UNIQUENESS OF HAHN–BANACH EXTENSIONS
AND LIFTINGS OF LINEAR DEPENDENCES

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Abstract.

We study intersection properties of balls for a subspace $M$ of a Banach space $E$ which ensures either that each linear functional on $M$ has a unique norm-preserving extension to $E$ or that if $f_1, \ldots, f_n \in M^*$ are such that $\sum_{i=1}^n f_i = 0$, then every $f_i$ has a norm-preserving extension $g_i \in E^*$ such that $\sum_{i=1}^n g_i = 0$. We relate these properties to the existence of norm-1 projection in $E^*$ with kernel $M^\perp$.

1. Introduction.

Let $E$ be a real Banach space and let $M$ be a closed subspace. The dual space of $E$ is denoted $E^*$ and the annihilator of $M$ in $E^*$ is denoted $M^\perp$. $B(x, r)$ denotes the closed ball in $E$ with center $x$ and radius $r$. The closure of a set $S$ is denoted $\bar{S}$, its convex hull $\text{conv}(S)$ and the distance from $y$ to $S$ by $d(y, S)$. The unit ball of $E$ is written $E_1$, and the set of extreme points of a set $S$ is denoted $\delta S$.

We shall study extensions of linear functionals from $M$ to $E$ and we write for $f \in E^*$, $\|f\|_M$ for the norm of the restriction $f|_M$ of $f$ to $M$. By $M^\sharp$ we mean

$$M^\sharp = \{f \in E^* : \|f\| = \|f\|_M\}$$

$L(E, F)$ (respectively $K(E, F)$) denotes the space of bounded (respectively compact) linear operators from $E$ into $F$.

$M$-ideals were first studied by Alfsen and Effros in [1]. They called $M$ an $M$-ideal if there exists a projection $P$ in $E^*$ such that $P(E^*) = M^\perp$ and for all $f \in E^*$:

$$\|f\| = \|Pf\| + \|f - Pf\|.$$

One characterization of $M$-ideals is as follows:

$M$ is an $M$-ideal in $E$ if and only of whenever $\{B(a_i, r_i)\}_{i=1}^n$ is a finite family of balls in $E$ such that

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\[
\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset \quad \text{and} \quad M \cap B(a_i, r_i) \neq \emptyset \quad \text{for all } i,
\]
then \(M \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset\) for all \(\varepsilon > 0\). \[1, \[10].

For example, \(c_0\) is an \(M\)-ideal in \(l_\infty\). In this paper we are looking for weaker intersection properties which characterize those subspaces \(M\) of \(E\) such that \(M^\perp\) is the kernel of a norm-1 projection in \(E^*\).

One direction of weakening the intersection properties is to start with the characterization of semi \(M\)-ideals as defined in \[10\]. This leads us to characterizations of subspaces \(M\) such that if \(f \in M^*\), then \(f\) has a unique norm-preserving extension to \(E\). An example of this is Theorem 2.2 which says that if \(M\) is a closed subspace of \(E\), then we have:

Every \(f \in M^*\) which attains its norm on \(M_1\) has a unique norm-preserving extension to \(E\) if and only if whenever \(x \in M\), \(y \in E\) with \(\|x\| = \|y\| = 1\) and \(\varepsilon > 0\), there exists \(r \geq 1\) such that

\[
M \cap B(y + rx, r + \varepsilon) \cap B(y - rx, r + \varepsilon) \neq \emptyset.
\]

This intersection property characterize semi \(M\)-ideals if we can take \(r = 1\).

In the other direction we generalize the intersection property characterizing \(M\)-ideals in that we require that the centers of the balls are in \(M\). Then we get a result that ensure that we can obtain simultaneous norm-preserving extensions of several linear functionals. For instance, Theorem 3.1 implies that the following statements are equivalent:

(i) If \(\{B(a_i, r_i)\}_{i=1}^n\) are balls with centers in \(M\) and \(\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset \) in \(E\), then \(M \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) = \emptyset\) for all \(\varepsilon > 0\).

(ii) If \(f_1, \ldots, f_n \in M^*\) are such that \(f_1 + \ldots + f_n = 0\), then there exist norm-preserving extensions \(g_i\) of \(f_i\) such that \(g_1 + \ldots + g_n = 0\).

As shown in Corollary 4.9, if \(E\) is a smooth Banach space, then (i) with \(n = 3\) is equivalent to \(M^\perp\) being the kernel of a norm-1 projection in \(E^*\).

In the course of these investigations, we also get characterizations of HB-subspaces. HB subspaces were defined by Hennefeld in [6]. He said that \(M\) is an HB subspace of \(E\) if \(M^\perp\) is complemented by a subspace \(M_*\) in \(E^*\) such that whenever \(f_\ast \in M_*\) and \(f^\perp \in M^\perp \setminus \{0\}\), then \(\|f_\ast + f^\perp\| \geq \|f^\perp\|\) and \(\|f_\ast + f^\perp\| > \|f_\ast\|\). In Theorem 4.1 we show that \(M\) is an HB-subspace of \(E\) if and only if \(M\) has property (i) above and every \(f \in M^*\) has a unique norm-preserving extension to \(E\).

We follow Sullivan [17] and say that \(M\) is (weakly) Hahn–Banach smooth in \(E\) if every \(f \in M^*\) (which attains its norm on \(M_1\)) has a unique norm-preserving extension to \(E\). By Phelps [14] and others, this has been called
property $U$. With our notation we get that $E$ is smooth if and only if every subspace of $E$ is weakly Hahn–Banach smooth, and $E^*$ is strictly convex if and only if every subspace of $E$ is Hahn–Banach smooth. We call the intersection property in (i) the $n.E.$ intersection property ($n.E.I.P.$) This resemble the $n.k.$ intersection property as defined in [12].

