta - 1} F(x) \, dx \right)^{\chi(\alpha, \beta)}
\]
and
\[
\eta = \left( \int_0^\infty x^{1/\beta - 1} G(x) \, dx \right)^{\chi(\alpha, \beta)}
\]
and a simple computation gives (5.1).

**Step 3.** \( \alpha < 0, \beta > 0 \).

**Proof of Step 3.** By making some minor changes in the proof of Step 1, the result follows at once. We omit the details here.

**Step 4.** \( \alpha = 0, \beta > 0 \).

**Proof of Step 4.** The inequality (5.1) results from the previous step using an obvious limit argument.

This concludes the proof of Lemma 5.1.

**Proof of Theorem 5.1.** For each \( A \in \langle C \rangle \) the indicator function \( 1_A : (E, C) \to \mathbb{R} \) is non-negative and decreasing and, hence, it is enough to prove that
\[
\int h \, d\tau \geq M^{\theta}_{\alpha, \beta} \left( \int f \, d\mu, \int g \, dv \right).
\]
To this end, first suppose \( \beta \neq 0 \). Then, if \( s, t > 0 \),
\[
[h \geq (\theta s + (1 - \theta)t)^{1/\beta}] \geq \theta[f \geq s^{1/\beta}] + (1 - \theta)[g \geq t^{1/\beta}]
\]
where all the involved sets are \( C \)-invariant. Accordingly,
\[
\tau(h \geq (\theta s + (1 - \theta)t)^{1/\beta}) \geq M^{\theta}_{\alpha}(\mu(f \geq s^{1/\beta}), \nu(g \geq t^{1/\beta}))
\]
and the desired inequality is obvious from Lemma 5.1.

Finally, the case \( \beta = 0, \alpha > 0 \) follows from the case already proved and the case \( \alpha = \beta = 0 \) is a direct consequence of Theorem 2.1.

**Example 5.1.** Suppose \( \mu \in \mathcal{M}_c(E; C) (\alpha \geq 0) \) is concentrated on \(-C\) and let \( c(\alpha, p) = 1, \alpha = 0; = \Gamma(\alpha^{-1} + p + 1), \alpha > 0 \). If \( \varphi : -C \to \mathbb{R}_+ \) is Borel measurable, concave, and decreasing, then the function
\[ p \sim \frac{c(x, p)}{\Gamma(p + 1)} \int_0^{+\infty} \varphi^p d\mu, \quad p > 0, \]

is 0-concave.

To prove this assertion there is no loss of generality assuming \( \mu \in \mathcal{M}_2(\mathbb{R}; \mathbb{R}^-) \), \( \varphi(x) = x \in \mathbb{R}_+ \) and the result follows exploiting the same line of reasoning as in the author's work [8], which treats the case \( x = 1/n, n \in \mathbb{N}_+ \).

For the case \( x = 0, p \geq 1 \), see also Marshall and Olkin [20, p. 494].

It is simple to settle variants of the above conclusion in the parameter interval \(-\infty < x < 0\) to the cost of some beauty.

We shall next discuss some examples of convexity in potential theory.

**Example 5.2.** Let \( a_1, \ldots, a_n \) be non-zero vectors in Euclidean \( \mathbb{R}^3 \) satisfying \( \langle a_i, a_j \rangle \geq 0 \), \( i, j = 1, \ldots, n \). Suppose \( \mu \in \mathcal{B}(\mathbb{R}^3) \) is concentrated on the union of the line segments \([0, a_i], i = 1, \ldots, n\), and assume \( \mu \) reduces to a linear measure on each individual line segment. Of course, \( \mu \) is 1-concave in \( \langle C \rangle \), where \( C \) is the convex cone spanned by the \( a_i \). From the above assumptions we conclude that the Newtonian potential of \( \mu \), that is \( \int d\mu(y)/|x - y| \), is a \( -\infty \)-convex function of \( x \) in \( -C^+ \).

