MANIFOLDS WITH A SPECIAL TYPE OF CONELIKE SINGULARITIES

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0. Introduction.

Manifolds with singularities have been studied by Sullivan [12], Baas [1], and others. In particular Levitt [10] considers a certain category of such manifolds, with the underlying structure of PL-manifolds.

The manifolds of Levitt's category are constructed inductively. The first step is to consider manifolds of the form $M = M_0 \cup \bigcup_i S_i$ where $M_0$ is a smooth manifold, $S_i$ is a PL-manifold with boundary, and where each $S_i$ is equipped with a standard PL-equivalence

$$f_i : V_i \times c\Sigma_i \to S_i$$

Here is $V_i$ a smooth manifold, and $c\Sigma_i$ the cone on a smooth sphere $\Sigma_i$. Some of these manifolds have the underlying structure of a PL-sphere.

We can now inductively consider manifolds $M = M_0 \cup \bigcup_i S_i$, where $M_0$ is a manifold with singularities constructed in a previous step, and each $S_i$ is equipped with a PL-equivalence

$$f_i : V_i \times c\Sigma_i \to S_i,$$

where $V_i$ is smooth, and $c\Sigma_i$ is the cone on a previously constructed manifold, with the underlying structure of a PL-sphere.

We imitate this procedure, to construct our category of $C$-manifolds as follows.

Let $bP_n$ denote the group of concordance classes of homotopy spheres that bound $n$-dimensional framed manifolds.

As a preliminary step we construct a category of $C_1$-manifolds. A $C_1$-manifold is a manifold of form $M = M_0 \cup \bigcup_i S_i$ where each $S_i$ is provided with a PL-isomorphism

$$f_i : V_i \times c\Sigma_i \to S_i$$

where $V_i$ is a smooth manifold, and $\Sigma_i$ is a homotopy sphere that represents an element in $bP_*$.  

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Let $S_{C_1}(M)$ denote the set of concordance classes of $C_1$-manifolds with underlying PL-manifold $M$. There is a forgetful map

$$i : S_O(M) \to S_{C_1}(M)$$

from the set of smooth structures $S_O(M)$ on $M$ to $S_{C_1}(M)$.

In particular there is a forgetful map $i : S_O(S^n) \to S_{C_1}(S^n)$ which factors over $S_O(S^n)/bP_{n+1}$.

Next we extend the $C_1$-category inductively to a category of manifolds with singularities, the $C$-manifolds, such that the composite $S_O(S^n)/bP_{n+1} \to S_{C_1}(S^n) \to S_C(S^n)$ is a close as possible to an isomorphism.

In order to do so, we construct a natural transformation $\chi_1 : S_{C_1}(M) \to N(M)$, which for a given smooth manifold $M$ maps $S_{C_1}(M)$ into the group of cobordism classes of normal maps $M' \to M$.

We can introduce a bundle category containing "tangent bundles" of $C_1$-manifolds. The set of $C_1$-bundles of a finite complex $X$ is classified by a space $BC_1$.

One can show by a formal argument that the set of $C_1$-structures on a smooth manifold $M$ is classified by homotopy classes of maps of $M$ into the fibre $PL/C_1$ of the natural map $BC_1 \to BPL$. Again by formal arguments we can show that $\chi_1$ induces a map $\chi_1 : PL/C_1 \to G/O$.

Using this map we construct the category of $C_2$-manifolds as a category of manifolds with singularities containing $C_1$. In sections 3–4 we continue this process. We obtain inductively the categories $C_i$. At each step the definition of the next category involves choices.

Let $C$ be the union of the categories $C_i$. There is a corresponding bundle category with classifying space $BC$. The set of $C$-structures on a smooth manifold $M$ are classified by the set of homotopy classes $[M, PL/C]$.

The homotopy types of $PL/C$ and $BC$ depend in a curious way on the (unresolved) Kervaire invariant conjecture.

Let $A \subset \mathbb{Z}$: be the set of numbers of form $n = 2^r - 2$ such that there exists a framed manifold of dimension $n$ with Kervaire invariant 1.