2. Uniqueness of Hahn–Banach extensions.

We shall say that a subspace $M$ of $E$ is Hahn–Banach smooth in $E$ if every functional on $M$ has a unique norm-preserving extension to $E$. Moreover, $M$ is weakly Hahn–Banach smooth in $E$ if every functional on $M$ which attains its norm on the unit ball of $M$ has a unique norm-preserving extension to $E$.

M. Smith and F. Sullivan studied in [16] spaces $E$ which are Hahn–Banach smooth or weakly Hahn–Banach smooth in $E^{**}$. They showed that if a space $E$ is weakly Hahn–Banach smooth in $E^{**}$, then $E^*$ has the Radon–Nikodym property.

A. E. Taylor [18] and S. R. Foguel [4] have shown that every subspace of $E$ is Hahn–Banach smooth in $E$ if and only if $E^*$ is strictly convex.

From R. R. Phelps [14], we get the following theorem. We use the notation

$$M^* = \{ f \in E^* : \|f\| = \|f\|_M \}.$$ 

**Theorem 2.1.** Let $M$ be a closed subspace of $E$. The following statements are equivalent:

1) $M$ is Hahn–Banach smooth in $E$.
2) $M^\perp$ is a Haar-subspace of $E^*$, i.e. if $x \in E^*$, then there exists a unique $y \in M^\perp$ such that $\|x - y\| = d(x, M^\perp)$.
3) If $x, y \in M^*$ and $x + y \in M^\perp$, then $x + y = 0$.
4) Every element in $E^*$ can be written in a unique way as a sum of elements from $M^*$ and $M^\perp$.

It is known that semi $M$-ideals are Hahn–Banach smooth [10]. Recall from [10] that $M$ is a semi $M$-ideal in $E$ if and only if whenever $x \in M$, $y \in E$ with $\|x\| = 1 = \|y\|$ and $\varepsilon > 0$, then there exists $z \in M$ such that $\max \|x \pm (y - z)\| \leq 1 + \varepsilon$. We can generalize this result as follows.

**Theorem 2.2.** The following statements are equivalent for a closed subspace $M$ of $E$.

1) $M$ is weakly Hahn–Banach smooth in $E$.
2) If $x \in M$, $y \in E$ with $\|x\| = 1 = \|y\|$ and $\varepsilon > 0$, then there exist $r \geqslant 1$ and $z \in M$ such that
\[
\max \| rx \pm (y-z) \| \leq r + \varepsilon. \\
\]

**Proof.** 2) \(\Rightarrow\) 1). Let \(f \in M^*\) be such that \(\| f \| = f(x)\) for some \(x \in M\) with \(\| x \| = 1\). Let \(g, h \in E^*\) be norm-preserving extensions of \(f\) and let \(\varepsilon > 0\). Then it suffices to show that \((g-h)(y) \leq \varepsilon 2 \| f \|\) for each \(y \in E\) with \(\| y \| = 1\).

Let \(r \geq 1\) and let \(z \in M\) be such that \(\max \| rx \pm (y-z) \| \leq r + \varepsilon\). Then we have \(\pm\)

\[
(g-h)(y) + 2r \| f \| = (g-h)(y) + 2r f(x) \\
= (g-h)(y) + g(rx) + h(rx) \\
= g(rx + y - z) + h(rx - y + z) \\
\leq \| g \| \cdot \| rx + y - z \| + \| h \| \cdot \| rx - y + z \| \\
\leq 2 \| f \| (r + \varepsilon). 
\]

Hence \((g-h)(y) \leq \varepsilon 2 \| f \|\).

1) \(\Rightarrow\) 2). Assume 2) is false. Then there exist \(x \in M\), \(y \in E\) with \(\| x \| = 1 = \| y \|\) and \(\varepsilon > 0\) such that

\[
M \cap B(y+rx, r+\varepsilon) \cap B(y-rx, r+\varepsilon) = \emptyset \quad \text{for all } r \geq 1.
\]

Let

\[
A = \bigcup_{r \geq 1} B(y+rx, r) \quad \text{and} \quad B = \bigcup_{r \geq 1} B(y-rx, r).
\]

Let \(A_M = \{(x, x) \in M \times M\}\). \(A\) and \(B\) are convex, and

\[
A_M \cap [(A \times B) + B(0, \varepsilon)] = \emptyset
\]

in \(E \oplus_\infty E\). By the Hahn–Banach theorem, there exist \(\lambda \in \mathbb{R}\) and \(g_1, g_2 \in E^*\) such that

\[
\sup_{x \in M} (g_1 + g_2)(x) < \lambda < \inf_{(u, v) \in A \times B} (g_1(u) + g_2(v)).
\]

Since \(M\) is a subspace, we get \(g_1 + g_2 \in M^\perp\). If \((u, v) \in A \times B\), then we have

\[
\| u - (y+rx) \| \leq r \quad \text{and} \quad \| v - (y-rx) \| \leq r
\]

for all sufficiently large \(r\). Hence

\[
0 < \lambda < \inf_{r \geq 1} [g_1(y+rx) + g_2(y-rx) - r \| g_1 \| - r \| g_2 \|].
\]

From this we get

\[
r \| g_1 \| + r \| g_2 \| + \lambda < g_1(y+rx) + g_2(y-rx).
\]

\[
\]
We divide by \( r \) and let \( r \to \infty \). Hence
\[
\|g_1\| + \|g_2\| \leq g_1(x) + g_2(-x) \leq \|g_1\| + \|g_2\|.
\]
Thus \( g_1 \) and \( -g_2 \) are norm-preserving extensions of \( f = g_1|_M \). Moreover,
\[
r\|g_1\| + r\|g_2\| + \lambda \leq g_1(y) + r\|g_1\| + g_2(y) + r\|g_2\|
\]
so that
\[
0 < \lambda \leq g_1(y) + g_2(y).
\]
Thus \( g_1 \neq -g_2 \).