**Example 5.3.** Let \( \Gamma \) be a closed convex cone in \( \mathbb{R}^n \) and suppose \( f; (\mathbb{R}^n; \Gamma) \rightarrow (\mathbb{R}^n; \Gamma) \) is an increasing, convex, and uniformly Lipschitz continuous function. Below we let \( X \) denote the Brownian motion in \( \mathbb{R}^n \) with the drift vector \( f \), that is

\[ dX(t) = dB(t) + f(X(t)) dt, \quad t \geq 0, \]

where \( (B(t), t \geq 0) \) stands for the standard Brownian motion in \( \mathbb{R}^n \). It is natural that \( X \) inherits suitable convexity properties from those of the drift vector and the Brownian motion. To explain this, let \( \Omega = (\mathcal{C}(\mathbb{R}_+))^n \), \( \Omega_{\Gamma} = \{ \omega \in \Omega; \omega(t) \in \Gamma, t \geq 0 \} \), and \( \mu_x = P_x[\cdot | X(0) = x] \), respectively. We claim that

\[ \mu_{\theta x + (1 - \theta)y} \in \mathcal{M}_0(\mu_x, \mu_y; \Omega_{\Gamma}). \]

This is evident if \( f = 0 \). To prove the general case, suppose \( \omega \in \Omega \) is fixed and define

\[
\begin{cases}
X_0(\omega, t) = \omega(t) \\
X_{k+1}(\omega, t) = \omega(t) + \int_0^t f(X_k(\omega, s)) ds, \quad t \geq 0.
\end{cases}
\]

Here each map \( X_k: (\Omega; \Omega_{\Gamma}) \rightarrow (\Omega; \Omega_{\Gamma}) \) is (increasing and) convex and applying Theorem 3.1, we have
Now using Corollary 4.1, the claim above follows by letting \( k \) tend to plus infinity.

Suppose \( g : (\mathbb{R}^n; \Gamma) \to \mathbb{R} \) is bounded from below, increasing, and convex and let \( A \in \langle \Gamma \rangle \) be convex. As is well-known the physical solution of the initial-value problem

\[
\begin{cases}
\frac{1}{2}Au + f \cdot \nabla u - gu = \partial u / \partial t, & t > 0 \\
u(\cdot, 0) = 1_A
\end{cases}
\]

is given by the Feynman–Kac formula

\[
u(x, t) = \int_{\omega(t) \in A} \exp \left( - \int_0^t g(\omega(s)) \, d\mu_x(\omega) \right).
\]

Consequently, \( u(\cdot, t) \) is 0-concave for each fixed \( t > 0 \) and, of course, the same function decreases as a mapping of \( (\mathbb{R}^n; \Gamma) \) into \( \mathbb{R} \).

**Theorem 5.2.** For each \( i \in \{1, 2\} \), let \( (E_i; C_i) \) be semi-ordered, locally convex Hausdorff spaces over \( \mathbb{R} \) and suppose \( \tau_i \in \mathcal{M}_x^\theta(\mu_i, v_i; C_i) \), where \( \alpha_i \in \mathbb{R} \) and \( \alpha_1 + \alpha_2 \geq 0 \). Then \( \tau_1 \otimes \tau_2 \in \mathcal{M}_x^\theta(\mu_1 \otimes \mu_2, v_1 \otimes v_2; C_1 \times C_2) \). In particular, if \( E_1 = E_2 = E \), then \( \tau_1 \ast \tau_2 \in \mathcal{M}_x^\theta(\mu_1 \ast \mu_2, v_1 \ast v_2; C_1 + C_2) \).

**Proof.** For every \( M \subseteq E_1 \times E_2 \) and \( x_1 \in E_1 \), set

\[
M(x_1) = \{ x_2 \in E_2; (x_1, x_2) \in M \}.
\]

Now choose \( A, B \in \langle C_1 \times C_2 \rangle \) arbitrarily but fixed and note that for all \( x_1, y_1 \in E_1 \),

\[
(\theta A + (1 - \theta)B)(\theta x_1 + (1 - \theta)y_1) \geq \theta A(x_1) + (1 - \theta)B(y_1)
\]

where each individual set is \( C_2 \)-invariant. Hence

\[
\tau_2((\theta A + (1 - \theta)B)(\theta x_1 + (1 - \theta)y_1)) \geq M^\theta_{x_2}(\mu_2(A(x_1)), v_2(B(y_1)))
\]

and since for each \( c_1 \in C_1, A(x_1 - c_1) \supseteq A(x_1) \), and \( B(y_1 - c_1) \supseteq B(y_1) \), the Fubini theorem and Theorem 5.1 imply that

\[
(\tau_1 \otimes \tau_2)((\theta A + (1 - \theta)B) \geq M^\theta_{x_1, x_2}((\mu_1 \otimes \mu_2)(A), (v_1 \otimes v_2)(B))
\]

Finally, the last statement in Theorem 5.2 follows by combining Theorem 3.1 and the first part of Theorem 5.2.