**Theorem A.** There is a category $C$ of manifolds with singularities containing the category $C_1$ constructed by the inductive procedure outlined above.

(i) Let $C'$ be another category, constructed by the same inductive procedure, but possibly by making different choices.

There is a natural equivalence

$$E : S_C(-) \to S_{C'}(-).$$

(ii) There is a fibration
\[
\text{PL/}C \rightarrow G/O \rightarrow \prod_{n \in A} K(\mathbb{Z}/2, n),
\]

(iii) \textit{There is a fibration}

\[
BC \rightarrow G/\text{PL} \times BO \rightarrow \prod_{n \in A} K(\mathbb{Z}/2, n).
\]

By modifying our argument we construct other similar categories. In particular we give an interpretation of Coker \(J\) and of \(BSO\) localized at the odd primes.

\textbf{Theorem B.} \textit{There is a category} \(\bar{C}\) \textit{of manifolds with singularities, unique in the same sense as} \(C\) \textit{above, such that there are homotopy equivalences}

\[
\iota: \text{PL}/\bar{C} \rightarrow (\text{Coker } J)[\frac{1}{2}]
\]
\[
\varrho: \text{PL}/\bar{C} \rightarrow BSO[\frac{1}{2}].
\]

We finally want to thank our advisor Ib Madsen for many helpful discussions.

1. \textbf{The} \(C_1\)-\textbf{category.}

In this paragraph the \(C_1\)-category will be defined and studied. This represents the first step in the inductive construction of the \(C\)-category.

A \(C_1\)-manifold \(M\) is a PL-manifold with extra structure. This extra structure consists of two elements: A smooth, codimension 0 submanifold \(M_0\) with boundary, and certain conelike singularities in the complement. More precisely, the complement will be the disjoint union of manifolds, each PL-homeomorphic to \(V \times D\). Here the \(V\)'s are smooth manifolds, and the \(D\)'s are PL-discs. The boundary \(V \times S\) is assumed to be contained in the boundary of the smooth submanifold \(M_0\), so it inherits a smooth structure. We demand that as a smooth manifold it is a product \(V \times \Sigma\) where \(\Sigma\) is a smooth manifold, PL-equivalent to a sphere.

We can think of a \(C_1\)-manifold as a smooth manifold, with cone-singularities glued onto the boundary.

The next inductive step will be to glue singularities onto a \(C_1\)-manifold. If we do not put any restrictions on the singularities, we will essentially end up with the PL-category. This is a theorem of Levitt [10].

There are various ways to specify allowable sets of singularities. In this paper, we shall allow singularities of Milnor- and Kervaire-spheres. Specifically, recall the cyclic group \(bP_{2i}\) of \((2i-1)\)-dimensional homotopy spheres which bound parallelizable manifolds. The elements of \(bP\) appear to
be the simplest, and are the best understood exotic spheres. We allow only
singularities of the form $V \times c\Sigma$, where $\Sigma \in bP_\#$, and $c\Sigma$ denotes the cone on $\Sigma$.
For each concordance class in $bP_\#$, choose a particular representative.

**Definition 1.1.** A $C_1$-manifold $M$ is a quadruple $(M_0, V_i, \Sigma_i, f_i)$ consisting of

(i) A smooth manifold with boundary $M_0$.
(ii) Smooth compact manifolds $V_i$, $i = 1, \ldots, p$.

$V_i$ may or may not have boundary.
(iii) Smooth spheres $\Sigma_i$ which are among the particular representatives chosen.
(iv) Disjoint inclusions $f_i: V_i \times \Sigma_i \to \partial M_0$ of $V_i \times \Sigma_i$ in $\partial M_0$ as codimension 0 submanifolds.

This definition only mentions the smooth part of the manifold. In order to get the underlying PL-manifold of $M$, we glue in the singularities:

**Definition 1.2.** The underlying PL-manifold $M_{PL}$ of a $C_1$-manifold $M$ is the union

$$M_{PL} = M_0 \cup \bigcup_i (V_i \times c\Sigma_i).$$

The $C_1$-category does not possess a natural Cartesian product. There is, however, a natural product of a smooth manifold and a $C_1$-manifold.