From Taylor [18], Foguel [4] and Phelps [14], we have

**Theorem 2.3.** The following statements are equivalent:

1) \( E^* \) is strictly convex.
2) Every closed subspace of \( E \) is Hahn–Banach smooth in \( E \).
3) Every closed hyperplane through 0 in \( E \) is Hahn–Banach smooth in \( E \).

The following theorem is easy.

**Theorem 2.4.** The following statements are equivalent:

1) \( E \) is smooth.
2) Every one dimensional subspace of \( E \) is weakly Hahn–Banach smooth in \( E \).
3) Every closed subspace of \( E \) is weakly Hahn–Banach smooth in \( E \).
4) Every closed hyperplane through 0 in \( E \) is weakly Hahn–Banach smooth in \( E \).

**Proof.** 1) \( \Rightarrow \) 3) \( \Rightarrow \) 4) and 3) \( \Rightarrow \) 2) \( \Rightarrow \) 1) are trivial.

4) \( \Rightarrow \) 1). Assume 1) is false. Then there exist \( x \in E \), \( \|x\| = 1 \) and \( f, g \in E^* \) with \( f \neq g \) such that \( \|f\| = f(x) = 1 = g(x) = \|g\| \). Let \( M = (\ker f \cap \ker g) + R \cdot \{x\} \). Then \( M \) is a closed hyperplane such that \( f = g \) on \( M \) and \( \|f\| = \|f\|_M \). Thus \( f \) and \( g \) are norm-preserving extensions of \( f|_M \) and \( M \) is not weakly Hahn–Banach smooth.

Lima and Uttersrud [20] have given a characterization of smooth Banach spaces as follows: \( E \) is smooth if and only if \( \bigcup_{n=1}^{\infty} B(nx, n) \) is a half-space whenever \( \|x\| = 1 \).

This is related to Vlasov's theorem characterizing preduals of strictly convex spaces [19]. Taking Vlasov's theorem as a starting point, we can find a characterization of Hahn–Banach smooth subspaces of \( E \) similar to Theorem 2.2.
We shall use this result in the proof of Theorem 4.5. There we prove that if $K(E)$ is Hahn–Banach smooth in $L(E)$, then $E$ is Hahn–Banach smooth in $E^{**}$.

THEOREM 2.5. Let $M$ be a closed subspace of $E$. The following statements are equivalent:

1) $M$ is Hahn–Banach smooth in $E$.
2) If $\varepsilon \geq 0$, $y \in E \setminus M$ and $(a_n)_{n=1}^{\infty}$ is a sequence in $M$ such that $\|a_1\| \leq 1 + \varepsilon$ and

$$\|a_{n+1} - a_n\| \leq 1 + \frac{\varepsilon}{2^{n+1}} \quad \text{for all } n \geq 1,$$

then $M \cap A_1 \cap A_2 \neq \emptyset$, where

$$A_i = \bigcup_{n=1}^{\infty} B \left( y + (-1)^i a_n, n + 2 \varepsilon - \frac{\varepsilon}{2^n} \right); \quad i = 1, 2.$$

PROOF. 1) $\Rightarrow$ 2). Assume there exist $\varepsilon \geq 0$, $y \in E \setminus M$ and a sequence $(a_n)_{n=1}^{\infty}$ as in 2) such that $M \cap A_1 \cap A_2 = \emptyset$. Define

$$B_i = \bigcup_{n=1}^{\infty} B \left( y + (-1)^i a_n, n + \frac{3}{2} \varepsilon - \frac{\varepsilon}{2^n} \right).$$

Then $A_i = B_i + B(0, \varepsilon/2)$. Since $\|a_1\| \leq 1 + \varepsilon$ and $\|a_{n+1} - a_n\| \leq 1 + \varepsilon/2^{n+1}$, we get

$$y \in B \left( y + (-1)^i a_n, n + \frac{3}{2} \varepsilon - \frac{\varepsilon}{2^n} \right) \subseteq B \left( y + (-1)^i a_{n+1}, n + 1 + \frac{3}{2} \varepsilon - \frac{\varepsilon}{2^{n+1}} \right).$$

Thus $B_i$ is convex for $i = 1, 2$.

Let $A_M$ be as in the proof of Theorem 2.2. Then $A_M$ and $B_1 \times B_2$ can be strongly separated. Thus as in the proof of Theorem 2.2 there exist $g, h \in E^*$ and $\lambda > 0$ such that $g + h \in M^*$ and

$$\lambda \leq \inf_{b_i \in B_i} (g(b_1) + h(b_2)).$$

Thus we get

$$\lambda \leq \inf_n \left( g(y - a_n) + h(y + a_n) - (\|g\| + \|h\|) \left( n + \frac{3}{2} \varepsilon - \frac{\varepsilon}{2^n} \right) \right).$$

Since $\|a_n\| \leq n + \frac{3}{2} \varepsilon - \varepsilon/2^n$, we find by dividing by $n$ and then letting $n \to \infty$, that

$$\|g\| + \|h\| \leq \lim_{n \to \infty} (h - g) \left( \frac{a_n}{n} \right) \leq \|g - h\|.$$

Thus $\|g\| + \|h\| = \|g - h\|$. Hence it follows that
\[ (\|g\| + \|h\|) \left( n + \frac{3}{2} \varepsilon - \frac{\varepsilon}{2^n} \right) + \lambda \leq (g + h)(y) + (h - g)(a_n) \]
\[ \leq (g + h)(y) + \|h - g\| \cdot \left( n + \frac{3}{2} \varepsilon - \frac{\varepsilon}{2^n} \right). \]

and

\[ 0 < \lambda \leq (g + h)(y). \]

Thus shows that \( M \) is not Hahn–Banach smooth in \( E \).