**Corollary 5.1.** If \( \alpha, \beta \in \mathbb{R} \) and \( \alpha + \beta \geq 0 \), then

\[
\mathcal{M}_\alpha(E; C) \ast \mathcal{M}_\beta(E; C) \subseteq \mathcal{M}_{x(\alpha, \beta)}(E; C).
\]
Corollary 5.1 is known in at least one special case for which \( C = E \) is a proper cone. In fact, the inclusion
\[
\mathcal{M}_0(\mathbb{R}; \mathbb{R}_-) \ast \mathcal{M}_0(\mathbb{R}; \mathbb{R}_-) \subseteq \mathcal{M}_0(\mathbb{R}; \mathbb{R}_-)
\]
is frequently used in the theory of reliability [1].

We will end this section by proving some complements of the results obtained so far. Below \( X \) is a real-valued random variable and \( X_1, \ldots, X_n \) stand for stochastically independent copies of \( X \).

First note that
\[
P_X \in \mathcal{M}_\alpha(\mathbb{R}; \mathbb{R}_+) \Rightarrow P_{\max_{1 \leq i \leq n} X_i} \in \mathcal{M}_{\frac{\alpha}{n}}(\mathbb{R}; \mathbb{R}_+)
\]
for each \(-\infty \leq \alpha < +\infty\). Here the special case \( 0 \leq \alpha < +\infty \), in fact, is included in Theorem 5.2. More interesting, we have

**Theorem 5.3.** Assume \(-\infty < \alpha < +\infty \) and let \( \beta = \beta(\alpha) \) be the largest member \(-\infty \leq \beta < +\infty \) having the following property:
\[
(\forall n \in \mathbb{N}_+)(P_X \in \mathcal{M}_\alpha(\mathbb{R}; \mathbb{R}_-) \Rightarrow P_{\max_{1 \leq i \leq n} X_i} \in \mathcal{M}_\beta(\mathbb{R}; \mathbb{R}_-)).
\]

Then \( \beta(\alpha) > -\infty \). Moreover, \( \beta(\alpha) \leq \alpha \), where equality occurs if and only if \( \alpha \geq -1 \).

Theorem 5.3 is well-known if \( \alpha = 0 \) [1, p. 38]. The general case follows at once from the next

**Lemma 5.2.** Suppose \( n \in \mathbb{N}_+, \alpha, \beta \in \mathbb{R} \setminus \{0\}, \alpha \beta > 0 \), and \( f(x) = (1 - (1 - x^{1/\alpha} n)^\beta), \ x > 0, \ x^{1/\alpha} < 1. \) Then for any \( \alpha > 0 \) \(-1 \leq \alpha < 0\) the largest \( \beta \) such that \( f \) is concave [convex] equals \( \alpha \). If \( \alpha < -1 \), then there exists a \( \beta \), independent of \( n \), such that \( f \) is convex for every \( n \geq 1 \). The largest \( \beta \) with this property is strictly smaller than \( \alpha \).

**Proof.** The second derivative of \( f(x) \) equals \( \alpha \) times a strictly positive function times
\[
g(y) = 1 - n + (n - \alpha)y + (n\beta - 1)y^n + (\alpha - n\beta)y^{n+1}, \quad y = 1 - x^{1/\alpha}.
\]
Since \( g(1-) = 0 \) and \( g'(1-) = n(\alpha - \beta) \), necessarily, \( \beta \leq \alpha \) if \( g \leq 0 \). Moreover, note that \( g'' \) has at most one change of sign and that \( g(0+) \leq 0 \). Also, if \( \alpha = \beta \), then \( g''(1-) < 0 \) (respectively \( > 0 \)) if and only if \( (n-1)(\alpha + 1) > 0 \) (respectively \( < 0 \)). Consequently,
\[
\alpha = \beta \geq -1 \Rightarrow g \leq 0
\]
and
\[ \alpha = \beta < -1 \Rightarrow \neg (g \leq 0, \text{ all } n). \]

In the following we suppose that the parameter \( \alpha \) is strictly smaller than \(-1\).

