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A NON-EXISTENCE THEOREM FOR TRANSLATION INVARIANT OPERATORS ON WEIGHTED L_p-SPACES

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Abstract.

It is proved that there exists no non-trivial translation invariant operators on $L_p(w)$, if w belongs to a class of rapidly varying weight functions, including for instance $w(x)=\exp(\pm|x|^\alpha)$, $\alpha>1$.

0. Introduction.

Translation invariant operators are frequently used in several branches of analysis, for instance in approximation- and interpolation theory or in the study of partial differential operators. On the Lebesgue space L_p, translation invariant operators can be represented as convolutions with a tempered distribution. After Fourier transformation the translation invariant operator will then be disguised as a "Fourier multiplier". See Hörmander [1]. The L_p-theory for translation invariant operators can be extended, at least partly, to weighted L_p-spaces, (which we denote $L_p(w)$) for certain weight functions w closely connected to positive polynomials. See e.g. Triebel [3] and Löfström [2]. In this note we shall consider weight functions w with a "bad" behaviour of infinity. As an example we mention

$$w(x) = \exp(\pm|x|^\alpha), \ \alpha>1 .$$

We prove that there are no translation invariant operators T on $L_p(w)$ other than the obvious one

$$Tf = cf \quad (c \text{ a constant}) .$$

1. Representation of translation invariant operators.

A given function w on $\mathbf{R}^d$ is called a weight function if it is non-negative and locally integrable with respect to the Lebesgue measure. We denote by $L_p(w)$, $1\leqq p\leqq\infty$ the Banach space defined by the norm

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$$\|wf\|_p = \left(\int_{\mathbf{R}^d} (w(x)|f(x)|)^p \, dx \right)^{1/p} .$$

A translation invariant operator T on $L_p(w)$ is a linear operator that commutes with all translations. The following lemma, essentially due to Hörmander [1], represents a bounded translation invariant operator as a convolution.

LEMMA. *Suppose that the weight function is locally bounded away from* 0 *and* ∞, *that is for every compact set* K *there is a constant* $M>0$ *such that*

$$M^{-1} \leqq w(x) \leqq M \quad \text{for all } x \in K . \tag{1.1}$$

Then every bounded translation invariant operator T *on* $L_p(w)$ *can be uniquely represented as a convolution with a distribution* k, *in the sense that*

$$Tf = k * f , \tag{1.2}$$

for all f *in the class* $\mathscr{D}$ *of all infinitely differentiable functions with compact support.*

The proof of this lemma for the case $w=1$ is given in Hörmander [1], but the same argument works with only minor modifications in our more general case. For the convenience of the reader we reproduce the arguments here.

PROOF. First we shall prove that

$$D^\alpha(Tf) = T(D^\alpha f), \quad f \in \mathscr{D} . \tag{1.3}$$

Here D^α denotes an arbitrary derivative in the distribution sense, but it is clearly enough to prove the formula in the case $D^\alpha = D_1 = \partial/\partial x_1$. Thus let us write

$$f_h(x_1, x_2, \ldots, x_d) = f(x_1+h, x_2, \ldots, x_d) ,$$

$$\Delta_h = D_1 f - h^{-1}(f_h - f) = -h^{-1} \int_0^h (h-t)(D_1^2 f)_t \, dt .$$

We shall prove that $\Delta_h \to 0$ in $L_p(w)$ as $h \to 0$.

Let K_0 be the support of f and let K be the set of all x at distance not more than 1 from K_0. From (1.1) we then conclude that

$$w(x_1, x_2, \ldots, x_d) \leqq M^2 w(x_1+t, x_2, \ldots, x_d) ,$$

if $x \in K$, $|t| \leqq 1$. Thus

$$w|\Delta_h| \leqq \left| h^{-1} \int_0^h |h-t| \, |(wD_1^2 f)_t| \, dt \right| .$$

Hence

$$\|w\Delta_h\|_p \leqq \left|h^{-1}\int_0^h |h-t|\,dt\right| \|wD_1^2 f\|_p ,$$

which implies that $\Delta_h \to 0$ in $L_p(w)$.

Since T is continuous and translation invariant we conclude that

$$h^{-1}((Tf)_h-(Tf)) = T(h^{-1}(f_h-f)) \to T(D_1 f)$$

in $L_p(w)$. Thus $D_1(Tf)=T(D_1 f)$ from which (1.3) follows. As a consequence $D^\alpha(Tf) \in L_p(w)$ for all α. But then (1.1) implies that $D^\alpha(Tf)$ belongs locally to $L_p = L_p(1)$ for all α. Therefore Sobolev's lemma implies that Tf, after correction on a set of measure zero, is a continuous function. Moreover if B denotes the unit ball we have the estimate

$$|(Tf)(0)| \leqq C \sum_{|\alpha|\leqq d}\left(\int_B |T(D^\alpha f)|^p\,dx\right)^{1/p} .$$

A new application of (1.1) implies

$$|(Tf)(0)| \leqq C \sum_{|\alpha|\leqq d} \|wD^\alpha f\|_p .$$

This implies that $(Tf)(0)=(k*f)(0)$ for some uniquely determined distributions k. By translation invariance of the operators T and $f \to (k*f)$ we conclude that (1.2) holds.