2) \( \Rightarrow \) 1). Assume for contradiction that there exists \( f \in M^* \), \( \|f\| = 1 \) such that \( f \) has two different norm-preserving extensions \( g, h \in E^* \). Let \( y \in E \setminus M \) be such that \( g(y) \neq h(y) \). Without loss of generality, we may assume \( E = M \oplus \mathbb{R} \cdot \{y\} \).

Define \( N = \ker g \cap \ker h \subseteq E \). Clearly \( N \subseteq M \) and \( \dim E/N = 2 \). Now \( g, h \in N^1 \) and \( \|g\| + \|h\| = \|g + h\| = 2 \). Choose \( c \in E/N \) such that \( 1 = \|c\| = g(c) = h(c) \).

Notice that if \( z \in B(c, 1) \), then \( g(z) \geq 0 \) and \( h(z) \geq 0 \).

We now follow Vlasov’s reasoning:

Put \( c_n = nc \). Let \( Q \) be the quotient map onto \( E/N \). Let \( C_n = Q^{-1}(c_n) \). Since \( g = h \) exactly on \( M \), we get that \( C_n \subseteq M \). Let \( r_n = n - \varepsilon/2^n \). First choose \( a_1 \in C_1 \) with \( \|a_1\| \leq 1 + \varepsilon \). Next assume that \( a_1, \ldots, a_n \) has been found such that \( a_k \in C_k \) and \( \|a_{k+1} - a_k\| \leq r_{k+1} - r_k \), for \( k = 1, 2, \ldots, n - 1 \). Since \( a_n \in C_n \), we have

\[ d(a_n, C_{n+1}) = \|c_{n+1} - c_n\| = 1 < r_{n+1} - r_n. \]

Thus we can find \( a_{n+1} \in C_{n+1} \) such that \( \|a_{n+1} - a_n\| \leq r_{n+1} - r_n \). Since \( r_{n+1} - r_n = 1 + \varepsilon/2^{n+1} \), we have found a sequence in \( M \) as in 2). By 2) there exist \( z \in M \) and \( n \) such that for \( i = 1, 2 \).

\[ \|y + (-1)^ia_n - z\| \leq n + 2\varepsilon - \frac{\varepsilon}{2^n} \leq n + 2\varepsilon. \]

This can be written as

\[ \max_{\pm} \|a_n \pm (y - z)\| \leq n + 2\varepsilon. \]

Hence

\[ n + 2\varepsilon \geq \max_{\pm} |g(a_n) \pm g(y - z)| \]
\[ = \max_{\pm} |g(c_n) \pm g(y - z)| \]
\[ = \max_{\pm} |h \pm (y - z)| \]
\[ = n + |g(y - z)|. \]
Thus $2\varepsilon \geq |g(y - z)|$.
Similarly $2\varepsilon \geq |h(y - z)|$.
Since $z \in M$, we have $g(z) = h(z) = f(z)$. Thus
$$|g(y) - h(y)| \leq 4\varepsilon.$$ Starting with a sufficiently small $\varepsilon > 0$, we obtain a contradiction.

We shall use Theorem 2.5 in section 4. But first we need some results about another intersection property.

### 3. Liftings and intersections of balls.

We shall assume $M$ is a closed subspace of $E$. Let $f_1, \ldots, f_n \in M^*$ with $f_1 + \ldots + f_n = 0$. We shall find conditions on $M$ which ensure the existence of norm-preserving extensions $\tilde{f}_i$ such that $\tilde{f}_1 + \ldots + \tilde{f}_n = 0$ in $E^*$.

**Definition.** Let $n \geq 3$ be a natural number. We shall say that $M$ has the $n.E.$ intersection property ($n.E.I.P.$) if whenever $\{B(a_i, r_i)\}_{i=1}^n$ are $n$ closed balls in $M$ with $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$ in $E$, then $M \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$.

The following result is the main theorem.

**Theorem 3.1.** Let $n \geq 3$. The following statements are equivalent:

1) $M$ has the $n.E.I.P.$
2) $M^{\perp\perp}$ has the $n.E^{**}.I.P.$
3) If $f_1, \ldots, f_n \in M^*$ are such that $f_1 + \ldots + f_n = 0$, then there exist norm-preserving extensions $\tilde{f}_i$ to $E$ such that $\tilde{f}_1 + \ldots + \tilde{f}_n = 0$.
4) If $f_1, \ldots, f_n \in E^*$ with $f_1 + \ldots + f_n = f \in M^\perp$ and $r_i = d(f_i, M^\perp)$, then there exist $h_i \in M^\perp \cap B(f_i, r_i)$ such that $h_1 + \ldots + h_n = f$.

**Proof.** 2) $\Rightarrow$ 1) follows from the “principle of local reflexivity” [13] since we can identify $M^{\perp\perp}$ with $M^{**}$.

3) $\Rightarrow$ 1). Let $\{B(a_i, r_i)\}_{i=1}^n$ be $n$ balls in $M$ such that $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$ in $E$. Let $f_1, \ldots, f_n \in M^*$ be such that $f_1 + \ldots + f_n = 0$. By 3) there exist norm-preserving extensions $\tilde{f}_i$ such that $\tilde{f}_1 + \ldots + \tilde{f}_n = 0$.

Let $a \in \bigcap_{i=1}^n B(a_i, r_i)$.