If \( \beta(\alpha) \) has the same meaning as in Theorem 5.3, then
\[ -\beta(\alpha) = \sup \left\{ \frac{(1 + n + (n - \alpha) y - y^n + \alpha y^{n+1})}{(n y^n - y^{n+1})} : 0 < y < 1, n \in \mathbb{N}_+ \right\}. \]

Thus, \( \beta(\alpha) > -\infty \) if and only if
\[ \sup \left\{ \frac{(1 + n + (n - \alpha) y + (\alpha - 1) y^n)}{(n y^n + (1 - y))} : 0 < y < 1, n \in \mathbb{N}_+ \right\} < +\infty. \]

Now setting
\[ h_\alpha(z) = \frac{(1 - \alpha - (n - \alpha) z + (\alpha - 1)(1 - z)^n)}{(n z(1 - z)^n)}, \quad 0 < z < 1, \]

and noting that \( h_{\alpha}(1) = 0, \alpha, 1 \leq 0, \), we conclude that \( \beta(\alpha) > -\infty \) if and only if
\[ \sup \left\{ h_\alpha(z) : 0 < z < (1 - \alpha)/n, n \in \mathbb{N}_+ \right\} < +\infty. \]

This, however, follows at once from the formula
\[ 2h_\alpha(z) = (1 - \alpha)h_{-1}(z) - (1 - 1/n)(\alpha + 1)/(1 - z)^n \]

and the already proved fact that the quantity \( h_{-1}(z) = h_{-1}(z, n) \) is uniformly bounded from above. Lemma 5.2 is thereby completely proved.

6. Examples of stochastic processes with increasing paths inducing 0-concave measures in suitable convex cones.

Throughout the present section \( I \) is assumed to be a fixed subinterval of the real line and \( R_n^I \) means \( R^I \) equipped with the topology of pointwise convergence.

As is well-known and easy to see each real-valued stochastic process \( X = (X(t), t \in I) \) satisfying
\[ P[X(s) \leq X(t)] = 1, \quad s \leq t, \]

induces a Radon probability measure \( P_X \) on \( R_n^I \) such that the closed convex cone of all increasing functions on \( I \) supports \( P_X \). For additional information, see e.g. Tjur [29, p. 170].

Now suppose \( Q: R \rightarrow ]-\infty, +\infty] \) is a decreasing function such that \( Q(x)^{\uparrow} + \infty, x \downarrow -\infty \), and \( Q(x)^{\downarrow} 0, x^{\uparrow} + \infty \). The extremal-\( Q \) process \( X = (X(t), t > 0) \), introduced by Dwass [14] and Lamperti [19], is a real-valued stochastic process characterized by the following equation
\[
\left\{ P[X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n] = \exp \left[ - \sum_{k=1}^{n} (t_k - t_{k-1}) Q(x_k \wedge \ldots \wedge x_n) \right] \right\}
\]

all \(0 = t_0 < t_1 < \ldots < t_n, x_1, \ldots, x_n \in \mathbb{R}, n \in \mathbb{N}_+\).

If \(0 = t_0 < t_1 < \ldots < t_n\) and \(U_1, \ldots, U_n\) are real-valued stochastically independent random variable with

\[
P[U_k \leq x] = \exp \left[ - (t_k - t_{k-1}) Q(x) \right], \quad k = 1, \ldots, n,
\]

then the random vectors \((X(t_1), \ldots, X(t_n))\) and \((U_1, U_1 \lor U_2, \ldots, U_1 \lor \ldots \lor U_n)\) obey the same probability law. Thus, combining Theorems 3.1 and 5.2, we have

**Theorem 6.1.** An extremal-Q process induces a 0-concave measure in \(\langle \mathbb{R}_+^{10}, +\infty \rangle\) if and only if \(Q\) is convex.