2. Definition of a class of weight functions.

We now consider weight functions w which are locally bounded away from 0 and ∞ in the sense explained in the lemma of the previous section. Then we let $T_p(w)$ denote the space of distributions k which represent bounded translation invariant operators on $L_p(w)$, via the formula (1.2). We equip $T_p(w)$ with the operator norm.

Obviously $T_p(w)$ always contains the trivial elements $c\delta_0$, c constant. In many cases there are other members of $T_p(w)$. Let us consider an example where $w \neq 1$.

Take $w(x)=\exp(|x|^\alpha)$, $0<\alpha\leqq 1$. Since $|x|^\alpha \leqq |x-y|^\alpha+|y|^\alpha$ we have $w(x)\leqq w(x-y)w(y)$. Assuming that k is a locally integrable function we conclude that

$$w(x)|k*f(x)| \leqq \int_{\mathbf{R}^d} w(y)|k(y)|w(x-y)|f(x-y)|\,dy, \qquad f \in \mathscr{D} .$$

Hence, if $k \in L_1(w)$, we conclude that

$$\|w(k*f)\|_p \leqq \|wk\|_1 \|wf\|_p .$$

Thus we have proved that $L_1(w) \subset T_p(w)$, $1 \leqq p \leqq \infty$.

This example can be generalized in the following way. Assume that there is a weight function w^* such that

$$w(x) \leqq w(x-y)w^*(y) . \tag{2.1}$$

Consider a measure μ such that

$$A = \int_{\mathbf{R}^d} w^*(y)\, d|\mu|(y) < \infty .$$

Then $\mu \in T_p(w)$ for $1 \leqq p \leqq \infty$, since (as above)

$$w|k*f| \leqq (w^*|\mu|)*(wf) ,$$

and hence

$$\|w(k*f)\|_p \leqq A\|wf\|_p .$$

The objective of this note is however to consider weight functions for which (2.1) is *not* satisfied. An example of this is $w(x) = \exp(|x|^\alpha)$ with $\alpha > 1$. We shall see that in this example there are no non-trivial distribution in $T_p(w)$. (For further results on translation invariant operators on $L_p(w)$, where w is locally bounded away from 0 and ∞ and (2.1) is satisfied, see Löfström [2], Triebel [3]. Note that weight functions like $|x|^s$ are not treated here, since they are not bounded away from 0. They do not satisfy (2.1) either, but nevertheless there can be non-trivial translation invariant operators. See Young [4]).

A weight function for which (2.1) is not satisfied must clearly satisfy

$$\sup_x \frac{w(x)}{w(x-x_0)} = +\infty$$

for some $x_0 \neq 0$. This means that there must be a sequence $(x_n)_1^\infty$ such that

$$\frac{w(x_n)}{w(x_n+x_0)} \to +\infty, \qquad n \to \infty . \tag{2.2}$$

If w is locally bounded away from 0 we must have $|x_n| \to \infty$. We shall now assume that (2.2) holds not just for one particular x_0 but for all $x_0 \neq 0$. We shall also assume that the sequence $(x_n)_1^\infty$ can be changed into another sequence $(x'_n)_1^\infty$ for which (2.2) is still valid provided that $|x'_n - x_n| \leqq \varepsilon$ for all n. Finally we shall assume that the same sequence $(x_n)_1^\infty$ will work not just for x_0 but also for x'_0, if $|x'_0 - x_0| < \varepsilon$, ($\varepsilon$ being small enough). Thus we shall assume that

$$\inf \frac{w(x'_n)}{w(x'_n+x'_0)} \to \infty$$

where the infimum is taken over the set $|x_n'-x_n|\leqq\varepsilon$, $|x_0'-x_0|\leqq\varepsilon$. Now it is convenient to write $x_n'=x_n+x$, $x_0'=x_0-y-x$ and

$$\omega_n(x,y) = \frac{w(x_n+x)}{w(x_n+x_0-y)}. \tag{2.3}$$

We shall consider the class of weight functions given in the following definition.