Then we have
\[
\left| \sum_{i=1}^{n} f_i(a_i) \right| = \left| \sum_{i=1}^{n} \hat{f}_i(a_i) \right| \\
= \left| \sum_{i=1}^{n} \hat{f}_i(a_i - a) \right| \\
\leq \sum_{i=1}^{n} r_i \| \hat{f}_i \| \\
= \sum_{i=1}^{n} r_i \| f_i \| .
\]

By Theorem 1.1 in [10], we get that
\[ M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset \quad \text{for all} \quad \varepsilon > 0. \]

1) \(\Rightarrow\) 3). We introduce sets \( A \subseteq (M^* \oplus \ldots \oplus M^*)_{\ell_1} \) and \( B \subseteq (E^* \oplus \ldots \oplus E^*)_{\ell_1} \) as follows:
\[ A = \left\{ (f_1, \ldots, f_n) : \sum_{i=1}^{n} f_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \| f_i \| \leq 1 \right\} \]
and
\[ B = \left\{ (g_1, \ldots, g_n) : \sum_{i=1}^{n} g_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \| g_i \| \leq 1 \right\}. \]

Let \( Q : (E^* \oplus \ldots \oplus E^*)_{\ell_1} \rightarrow (M^* \oplus \ldots \oplus M^*)_{\ell_1} \) be defined by
\[ Q(g_1, \ldots, g_n) = (g_1|_M, \ldots, g_n|_M). \]

\(Q(B)\) is a convex \(w^*\)-compact subset of \(A\). Clearly it suffices to show that \(Q(B) = A\). Assume for contradiction that there exists \((f_1, \ldots, f_n) \in A \setminus Q(B)\). By the Hahn–Banach theorem there exist \(a_1, \ldots, a_n \in M\) such that
\[ \sum_{i=1}^{n} f_i(a_i) > 1 = \sup_{(g_1, \ldots, g_n) \in B} \sum_{i=1}^{n} g_i(a_i). \]

By Theorem 1.1 in [10], we have \( \bigcap_{i=1}^{n} B(a_i, 1 + \varepsilon) \neq \emptyset \) in \(E\) for all \( \varepsilon > 0 \), and
\[ M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) = \emptyset \quad \text{for some} \quad \varepsilon > 0. \]

3) \(\Leftrightarrow\) 4) is trivial.

4) \(\Rightarrow\) 2) follows by using Theorem 1.2 in [10].

Note that it follows from the proof of 1) \(\Rightarrow\) 3) that we can take all \( r_i = 1 \) in the definition of the \(n\).E.I.P. This also follows from Theorem 4.3 in [12].
REMARKS.

a) Let $E = C[0,1]$ and let $M$ be a subspace of $E$ isometric to $l_1$. Since $l_1$ has the 3.2.I.P. but not the 4.2.I.P., it follows that $M$ has the 3.E.I.P. but not the 4.E.I.P.

b) Let $E = l_1^3$ and let $M = \{(x,y,z) \in E : x + y + z = 0\}$. It is easy to see that $M$ does not have the 3.E.I.P.

c) From the "principle of local reflexivity", it easily follows that every Banach space $M$ has the $n.M^{**}$-I.P. for all $n$.

From Theorem 3.1 and the proof of Theorem 5.9 in [12] we get the following result.

**PROPOSITION 3.2.** The statements below are related as follows
1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Leftrightarrow$ 4):

1) There exists a norm 1 projection in $E$ with range $M$.
2) There exists a norm 1 projection in $E^*$ with kernel $M^\perp$.
3) $M$ has the n.E.I.P. for all $n$.
4) For each Banach space $Y$ such that $M^\perp \subseteq Y \subseteq E^{**}$ and $\dim Y/M^\perp = 1$, there is a norm 1 projection from $Y$ onto $M^\perp$.

REMARKS.

a) Clearly 2) $\not\Rightarrow$ 1) in Proposition 3.2, but we do not know if 3) $\Rightarrow$ 2).

b) We do not know if there exists a number $k \geq 4$ such that if $M$ has the $k.E.I.P.$, then $M$ has the n.E.I.P. for all $n \geq k$.

c) Using Helly's theorem [5], we get that if $\dim M = k < \infty$ and $M$ has the $(k + 1).E.I.P.$, then $M$ has the n.E.I.P. for all $n$.

d) From Proposition 3.2 and [8], we get that $E$ is isometric to a Hilbert space if and only if every two-dimensional subspace of $E$ has the 3.E.I.P. This result was first proved by Comfort and Gordon in [2].

We refer to [10] for the definition of $M$-ideals and semi $M$-ideals. An easy corollary of Theorem 3.1 is the following result.

**COROLLARY 3.3.** Assume $M$ is a semi $M$-ideal in $E$. Then the following statements are equivalent:

1) $M$ is an $M$-ideal in $E$.
2) $M$ has the n.E.I.P. for all $n$.
3) $M$ has the 3.E.I.P.
An easy corollary of this result and the Remarks above, is the following result of Saatkamp [15].

**Corollary 3.4.** If $M$ is a semi $M$-ideal in $M^{**}$, then $M$ is an $M$-ideal in $M^{**}$.

From a result of J. Johnson [7], we get:

**Proposition 3.5.** If $F$ or $E^*$ has the metric approximation property, then $K(E,F)$ has the n.L$(E,F)$.I.P. for all $n$. Moreover, if also $K(E,F)$ is a semi $M$-ideal in $L(E,F)$, then $K(E,F)$ is an $M$-ideal.

We shall end this section by considering which subspaces of $L_1(\mu)$-spaces and predual $L_1(\mu)$-spaces have the n.E.I.P.

**Proposition 3.6.** Let $E = L_1(\mu)$ and let $M$ be a closed subspace of $M$. Then $M$ has the 3.E.I.P. if and only if $M$ is the range of a norm-1 projection in $E$.