**Example 6.1.** Consider a real-valued homogeneous Lévy process \(X = (X(t), t > 0)\), where

\[
E[\exp (i\zeta X(1))] = \exp \left( \int_{-\infty}^{+\infty} (e^{i\theta x} - 1 - i\zeta \sin x) d\tau(x) \right)
\]

and \(\tau\) is a positive Borel measure on \(\mathbb{R}\) such that \(\{x \sim x^2\} \in L_{1,\text{loc}}(\tau)\) and \(\tau(\mathbb{R} \setminus [-x, x]) < +\infty, x > 0\). By a theorem of Dwass [15, p. 382], the stochastic process

\[
Y(t) = \sup_{0 < s \leq t} (X(s+) - X(s-))^+, \quad t > 0,
\]

is an extremal-Q process with \(Q(x) = +\infty, x < 0; = \tau(\{x, +\infty\}, x > 0\) (see also Resnick and Rubinovitch [26, Th. 1]). In particular, if \(0 < \alpha < 2\) and \(X\) is an \(\alpha\)-stable, symmetric, and homogeneous Lévy process, then \(\tau(\{x, +\infty\}) = \text{const.} x^{-\alpha}, x > 0\), and, hence, \(P_Y\) is 0-concave in \(\langle \mathbb{R}_+^{10}, +\infty \rangle\).

Recall that a real-valued stochastic process \(X = (X(t), t \in I)\) is called additive if the increments \(X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})\) are stochastically independent for all points of time \(t_1 < \ldots < t_n, n \in \mathbb{N}_+\). Below \(D_+(I)\) denotes the set of all non-negative increasing functions on \(I\).

**Theorem 6.2.** Any increasing and additive stochastic process \(X = (X(t), t \in I)\), processing \(\mathcal{M}_0(\mathbb{R}; \mathbb{R}_+)\) distributed increments, induces a 0-concave measure in \(\langle D_+(I) \rangle\).

**Proof.** Suppose \(\xi_1, \ldots, \xi_m \in (D_+(I))^+\) and choose \(t_1 < \ldots < t_n\) such that each \(\xi_j\) only depends on the coordinates \(x(t_1), \ldots, x(t_n)\). Then, from Theorem 5.2,
\[ P_{x(t_1), x(t_2) - x(t_1), \ldots, x(t_n) - x(t_{n-1})} \in \mathcal{M}_0(\mathbb{R}^n; \mathbb{R}_+^n) \]

and using Theorem 3.1 we conclude that \( P_{x(t_1), \ldots, x(t_n)} \) is 0-concave in \( \langle \{x \in \mathbb{R}^n; 0 \leq x_1 \leq \ldots \leq x_n \} \rangle \). Hence

\[ P_{\xi_1(x), \ldots, \xi_n(x)} \in \mathcal{M}_0(\mathbb{R}^m, \mathbb{R}_+^m) \]

and the result follows from Theorem 4.1.

**Example 6.2.** Let \( X = (X(t), t \geq 0) \) be an one-sided, stable, and homogeneous Lévy process. Remembering Example 3.1 we have that \( P_X \) is 0-concave in \( \langle D_+(\mathbb{R}_+) \rangle \).

Now suppose \( B \) denotes a standard Brownian motion in \( \mathbb{R} \) with \( B(0)=0 \) and let \( \tau_x \) be the first time \( B \) hits \( x \neq 0 \). Since \( (\tau_x)_x>0 \) is a one-sided \( \frac{1}{2} \)-stable homogeneous Lévy process it follows that the probability

\[ P \left[ \max_{0 \leq t \leq \tau_k} B(t) \geq x_k, \ k=1, \ldots, n \right] \]

is a 0-concave function of \( (t_1, \ldots, t_n)>0 \) for all fixed \( x_1, \ldots, x_n>0 \).

**Example 6.3.** Consider an extremal-\( Q \) process \( X = (X(t), t > 0) \) such that \( a = \inf \{ x; Q(x) < +\infty \} \) and \( b = \sup \{ x; Q(x) > 0 \} \) do not coincide. Set \( X^{-1}(x) = \inf \{ t; X(t) > x \} \), \( a < x < b \). From Resnick [25, Th. 1], we know that the stochastic process \( X^{-1} \) is increasing and additive. Moreover, for arbitrary \( a < x < y < b \),

\[ P[X^{-1}(x) \leq t] = 1 - \exp(-tQ(x)), \quad t > 0, \]

and

\[ P[X^{-1}(y) - X^{-1}(x) \leq t] = \theta + (1 - \theta)(1 - \exp(-tQ(y))), \quad t > 0, \]

for a suitable \( 0 < \theta = \theta(x, y) < 1 \). Consequently, \( P_{X^{-1}} \) is 0-concave in \( \langle D_+([a, b]) \rangle \).

