A NON-EXISTENCE THEOREM
FOR TRANSLATION INVARIANT OPERATORS
ON WEIGHTED $L^p$-SPACES

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Abstract.
It is proved that there exists no non-trivial translation invariant operators on $L^p(w)$, if $w$ belongs to a class of rapidly varying weight functions, including for instance $w(x) = \exp(\pm |x|^\alpha)$, $\alpha > 1$.

0. Introduction.
Translation invariant operators are frequently used in several branches of analysis, for instance in approximation- and interpolation theory or in the study of partial differential operators. On the Lebesgue space $L^p$, translation invariant operators can be represented as convolutions with a tempered distribution. After Fourier transformation the translation invariant operator will then be disguised as a "Fourier multiplier". See Hörmander [1]. The $L^p$-theory for translation invariant operators can be extended, at least partly, to weighted $L^p$-spaces, (which we denote $L^p(w)$) for certain weight functions $w$ closely connected to positive polynomials. See e.g. Triebel [3] and Löfström [2]. In this note we shall consider weight functions $w$ with a "bad" behaviour of infinity. As an example we mention

$$w(x) = \exp(\pm |x|^\alpha), \quad \alpha > 1.$$ 

We prove that there are no translation invariant operators $T$ on $L^p(w)$ other than the obvious one

$$Tf = cf \quad (c \text{ a constant}).$$

1. Representation of translation invariant operators.
A given function $w$ on $\mathbb{R}^d$ is called a weight function if it is non-negative and locally integrable with respect to the Lebesgue measure. We denote by $L^p(w)$, $1 \leq p \leq \infty$ the Banach space defined by the norm

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\[ \|wf\|_p = \left( \int_{\mathbb{R}^d} (w(x)|f(x)|^p) \, dx \right)^{1/p}. \]

A translation invariant operator \( T \) on \( L_p(w) \) is a linear operator that commutes with all translations. The following lemma, essentially due to Hörmander [1], represents a bounded translation invariant operator as a convolution.

**Lemma.** Suppose that the weight function is locally bounded away from 0 and \( \infty \), that is for every compact set \( K \) there is a constant \( M > 0 \) such that

\[ M^{-1} \leq w(x) \leq M \quad \text{for all } x \in K. \]

Then every bounded translation invariant operator \( T \) on \( L_p(w) \) can be uniquely represented as a convolution with a distribution \( k \), in the sense that

\[ Tf = k * f, \]

for all \( f \) in the class \( \mathcal{D} \) of all infinitely differentiable functions with compact support.

The proof of this lemma for the case \( w = 1 \) is given in Hörmander [1], but the same argument works with only minor modifications in our more general case. For the convenience of the reader we reproduce the arguments here.

**Proof.** First we shall prove that

\[ D^x(Tf) = T(D^x f), \quad f \in \mathcal{D}. \]

Here \( D^x \) denotes an arbitrary derivative in the distribution sense, but it is clearly enough to prove the formula in the case \( D^x = D_1 = \partial/\partial x_1 \). Thus let us write

\[ f_h(x_1, x_2, \ldots, x_d) = f(x_1 + h, x_2, \ldots, x_d), \]

\[ \Delta_h = D_1 f - h^{-1}(f_h - f) = -h^{-1} \int_0^h (h-t)(D_1^2 f)_t \, dt. \]

We shall prove that \( \Delta_h \to 0 \) in \( L_p(w) \) as \( h \to 0 \).

Let \( K_0 \) be the support of \( f \) and let \( K \) be the set of all \( x \) at distance not more than 1 from \( K_0 \). From (1.1) we then conclude that

\[ w(x_1, x_2, \ldots, x_d) \leq M^2 w(x_1 + t, x_2, \ldots, x_d), \]

if \( x \in K, |t| \leq 1 \). Thus

\[ w|\Delta_h| \leq h^{-1} \int_0^h |h-t| |(wD_1^2 f)_t| \, dt. \]
Hence

$$\|wA_h\|_p \leq \left| h^{-1} \int_0^h |h-t| \, dt \right| \|wD_t^2 f\|_p,$$

which implies that $A_h \to 0$ in $L_p(w)$. Since $T$ is continuous and translation invariant we conclude that

$$h^{-1}((Tf)_h - (Tf)) = T(h^{-1}(f_h - f)) \to T(D_1 f)$$
in $L_p(w)$. Thus $D_1 (Tf) = T(D_1 f)$ from which (1.3) follows. As a consequence $D^\alpha (Tf) \in L_p(w)$ for all $\alpha$. But then (1.1) implies that $D^\alpha (Tf)$ belongs locally to $L_p = L_p(1)$ for all $\alpha$. Therefore Sobolev's lemma implies that $Tf$, after correction on a set of measure zero, is a continuous function. Moreover if $B$ denotes the unit ball we have the estimate

$$|(Tf)(0)| \leq C \sum_{|\alpha| \leq d} \left( \int_B |T(D^\alpha f)|^p \, dx \right)^{1/p}.$$

A new application of (1.1) implies

$$|(Tf)(0)| \leq C \sum_{|\alpha| \leq d} \|wD^\alpha f\|_p.$$

This implies that $(Tf)(0) = (k \ast f)(0)$ for some uniquely determined distributions $k$. By translation invariance of the operators $T$ and $f \to (k \ast f)$ we conclude that (1.2) holds.