DEFINITION. The class $\mathscr{W}$ consists of all weight functions w that are locally bounded away from 0 and ∞, such that for every $x_0\neq 0$ there is a number $\varepsilon>0$ and a sequence $(x_n)_1^\infty$ such that

$$\inf_{\substack{|x|\leqq\varepsilon\\|y|\leqq\varepsilon}} \omega_n(x,y) \to \infty \quad \text{as } n\to\infty. \tag{2.4}$$

EXAMPLE. We shall give a fairely general example of a weight function in the class $\mathscr{W}$ starting from a quadratic form

$$\sum_{i,j} a_{ij}x_iy_j = (x,Ax),$$

given by a non-singular symmetric matrix A. Put

$$Q(x) = (x,Ax)$$

and consider first the weight function

$$w_1(x) = \exp Q(x).$$

We shall prove that $w_1\in\mathscr{W}$.

Thus let $x_0\neq 0$ be given. Then put $x_n=-nA^{-1}x_0$ and $\varepsilon=|x_0|/4$. Then $Q(x_n)=n^2(x_0,A^{-1}x_0)$ and thus

$$Q(x_n+v) = n^2(x_0,A^{-1}x_0)-2n(x_0,v)+Q(v).$$

As a consequence of this we have

$$\begin{aligned} R_n(x,y) &= Q(x_n+x)-Q(x_n+x_0-y)\\ &= 2n(|x_0|^2-(x_0,x+y))+Q(x)-Q(x_0-y), \end{aligned}$$

which is bounded from below by

$$2n(|x_0|^2-2\varepsilon|x_0|)-c(\varepsilon,|x_0|)$$

if $|x|\leqq\varepsilon$, $|y|\leqq\varepsilon$. In view of the choice of ε we conclude that

$$\omega_n(x,y) = \exp R_n(x,y) \geqq \exp(n|x_0|^2-c(\varepsilon,|x_0|)).$$

It follows that $w_1\in\mathscr{W}$.

Next we assume in addition that $Q(x)>0$ for $x\neq 0$. Put

$$q(x) = Q(x)^{\beta}, \qquad \beta > 1/2 ,$$

and

$$w_2(x) = \exp(q(x)) .$$

Then $w_2 \in \mathscr{W}$. In fact, an easy computation will show that

$$\begin{aligned} q(x_n+x) - q(x_n+x_0-y) &\geqq cn^{2(\beta-1)}(n|x_0|^2 - c(\varepsilon, |x_0|)) \\ &\geqq cn^{2\beta-1} , \end{aligned}$$

from which the result follows.

Finally we observe that if $w \in \mathscr{W}$ and w is symmetric, that is $w(-x) = w(x)$ for all x, then $w^{-1} \in \mathscr{W}$, and as a consequence $w^s \in \mathscr{W}$ for all real $s \neq 0$. This follows at once from the identity

$$\frac{1/w(y_n+x)}{1/w(y_n+x_0-y)} = \frac{w(-y_n-x_0+y)}{w(-y_n-x_0+x_0-x)} = \frac{w(x_n+y)}{w(x_n+x_0-x)}$$

where $y_n = -x_n - x_0$.

Now it follows that

$$\exp(s \cdot Q(w)^{\alpha/2}) \in \mathscr{W}$$

for every $\alpha > 1$ and every real $s \neq 0$. In particular we see that

$$\exp(\pm |x|^{\alpha}) \in \mathscr{W}, \qquad \alpha > 1 .$$

Another particular case yields

$$\exp\left(\pm \sum_{i \neq j} x_i x_i\right) \in \mathscr{W} .$$

3. A non-existence theorem.

Let $\mathscr{W}$ be the class of weight functions defined at the end of the preceding section.

THEOREM. *Suppose that $w \in \mathscr{W}$. Then there are no non-trivial bounded translation invariant operators on $L_p(w)$. More precisely, if $k \in T_p(w)$ then $k = c\delta_0$, (c a constant).*

PROOF. First we consider the case $p = \infty$. Then there is a constant B such that

$$w(x)|k * f(x)| \leqq B \sup_y w(y)|f(y)| , \tag{3.1}$$

for all $f \in \mathscr{D}$.