7. A zero-one law.

A non-empty subset \( G \) of \( E \) is said to be an additive subgroup of \( E \) if \( G - G = G \).

**Theorem 7.1.** Suppose \( \mu \in \mathcal{M}_a(E; C) \) and let \( G \) be a \( \mu \)-measurable additive subgroup of \( E \) with strictly positive \( \mu \)-measure.

a) If \( G \) is \( C \)-invariant, then \( \mu \) is supported on \( G \).

b) If \( \alpha > -\infty \), then \( \mu \) is supported on \( C + G \).
Here Part a) is a pure extension of the zero-one law for \(-\infty\)-convex measures [4].

**Proof.** We first choose a compact set \(K = -K \subseteq G\), with \(\mu(K) > 0\), and set

\[
A = C + \bigcup \left[ K + \ldots + K : n \in \mathbb{N}_+ \right].
\]

Now, because \(\mu(A \cap (K - C)) > 0\), there exists a compact set \(L \subseteq E \setminus [A \cup (K - C)]\) such that \(\mu(E \setminus (A \cup L)) < \mu(K - C)\). Moreover, for each \(n \in \mathbb{N}_+\),

\[
E \setminus (A \cup L) \supseteq \frac{1}{n + 1} \left[ E \setminus \{ A \cup (nK + (n + 1)L + C) \} \right] + \frac{n}{n + 1} (K - C)
\]

and as the complement of a \(-C\)-invariant set is \(C\)-invariant, we have

\[
\mu(E \setminus (A \cup L)) \geq \min \left( \mu(E \setminus \{ A \cup (nK + (n + 1)L + C) \}), \mu(K - C) \right).
\]

Thus

\[
\mu(E \setminus (A \cup L)) \geq \mu(E \setminus \{ A \cup (nK + (n + 1)L + C) \})
\]

and, hence,

\[
\mu(nK + (n + 1)L + C) \geq \mu(L), \quad \text{all } n \in \mathbb{N}_+.
\]

However, for any fixed compact \(M \subseteq E\), \(M \cap (nK + (n + 1)L + C) = \emptyset\) for an appropriate \(n \in \mathbb{N}_+\) and it follows that \(\mu(L) = 0\), which proves Part a).

To show Part b), first note that \(\mu(E \setminus A) < \mu(K - C)\). If \(\mu(E \setminus A) > 0\), then we may use the relation

\[
E \setminus A \supseteq \frac{1}{2}(E \setminus A) + \frac{1}{2}(K - C)
\]

and have

\[
\mu^x(-1)(E \setminus A) \leq \frac{1}{2} \mu^x(-1)(E \setminus A) + \frac{1}{2} \mu^x(-1)(K - C)
\]

that is, \(\mu(E \setminus A) \geq \mu(K - C)\), which is a contradiction. Thus \(\mu(E \setminus A) = 0\) and Part b) is proved, too.

**Corollary 7.1.** Let \(\mu \in \mathcal{M}_a(E; C) (a > -\infty)\). If \(a \in E\) is an atom of \(\mu\), then \(\mu\) is concentrated on \(a + C\).

8. Integrability of sublinear functions.

A function \(\varphi : E \to \mathbb{R} \cup \{ +\infty \}\) is said to be an extended valued sublinear function if
\[
\begin{cases}
\varphi(x+y) \leq \varphi(x) + \varphi(y), & x, y \in E, \\
\varphi(\lambda x) = \lambda \varphi(x), & \lambda > 0, x \in E.
\end{cases}
\]

Below, for any \( \varphi: E \to \mathbb{R} \cup \{+\infty\} \), we set \( \varphi_-(x) = \varphi(-x), x \in E \).