**Proof.** One way is trivial.

Assume $M$ has the 3.E.I.P. Then $M$ has the 3.2.I.P. By Theorem 4.3, Theorem 3.12, and Corollary 3.3 in [10], it follows that $M$ is isometric to an $L_1(\nu)$-space. By Theorem 6.3 in [9] it follows that $M$ is the range of a norm-1 projection in $E$.

**Proposition 3.7.** Assume $E^* = L_1(\mu)$ and that $M$ is a subspace of $E$. Then $M$ has the 4.E.I.P. if and only if $M^\perp$ is the kernel of a norm-1 projection in $E^*$.

**Proof.** Use proposition 3.8 and Theorem 2.17 in [10].

4. HB-subspaces.

Hennefeld [6] call a subspace $M$ of $E$ a HB-subspace if $M^\perp$ is complemented by a subspace $M_*$ such that whenever $f_* \in M_*$ and $f^\perp \in M^\perp \setminus \{0\}$, then $\|f_* + f^\perp\| \geq \|f^\perp\|$ and $\|f_* + f^\perp\| > \|f_*\|$.

We use the notation $M^s = \{f \in E^* : \|f\| = \|f\|_M\}$.

**Theorem 4.1.** The following statements are equivalent:

1) $M$ is a HB-subspace of $E$.
2) $M^s$ is a linear subspace.
3) If $f_1, f_2, f_3 \in M^s$ with $f_1 + f_2 + f_3 \in M^\perp$, then $f_1 + f_2 + f_3 = 0$.
4) $M$ is Hahn–Banach smooth in $E$ and has the 3.E.I.P.
5) $M$ is Hahn–Banach smooth in $E$ and has the n.E.I.P. for all $n \geq 3$. 

PROOF. 5) \(\Rightarrow\) 4) is trivial.

4) \(\Rightarrow\) 3) follows from Theorem 3.1 since \(f \in M^s\) implies that \(f\) is a norm-preserving extension of \(f|_M\).

3) \(\Rightarrow\) 2). Let \(f_1, f_2 \in M^s\). Then we can write \(f_1 + f_2 = -f_3 + f\) where \(f_3 \in M^s\) and \(f \in M^\perp\).

By 3) \(f_1 + f_2 = -f_3 \in M^s\).

2) \(\Rightarrow\) 5). Let \(f \in M^s\) and let \(g, h \in E^s\) be norm-preserving extensions of \(f\). Then \(g, h \in M^s\) and \(g - h \in M^\perp\). Thus \(g - h = 0\) and \(M\) is Hahn–Banach smooth in \(E\). Let \(P\) be the projection in \(E^s\) with range \(M^s\) and kernel \(M^\perp\). Then \(\|P\| = 1\) and \(M\) has the n.E.I.P. by Proposition 3.2.

1) \(\Rightarrow\) 5) follows from Lemma 1.2 and 1.3 in [6].

5) \(\Rightarrow\) 1). Define \(M_+ = M^s\). Clearly if \(f \in M^s\) and \(g \in M^\perp \setminus \{0\}\), then \(\|f + g\| \geq \|f + g\|_M = \|f\|\) and \(\|f + g\| > \|f\|\) since \(M\) is Hahn–Banach smooth.

COROLLARY 4.2. \(M\) is a HB-subspace of \(M^{**}\) if and only if \(M\) is Hahn–Banach smooth in \(M^{**}\).

COROLLARY 4.3. If \(M\) is a HB-subspace of \(M^{**}\), then \(M^s\) has the Radon–Nikodym property.

PROOF. It follows from [16] and Corollary 4.2.

COROLLARY 4.4. If \(E^s\) or \(F\) has the metric approximation property, then \(K(E, F)\) is a HB-subspace of \(L(E, F)\) if and only if \(K(E, F)\) is Hahn–Banach smooth in \(L(E, F)\).

In [11], we proved that if \(K(E)\) is an \(M\)-ideal in \(L(E)\), then \(E\) is an \(M\)-ideal in \(E^{**}\). A similar result is true for HB-subspaces.

THEOREM 4.5. Assume \(K(E)\) is a HB-subspace of \(L(E)\). Then \(E\) is a HB-subspace of \(E^{**}\). In particular \(E^s\) has the Radon–Nikodym property.

Note that similar results are true if we replace the word HB-subspace by Hahn–Banach smooth or by weakly Hahn–Banach smooth.

PROOF. By Proposition 3.6 and Theorem 4.1, it suffices to show that \(E\) is Hahn–Banach smooth in \(E^{**}\). To this end we use Theorem 2.5.

Let \(\varepsilon > 0\) and let \(y \in E^{**} \setminus E\). Clearly we may assume that \(\|y\| = y(f)\) for some \(f \in E^s\) with \(\|f\| = 1\). (We use the Bishop– Phelps theorem.) Let \((a_n)_{n=1}^\infty\) be
a sequence in $E$ such that $\|a_1\| \leq 1 + \varepsilon$ and $\|a_{n+1} - a_n\| \leq 1 + \varepsilon/2^n$. Define $S_n \in K(E)$ by

$$S_n(u) = f(u)a_n.$$ 

Then $\|S_1\| \leq 1 + \varepsilon$ and $\|S_{n+1} - S_n\| \leq 1 + \varepsilon/2^n$.  

By Theorem 2.5 there exist $T \in K(E)$ and $n$ such that

$$\max_{\pm} \|S_n \pm (I - T)\| \leq n + 2\varepsilon - \frac{\varepsilon}{2^n}.$$

Thus

$$n + 2\varepsilon - \varepsilon/2^n \geq \max_{\pm} \|S_n^{**} \pm (I - T^{**})\|$$

$$\geq \max_{\pm} \|S_n^{**}y \pm (y - T^{**}y)\|$$

$$= \max_{\pm} \|a_n \pm (y - T^{**}y)\|.$$

Since $T$ is compact, we have $T^{**}y \in E$. Thus $E$ is Hahn–Banach smooth in $E^{**}$ by Theorem 2.5.