Now take $x_0 \neq 0$ and choose $\varepsilon > 0$ and $(x_n)_1^\infty$ according to the definition of the class $\mathscr{W}$. Let τ_h denote the translation operator defined by

$$(\tau_h f)(x) = f(x-h).$$

Moreover put

$$\check{f}(x) = f(-x).$$

Then

$$k * f(x) = \langle k, \tau_x \check{f} \rangle = \langle k_0, \tau_{x+x_0} \check{f} \rangle,$$

where

$$k_0 = \tau_{x_0} k.$$

After these preparations we shall evaluate (3.1) in the point $x = x_n$, with f replaced by $\tau_{x_n+x_0}\check{\Psi}$. Here Ψ is an arbitrary function in $\mathscr{D}$ with support in $|y| < \varepsilon$. Then $\tau_{x_n+x_0}\check{f} = \Psi$, and (3.1) implies

$$w(x_n)|\langle k_0, \Psi \rangle| \leqq B \sup_y w(y)|\Psi(x_n+x_0-y)| \leqq B \sup_{|\eta|<\varepsilon} w(x_n+x_0-\eta) \cdot \|\Psi\|_\infty .$$

Hence k_0 is a bounded measure μ, on $|y| < \varepsilon$, such that

$$\inf_{|y|<\varepsilon} \omega_n(0,y) \cdot \int_{|y|<\varepsilon} d|\mu|(y) \leqq B .$$

By the assumption on w this is impossible unless μ vanishes on $|y| < \varepsilon$.

We have proved that for every $x_0 \neq 0$ there is a number $\varepsilon > 0$ such that $k = \tau_{-x_0} k_0$ vanishes on $|x - x_0| < \varepsilon$. Thus k is a linear combination of δ_0 and its derivatives. But then (3.1) implies that $k = c\delta_0$.

Next we consider the case $1 \leqq p < \infty$, which will be treated quite similarly. Since $k \in T_p(w)$ we now have the basic estimate

$$\int_{\mathbf{R}^d} (w(x)|k * f(x)|)^p \, dx \leqq B \int_{\mathbf{R}^d} (w(y)|f(y)|)^p \, dy , \tag{3.2}$$

for all $f \in \mathscr{D}$. Choose $x_0 \neq 0$, ε and $(x_n)_1^\infty$ as above, and localize (3.2) at the point x_n, replacing f by $\tau_{x_n+x_0}\check{\Psi}$ as above. Writing $x = x_n + \xi$ we have

$$k * f(x) = \langle k_0, \tau_\xi \Psi \rangle .$$

Thus (3.2) implies (for all $\delta > 0$)

$$\int_{|\xi|<\delta} (w(x_n+\xi)|\langle k_0, \tau_\xi \Psi \rangle|)^p \, d\xi \leqq \int_{|\eta|<\varepsilon} (w(x_n+x_0-\eta)|\Psi(\eta)|)^p \, d\eta . \tag{3.3}$$

We shall now prove that k vanishes on $|x - x_0| < \varepsilon$, i.e. that $\langle k_0, \Psi \rangle = 0$ for all

$\Psi \in \mathscr{D}$ with support in $|y| < \varepsilon$. Assume this were not the case. Then there is a $\Psi \in \mathscr{D}$ such that

$$\langle k_0, \Psi \rangle = 1 .$$

Now the function $\xi \to \langle k_0, \tau_\xi \Psi \rangle$ is continuous and has the value 1 at the origin. Thus, choosing $\delta < \varepsilon$ small enough, we have that $|\langle k_0, \tau_\xi \Psi \rangle| > 1/2$ for $|\xi| < \delta$. Since Ψ is a fixed bounded function we conclude from (3.3) that

$$\int_{|\xi| < \delta} (w(x_n + \xi))^p \, d\xi \leqq 2B \|\Psi\|_\infty \int_{|\eta| < \varepsilon} (w(x_n + x_0 - \eta))^p \, d\eta ,$$

and hence

$$\inf_{\substack{|\xi| < \delta \\ |\eta| < \varepsilon}} \omega_n(\xi, \eta) \leqq 2B \|\Psi\|_\infty .$$

This violates the assumption on w. Again we conclude that $k = c\delta_0$. The proof is complete.

The theorem implies that if $w \in \mathscr{W}$ then the only choice for k in the estimate

$$\|w(kf)\|_p \leqq B \|wf\|_p ,$$

is the trivial one $k = c\delta_0$. It is easily seen from the proof of the theorem that we have the following more general result.

COROLLARY. *Suppose that w_0 and w_1 are two weight functions such that w_0 and $1/w_1$ are locally bounded. For a given sequence $(x_n)_0^\infty$ write*

$$\omega_n^*(x, y) = \frac{w_0(x_n + x)}{w_1(x_n + x_0 - y)} .$$

Assume that for some $x_0 \neq 0$ there is a sequence $(x_n)_1^\infty$ and a number $\varepsilon > 0$ such that

$$\inf_{\substack{|x| \leqq \varepsilon \\ |y| \leqq \varepsilon}} \omega_n^*(x, y) \to \infty \quad \text{as } n \to \infty .$$

Then an estimate of the form

$$\|w_0(kf)\|_p \leqq B \|w_1 f\|_q, \quad f \in \mathscr{D} ,$$

implies that k vanishes in a neighbourhood of x_0.

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