**Theorem 8.1.** Suppose \( \mu \in \mathcal{M}_a(E; C) \) \((\alpha > -\infty)\) and let \( \varphi \) and \( \varphi_- \) be \( \mu \)-measurable extended valued sublinear functions such that \( \varphi|_C < +\infty \) and \( \mu(\varphi + \varphi_- < +\infty) > 0 \). Then \( \varphi < +\infty \) a.s. \([\mu]\). If \( \varphi \geq 0 \), \( \varphi|_C = 0 \), and

(i) \(-\infty < \alpha < 0\), then \( \varphi^p \in L_1(\mu), 0 < p < -1/\alpha \),

(ii) \( \alpha = 0 \), then \( \exp(\varepsilon \varphi) \in L_1(\mu) \) for some \( \varepsilon > 0 \),

(iii) \( \alpha > 0 \), then \( \varphi \in L_\infty(\mu) \).

In the special case \( C = \{0\} \), Theorem 8.1 is well-known [4]. For connections with integrability of Gaussian semi-norms, see e.g. [17].

**Proof.** The first part of Theorem 8.1 follows from Theorem 7.1. Now suppose \( \varphi \geq 0 \) and \( \varphi|_C = 0 \). Then, for all \( s > 0 \) and \( t > 1 \),

\[
[\varphi > s] \supseteq \frac{2}{t+1} [\varphi \geq st] + \frac{t-1}{t+1} [\varphi_- < s]
\]

where the sets in the right-hand side are \( C \)-invariant. Consequently,

\[
\mu(\varphi > s) \geq M_a^{2/(t+1)}(\mu(\varphi \geq st), \mu(\varphi_- < s)).
\]

**Case (i):** First choose an \( s > 0 \) satisfying the inequalities

\[
\mu^\alpha(\varphi > s) > 2\mu^\alpha(\varphi_- < s) > 0.
\]

Then \( \mu(\varphi \geq st) = O(t^{1/\alpha}) \) as \( t \to +\infty \) and thus \( \varphi^p \in L_1(\mu) \) for each \( 0 < p < -1/\alpha \).

**Case (ii) may be treated as Case (i).**

**Case (iii):** If \( \varphi \notin L_\infty(\mu) \), then for all large \( s > 0 \)

\[
\mu^\alpha(\varphi > s) \geq \frac{2}{3} \mu^\alpha(\varphi \geq 2s) + \frac{1}{3} \mu^\alpha(\varphi_- < s)
\]

which implies the contradiction \( 0 \geq (1/3) \mu^\alpha(\varphi_- < +\infty) \).

This completes the proof of Theorem 8.1.

Recall that a measure \( \mu \in \mathcal{R}(E) \) has a barycentre at the point \( e \in E \) if \( E' \subseteq L_1(\mu) \) and \( \xi(e) = \int \xi \, d\mu, \xi \in E' \). The next theorem is an example of an application of Theorem 8.1.
THEOREM 8.2. Assume $\mu \in \mathcal{M}_a(E; C)$ ($\alpha > -1$) has a barycentre $e \in E$. Moreover, suppose $G$ is an affine linear subspace of $E$ such that $\mu(K) > 0$ for a suitable compact and convex $K \subseteq G$. Then $e \in C + G$.

PROOF. Of course, there is no loss of generality to set $e = 0$. Now write $G = F - a$, where $a \in -G$ is fixed. If $0 \notin C + G$, that is, $a \notin C + F$, then we obtain a contraction as follows.

Suppose $L \subseteq F$ is a compact, convex, and symmetric set such that $\mu_a(L) = \mu(L - a) > 0$ and choose for each $n \in \mathbb{N}_+$ a $\xi_n \in E'$ such that $\xi_n(x) > \xi_n(a)$, $x \in C + nL$. Obviously, each $\xi_n \in C^+$ and without loss of generality we may assume that $\xi_n(a) = -1$. Set $\varphi = \sup_{n \in \mathbb{N}_+} \xi_n^-$. Then $\mu_a(\varphi + \varphi^- < +\infty) > 0$ and $\varphi|_C = 0$. Thus $\varphi \in L_1(\mu_a)$ by Theorem 8.1 and it follows that

$$\lim_{n \to +\infty} \int \xi_n^- d\mu_a = 0$$

since in view of Theorem 7.1, $\xi_n^- \to 0$ a.s. $[\mu_a]$ as $n \to +\infty$. But

$$\int \xi_n^- d\mu_a \geq \left( \int \xi_n d\mu_a \right)^- = 1$$

and we have got a contradiction.

REFERENCES