**Theorem 4.6.** Assume $M$ is a closed subspace of $E$ and that $E$ is smooth and reflexive. Then the following statements are equivalent:

1) $M$ is the range of a norm 1 projection in $E$.
2) $M$ has the n.E.I.P. for all $n \geq 3$.
3) $M$ has the 3.E.I.P.
4) $M$ is a HB-subspace of $E$.

**Proof.** Since $E$ is smooth and reflexive, it follows that $M$ is Hahn–Banach smooth in $E$. The theorem now follows from Theorem 4.1 and Proposition 3.2.

From [9], we now get:

**Corollary 4.7.** Let $E = L_p(\mu)$ for some measure $\mu$ and $1 < p < \infty$. A subspace $M$ of $E$ has the 3.E.I.P. if and only if $M$ is isometric to an $L_p(\nu)$ space.

**Theorem 4.8.** Assume $M$ has the 3.E.I.P. If $M$ is weakly Hahn–Banach smooth in $E$, then $M^\perp$ is the kernel of a norm-1 projection in $E^*$.

**Proof.** For each $f \in M^*$, let $P(f)$ denote the non-empty convex and $w^*$-compact set of norm-preserving extensions of $f$. Clearly it suffices to find a linear selection of the map $f \to P(f)$. 
If \( f \in M^* \) attains its norm on \( M_1 \), let \( \hat{f} \) be the unique norm-preserving extension of \( f \). Then \( P(f) = \{ \hat{f} \} \).

Assume \( f, g \in M^* \) both attain their norms on \( M_1 \). Then by Theorem 3.1, we get \( \|f - g\| = \|\hat{f} - \hat{g}\| \). By the Bishop–Phelps theorem [3], the norm-attaining functionals in \( M^* \) are norm-dense. Hence we get that if \( f \in M^* \), then there exists a unique \( \tilde{f} \in P(f) \) such that if \( f_a \to f \) in norm and each \( f_a \) attain its norm, then \( \tilde{f}_a \to \tilde{f} \) in norm. The selection \( f \to \tilde{f} \) is linear.

The projection is \( f \to (\tilde{f}_a|_M) \).

**Corollary 4.9.** Assume \( M \) is weakly Hahn–Banach smooth in \( E \). Then \( M \) has the 3.E.I.P. if and only if \( M^{\perp_1} \) is the range of a norm-1 projection in \( E^{**} \).

**Corollary 4.10.** Assume \( E \) is a smooth Banach space and that \( M \) is a closed subspace. If \( M \) has the 3.E.I.P., then \( M \) has the n.E.I.P. for all \( n \), and \( M^{\perp} \) is the kernel of a norm-1 projection in \( E^* \).

**Proof.** Use Theorem 4.8, Proposition 3.2, and Theorem 2.4.

In [21] Belobrov studied Banach spaces which are Hahn–Banach smooth in their biduals.

He showed the following result under the stronger hypothesis that \( E \) is Hahn–Banach smooth (rather than weakly Hahn–Banach smooth).

**Theorem 4.11.** Assume \( E \) is weakly Hahn–Banach smooth in \( E^{**} \). The following statements are true:

1) If \( M \) is a closed subspace of \( E \), then \( M \) is weakly Hahn–Banach smooth in \( M^{**} \).

2) If \( E \) is the range of a norm-1 projection in \( E^{**} \), then \( E \) is reflexive.

**Proof.** 1). Let \( f \in M^* \) and assume \( f \) attains its norm on \( M_1 \). Let \( f_1, f_2 \) be two norm-preserving extensions of \( f \) to \( E \). By 1) each \( f_i \) has a unique norm-preserving extension \( \tilde{f}_i \) to \( E^{**} \) defined by \( \tilde{f}_i(y) = y(f_i) \). If \( y \in M^{\perp_1} = M^{**} \) and \( (x_a)_a \) is a net in \( M \) converging weak* to \( y \), then

\[
\tilde{f}_1(y) = y(f_1) = \lim_a x_a(f_1) = \lim_a x_a(f_2) = y(f_2) = \tilde{f}_2(y).
\]

Thus \( \tilde{f}_1 = \tilde{f}_2 \) on \( M^{\perp_1} \).

Next let \( g, h \) be two norm-preserving extensions of \( f \) to \( M^{\perp_1} \). Then \( g \) and \( h \) have norm-preserving extensions \( \tilde{g} \) and \( \tilde{h} \) to \( E^{**} \). Clearly \( \tilde{g} = (\tilde{g}|_E)^\wedge \) and \( \tilde{h} = (\tilde{h}|_E)^\wedge \) and by the first part of the proof, if \( y \in M^{\perp_1} \), then \( g(y) = \tilde{g}(y) = \tilde{h}(y) = h(y) \). Thus \( f \) has a unique norm-preserving extension to \( M^{\perp_1} = M^{**} \).
2). Here we follow Belobrov’s argument. Assume \( P \) is a norm-1 projection in \( E^{**} \) with range \( E \). Assume there exists \( x^{**} \in \ker P \setminus \{0\} \). Let \( f \in E^* \) with \( \| f \| = 1 \) and \( 2x^{**}(f) > \| x^{**} \| \) and \( x^{**}(f) \neq Px^{**}(f) \). By the Bishop–Phelps theorem we may assume \( f \) attains its norm on \( E_1 \). \( P^* \) is a norm-1 projection in \( E^{***} \) with kernel \( E^1 \). Let \( \hat{f} \) be the unique norm-preserving extension of \( f \) to \( E^{**} \). Thus \( \hat{f} \in E^{***} \).