2. Definition of a class of weight functions.

We now consider weight functions $w$ which are locally bounded away from 0 and $\infty$ in the sense explained in the lemma of the previous section. Then we let $T_p(w)$ denote the space of distributions $k$ which represent bounded translation invariant operators on $L_p(w)$, via the formula (1.2). We equip $T_p(w)$ with the operator norm.

Obviously $T_p(w)$ always contains the trivial elements $c\delta_0$, $c$ constant. In many cases there are other members of $T_p(w)$. Let us consider an example where $w \neq 1$.

Take $w(x) = \exp (|x|^\alpha)$, $0 < \alpha \leq 1$. Since $|x|^\alpha \leq |x-y|^\alpha + |y|^\alpha$ we have $w(x) \leq w(x-y)w(y)$. Assuming that $k$ is a locally integrable function we conclude that

$$w(x)|k \ast f(x)| \leq \int_{\mathbb{R}^d} w(y)|k(y)||w(x-y)||f(x-y)| \, dy, \quad f \in \mathcal{D}.$$

Hence, if $k \in L_1(w)$, we conclude that

$$\|w(k \ast f)\|_p \leq \|wk\|_1 \|wf\|_p.$$
Thus we have proved that $L_1(w) \subset T_p(w)$, $1 \leq p \leq \infty$.

This example can be generalized in the following way. Assume that there is a weight function $w^*$ such that

\begin{equation}
\tag{2.1}
w(x) \leq w(x - y)w^*(y).
\end{equation}

Consider a measure $\mu$ such that

$$A = \int_{\mathbb{R}^d} w^*(y) d|\mu|(y) < \infty.$$ 

Then $\mu \in T_p(w)$ for $1 \leq p \leq \infty$, since (as above)

$$w|k*f| \leq (w^*|\mu|)^*(wf),$$

and hence

$$\|w(k*f)\|_p \leq A\|wf\|_p.$$

The objective of this note is however to consider weight functions for which (2.1) is not satisfied. An example of this is $w(x) = \exp(|x|^\alpha)$ with $\alpha > 1$. We shall see that in this example there are no non-trivial distribution in $T_p(w)$. (For further results on translation invariant operators on $L_p(w)$, where $w$ is locally bounded away from 0 and $\infty$ and (2.1) is satisfied, see Löfström [2], Triebel [3]. Note that weight functions like $|x|^\alpha$ are not treated here, since they are not bounded away from 0. They do not satisfy (2.1) either, but nevertheless there can be non-trivial translation invariant operators. See Young [4]).

A weight function for which (2.1) is not satisfied must clearly satisfy

$$\sup_x \frac{w(x)}{w(x - x_0)} = +\infty$$

for some $x_0 \neq 0$. This means that there must be a sequence $(x_n)_1^{\infty}$ such that

\begin{equation}
\tag{2.2}
\frac{w(x_n)}{w(x_n + x_0)} \to +\infty, \quad n \to \infty.
\end{equation}

If $w$ is locally bounded away from 0 we must have $|x_n| \to \infty$. We shall now assume that (2.2) holds not just for one particular $x_0$ but for all $x_0 \neq 0$. We shall also assume that the sequence $(x_n)_1^{\infty}$ can be changed into another sequence $(x'_n)_1^{\infty}$ for which (2.2) is still valid provided that $|x'_n - x_n| \leq \epsilon$ for all $n$. Finally we shall assume that the same sequence $(x_n)_1^{\infty}$ will work not just for $x_0$ but also for $x'_0$, if $|x'_0 - x_0| < \epsilon$, ($\epsilon$ being small enough). Thus we shall assume that

$$\inf \frac{w(x'_n)}{w(x'_n + x'_0)} \to \infty.$$
where the infimum is taken over the set $|x_n' - x_n| \leq \varepsilon$, $|x'_0 - x_0| \leq \varepsilon$. Now it is convenient to write $x'_n = x_n + x$, $x'_0 = x_0 - y - x$ and

$$\omega_n(x, y) = \frac{w(x_n + x)}{w(x_n + x_0 - y)}.$$  

(2.3)

We shall consider the class of weight functions given in the following definition.

**Definition.** The class $\mathcal{W}$ consists of all weight functions $w$ that are locally bounded away from 0 and $\infty$, such that for every $x_0 \neq 0$ there is a number $\varepsilon > 0$ and a sequence $(x_n)_{n=1}^\infty$ such that

$$\inf_{|x| \leq \varepsilon} \omega_n(x, y) \to \infty \quad \text{as} \quad n \to \infty.$$  

(2.4)

**Example.** We shall give a fairly general example of a weight function in the class $\mathcal{W}$ starting from a quadratic form

$$\sum_{i,j} a_{ij} x_i y_j = (x, Ax),$$

given by a non-singular symmetric matrix $A$. Put

$$Q(x) = (x, Ax)$$

and consider first the weight function

$$w_1(x) = \exp Q(x).$$

We shall prove that $w_1 \in \mathcal{W}$.

Thus let $x_0 \neq 0$ be given. Then put $x_n = -nA^{-1}x_0$ and $\varepsilon = |x_0|/4$. Then $Q(x_n) = n^2(x_0, A^{-1}x_0)$ and thus

$$Q(x_n + v) = n^2(x_0, A^{-1}x_0) - 2n(x_0, v) + Q(v).$$

As a consequence of this we have

$$R_n(x, y) = Q(x_n + x) - Q(x_n + x_0 - y) = 2n(|x_0|^2 - (x_0, x + y)) + Q(x) - Q(x_0 - y),$$

which is bounded from below by

$$2n(|x_0|^2 - 2\varepsilon|x_0| - c(\varepsilon, |x_0|))$$

if $|x| \leq \varepsilon$, $|y| \leq \varepsilon$. In view of the choice of $\varepsilon$ we conclude that

$$\omega_n(x, y) = \exp R_n(x, y) \geq \exp (n|x_0|^2 - c(\varepsilon, |x_0|)).$$

It follows that $w_1 \in \mathcal{W}$.

Next we assume in addition that $Q(x) > 0$ for $x \neq 0$. Put
\[ q(x) = Q(x)^\beta, \quad \beta > 1/2, \]

and

\[ w_2(x) = \exp(q(x)). \]

Then \( w_2 \in \mathcal{W} \). In fact, an easy computation will show that

\[
q(x_n + x) - q(x_n + x_0 - y) \geq cn^{2(\beta - 1)}(n|x_0|^2 - c(e, |x_0|)) \\
\geq cn^{2\beta - 1},
\]

from which the result follows.

Finally we observe that if \( w \in \mathcal{W} \) and \( w \) is symmetric, that is \( w(-x) = w(x) \) for all \( x \), then \( w^{-1} \in \mathcal{W} \), and as a consequence \( w^s \in \mathcal{W} \) for all real \( s \neq 0 \). This follows at once from the identity

\[
\frac{1/w(y_n + x)}{1/w(y_n + x_0 - y)} = \frac{w(-y_n - x_0 + y)}{w(-y_n - x_0 + x_0 - x)} = \frac{w(x_n + y)}{w(x_n + x_0 - x)}
\]

where \( y_n = -x_n - x_0 \).

Now it follows that

\[ \exp(s \cdot Q(w)^{s/2}) \in \mathcal{W} \]

for every \( s > 1 \) and every real \( s \neq 0 \). In particular we see that

\[ \exp(\pm |x|^\alpha) \in \mathcal{W}, \quad \alpha > 1. \]

Another particular case yields

\[ \exp\left(\pm \sum_{i \neq j} x_i x_j \right) \in \mathcal{W}. \]

3. A non-existence theorem.

Let \( \mathcal{W} \) be the class of weight functions defined at the end of the preceding section.

**Theorem.** Suppose that \( w \in \mathcal{W} \). Then there are no non-trivial bounded translation invariant operators on \( L_p(w) \). More precisely, if \( k \in T_p(w) \) then \( k = c \delta_0 \), (\( c \) a constant).

**Proof.** First we consider the case \( p = \infty \). Then there is a constant \( B \) such that

\[
(3.1) \quad w(x)|k \ast f(x)| \leq B \sup_y w(y)|f(y)|,
\]

for all \( f \in \mathcal{D} \).
Now take $x_0 \neq 0$ and choose $\varepsilon > 0$ and $(x_n)_1^\infty$ according to the definition of the class $\mathcal{W}$. Let $\tau_h$ denote the translation operator defined by

$$(\tau_h f)(x) = f(x - h).$$

Moreover put

$$\tilde{f}(x) = f(-x).$$

Then

$$k * f(x) = \langle k, \tau_x \tilde{f} \rangle = \langle k_0, \tau_{x + x_0} \tilde{f} \rangle,$$

where

$$k_0 = \tau_{x_0} k.$$

After these preparations we shall evaluate (3.1) in the point $x = x_n$, with $f$ replaced by $\tau_{x_n + x_0} \Psi$. Here $\Psi$ is an arbitrary function in $\mathcal{D}$ with support in $|y| < \varepsilon$. Then $\tau_{x_n + x_0} \tilde{f} = \Psi$, and (3.1) implies

$$w(x_n)|\langle k_0, \Psi \rangle| \leq B \sup_y w(y)|\Psi(x_n + x_0 - y)| \leq B \sup_{|\eta| < \varepsilon} w(x_n + x_0 - \eta) \cdot \|\Psi\|_\infty.$$