Then we have

\[
P^*\hat{f}(x^{**}) = \hat{f}(Px^{**}) = (Px^{**})(f) + x^{**}(f) = \hat{f}(x^{**}) .
\]

Moreover, if \( y \in E \) with \( \| y \| = 1 \) and \( f(y) = \| f \| \), then

\[
P^*\hat{f}(y) = \hat{f}(Py) = \hat{f}(y) .
\]

Thus \( P^*\hat{f} \) and \( \hat{f} \) are two different norm-preserving extensions of \( f \) to \( E^{**} \). This is a contradiction. Hence \( \ker P = \{0\} \).

5. More about liftings and intersections of balls.

We shall now dualize Theorem 3.1. We can prove the following result.

**Theorem 5.1.** Assume \( M \) is a closed subspace of \( E \). Let \( n \geq 3 \) be a natural number and let \( \varphi : E \to E/M \) be the quotient map. The following statements are equivalent:

1) \( M^1 \) has the n.E*-I.P.

2) If \( x_1, \ldots, x_n \in E/M \) with \( x_1 + \ldots + x_n = 0 \) and \( \varepsilon > 0 \), then there exist \( y_i \in E \) such that \( \varphi(y_i) = x_i \), \( y_1 + \ldots + y_n = 0 \), and \( \| y_i \| \leq \| x_i \| + \varepsilon \) for all \( i \).

3) If \( y_1, \ldots, y_n \in E \) with \( y = y_1 + \ldots + y_n \in M \) and \( \varepsilon > 0 \) and \( r_i = d(y_i, M) \), then there exist \( x_i \in M \cap B(y_i, r_i + \varepsilon) \) such that \( y = x_1 + \ldots + x_n \).

**Proof.** 3) \( \Rightarrow \) 2). Let \( x_1, \ldots, x_n \in E/M \) with \( x_1 + \ldots + x_n = 0 \) and let \( \varepsilon > 0 \). Choose \( z_i \in E \) with \( \varphi(z_i) = x_i \) and \( \| z_i \| \leq \| x_i \| + \varepsilon \). Let \( r_i = d(z_i, M) \) and let \( z = z_1 + \ldots + z_n \). Then \( z \in M \) and hence, there exist \( y_i \in M \cap B(z_i, r_i + \varepsilon) \) such that \( z = y_1 + \ldots + y_n \). Then we have \( \varphi(z_i - y_i) = x_i \), \( \| z_i - y_i \| \leq \| x_i \| + \varepsilon \), and \( (z_1 - y_1) + \ldots + (z_n - y_n) = 0 \).

2) \( \Rightarrow \) 3). Choose liftings \( z_i \) of \( \varphi(y_i) \) as in 2) and let \( x_i = y_i - z_i \).

2) \( \Rightarrow \) 1). Let \( \{B(f_i, r_i)\}_{i=1}^n \) be \( n \) balls in \( M^1 \) and assume there exists \( f \in E^* \) such that \( \| f - f_i \| \leq r_i \) for all \( i \). Let \( x_1, \ldots, x_n \in E/M \) with \( x_1 + \ldots + x_n = 0 \) and let \( \varepsilon > 0 \). Let \( y_i \) be as in 2). Then we have, since \( f_i \in M^1 \),
\[ \sum_{i=1}^{n} f_i(x_i) = \sum_{i=1}^{n} f_i(y_i) \]
\[ = \sum_{i=1}^{n} (f_i - f)(y_i) \]
\[ \leq \sum_{i=1}^{n} r_i \|y_i\| \]
\[ \leq \sum_{i=1}^{n} r_i (\|x_i\| + \varepsilon) . \]

Since \(\varepsilon > 0\) is arbitrary, we get

\[ \sum_{i=1}^{n} f_i(x_i) \leq \sum_{i=1}^{n} r_i \|x_i\| . \]

By Theorem 1.2 in [10], it follows that \(M^\perp \cap \bigcap_{i=1}^{n} B(f_i, r_i) \neq \emptyset\).

1) \(\Rightarrow\) 2). This is similar to the proof of 1) \(\Rightarrow\) 3) in Theorem 3.1.

Note that if \(M\) is proximinal, then we can take \(\varepsilon = 0\) in Theorem 4.1.

If \(M\) is the kernel of a norm 1 projection or \(M^\perp\) is the image of a norm 1 projection, then \(M^\perp\) has the n.E.*I.P. for all \(n\).

**Proposition 5.2.** Assume \(M\) has the Haar property, i.e. for each \(x \in E\), there is a unique \(y \in M\) such that \(\|x - y\| = d(x, M)\). Then \(M^\perp\) has the 3.E.*I.P. if and only if \(M\) is the kernel of a norm 1 projection.

**Proof.** Let \(P\) be a norm 1 projection in \(E\) with \(\ker P = M\). Then \(P^*\) is a norm 1 projection in \(E^*\) with range \(M^\perp\). Hence \(M^\perp\) has the n.E.*I.P. for all \(n\).

Assume conversely that \(M^\perp\) has the 3.E.*I.P. Let

\[ M^\Theta = \{ x \in E : \|x\| = d(x, M) \} . \]

Clearly \(M \cap M^\Theta = (0)\) and \(M + M^\Theta = E\). It suffices to show that \(M^\Theta\) is a linear subspace. Let \(x_1, x_2 \in M^\Theta\). Then we can write \(x_1 + x_2 = -x_3 + x\) where \(x_3 \in M^\Theta\) and \(x \in M\). Since we can take \(\varepsilon = 0\) in Theorem 5.1 and \(M \cap B(x_1, \|x_1\|) = \{0\}\), it follows from 3) in Theorem 5.1 that \(x = 0\). Thus \(M^\Theta\) is a linear subspace of \(E\). The projection in \(E\) onto \(M^\Theta\) with kernel \(M\) has norm 1.
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