Hence $k_0$ is a bounded measure $\mu$, on $|y| < \varepsilon$, such that

$$\inf_{|y| < \varepsilon} \omega_n(0, y) \cdot \int_{|y| < \varepsilon} d|\mu|(y) \leq B.$$

By the assumption on $w$ this is impossible unless $\mu$ vanishes on $|y| < \varepsilon$.

We have proved that for every $x_0 \neq 0$ there is a number $\varepsilon > 0$ such that $k = \tau_{-x_0} k_0$ vanishes on $|x - x_0| < \varepsilon$. Thus $k$ is a linear combination of $\delta_0$ and its derivatives. But then (3.1) implies that $k = c\delta_0$.

Next we consider the case $1 \leq p < \infty$, which will be treated quite similarly. Since $k \in T_p(w)$ we now have the basic estimate

$$(3.2) \quad \int_{\mathbb{R}^d} (w(x)|k * f(x)|^p \, dx \leq B \int_{\mathbb{R}^d} (w(y)|f(y)|^p \, dy,$$

for all $f \in \mathcal{D}$. Choose $x_0 \neq 0$, $\varepsilon$ and $(x_n)_1^\infty$ as above, and localize (3.2) at the point $x_n$, replacing $f$ by $\tau_{x_n + x_0} \Psi$ as above. Writing $x = x_n + \xi$ we have

$$k * f(x) = \langle k_0, \tau_\xi \Psi \rangle.$$

Thus (3.2) implies (for all $\delta > 0$)

$$(3.3) \quad \int_{|\xi| < \delta} (w(x_n + \xi)|\langle k_0, \tau_\xi \Psi \rangle|^p \, d\xi \leq \int_{|\eta| < \varepsilon} (w(x_n + x_0 - \eta)|\Psi(\eta)|^p \, d\eta.$$

We shall now prove that $k$ vanishes on $|x - x_0| < \varepsilon$, i.e. that $\langle k_0, \Psi \rangle = 0$ for all
$\Psi \in \mathcal{D}$ with support in $|y| < \varepsilon$. Assume this were not the case. Then there is a $\Psi \in \mathcal{D}$ such that

$$\langle k_0, \Psi \rangle = 1.$$ 

Now the function $\xi \to \langle k_0, \tau_\xi \Psi \rangle$ is continuous and has the value 1 at the origin. Thus, choosing $\delta < \varepsilon$ small enough, we have that $|\langle k_0, \tau_\xi \Psi \rangle| > 1/2$ for $|\xi| < \delta$. Since $\Psi$ is a fixed bounded function we conclude from (3.3) that

$$\int_{|\xi| < \delta} (w(x_n + \xi))^p \, d\xi \leq 2B \| \Psi \|_\infty \int_{|\eta| < \varepsilon} (w(x_n + x_0 - \eta))^p \, d\eta,$$

and hence

$$\inf_{|\xi| < \delta, |\eta| < \varepsilon} \omega_n(\xi, \eta) \leq 2B \| \Psi \|_\infty.$$

This violates the assumption on $w$. Again we conclude that $k = c\delta_0$. The proof is complete.

The theorem implies that if $w \in \mathcal{W}$ then the only choice for $k$ in the estimate

$$\|w(kf)\|_p \leq B \|wf\|_p,$$

is the trivial one $k = c\delta_0$. It is easily seen from the proof of the theorem that we have the following more general result.

**Corollary.** Suppose that $w_0$ and $w_1$ are two weight functions such that $w_0$ and $1/w_1$ are locally bounded. For a given sequence $(x_n)_0^\infty$ write

$$\omega_n^*(x, y) = \frac{w_0(x_n + x)}{w_1(x_n + x_0 - y)}.$$

Assume that for some $x_0 \neq 0$ there is a sequence $(x_n)_0^\infty$ and a number $\varepsilon > 0$ such that

$$\inf_{|x| \leq \varepsilon, |y| \leq \varepsilon} \omega_n^*(x, y) \to \infty \quad \text{as} \quad n \to \infty.$$

Then an estimate of the form

$$\|w_0(kf)\|_p \leq B \|w_1f\|_q, \quad f \in \mathcal{D},$$

implies that $k$ vanishes in a neighbourhood of $x_0$. 
REFERENCES