Let $X$ be a PL-manifold. A $C_1$-structure on $X$ is a pair $(M, h)$, where $M$ is a $C_1$-manifold, and $h$ is a PL-homeomorphism $h: M_{PL} \to X$. A concordance between the two $C_1$-structures $(M_0, h_0)$ and $(M_1, h_1)$ is a $C_1$-structure $W$ on $X \times I$, which restricts to $(M_i, h_i)$ on $X \times \{i\}$.

The usual version of smoothing theory uses tangent microbundles. This approach involves the smooth structure on the product of a manifold with itself. In the $C_1$-category, products are not available. For this reason we have to generalize smoothing theory using thickenings (see Wall [14]), not microbundles.

A $k$-dimensional thickening of the CW-complex $X$ is a compact $k$-dimensional manifold with boundary $M$, with $\pi_1(\partial M) = \pi_1(M)$ and a simple homotopy equivalence $i: X \to M$. Two thickenings are equivalent if there is a concordance $W$ between $M_1$ and $M_2$, and a homotopy commutative diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & (M_1)_{PL} \\
\downarrow & & \downarrow \\
(M_2)_{PL} & \rightarrow & W_{PL}
\end{array}
$$
where \((M_i)_{PL}\) and \(W_{PL}\) denote the underlying PL-manifolds.

If \(M\) is a thickening of \(X\), \(M \times I\) will also be a thickening of \(X\). This allows us to consider "stable thickenings".

We did not specify which structure \(M\) should have. We could choose \(M\) to be smooth, \(C_1\) or PL. Then we obtain the concepts of smooth thickening, \(C_1\)-thickening or PL-thickening.

The set of stable thickenings is in all three cases a representable functor on finite CW-complex. See e.g. Levitt [10] for the \(C_1\)-case.

In the smooth and the PL-case there is a natural equivalence between the stable thickening functor, and the corresponding bundle theory. Let \(i: X \rightarrow M\) be a thickening, and \(\tau(M)\) be the tangent bundle of \(M\). The equivalence maps the thickening \(i\) to the bundle \(i^*(\tau(M))\). The trivial bundle on a manifold \(N\) is represented by the thickening \(i: N \hookrightarrow D(\nu(N))\) where \(i\) is the inclusion of the zero section in the normal disc bundle.

This equivalence shows that the smooth and PL-thickening functors are classified by \(BO\), respectively \(BPL\). We let \(BC_1\) denote the classifying space for \(C_1\)-thickenings. A smooth thickening is also a \(C_1\)-thickening, and a \(C_1\)-thickening is also a PL-thickening. Thus we have maps \(BO \rightarrow BC_1\) and \(BC_1 \rightarrow BPL\). The composition is the natural map \(BO \rightarrow BPL\).

These maps are only defined on finite subcomplexes. In the rest of the paper we will not worry about these questions. Maps will only be specified on finite complexes.

Let \(X\) be a PL-manifold. We consider the set \(\mathcal{S}_{C_1}(X)\) of concordance classes of \(C_1\)-structures on \(X\).

**Theorem 1.2.** There is a 1–1 correspondence between \(\mathcal{S}_{C_1}(X)\) and homotopy classes of liftings of the map classifying the PL-tangent bundle of \(X\).

\[
\begin{array}{ccc}
& BC_1 \\
\downarrow & & \\
X & \hookrightarrow & BPL \\
& t_{PL}
\end{array}
\]

**Sketch of Proof.** Let \(f: M \rightarrow X\) be a \(C_1\)-structure. Then \(f^{-1}: X \rightarrow M\) is a \(C_1\)-thickening. This thickening determines a lifting of \(t_{PL}\). We need an inverse of this construction. Let \(t_{C_1}: X \rightarrow BC_1\) be a lifting of \(t_{PL}\). By stability we can represent it by a PL-embedding in a \(C_1\)-manifold \(X \hookrightarrow N\). \(X\) has trivial normal bundle in \(N\). Using the PL s-cobordism theorem, we conclude that \(N\) is PL-equivalent to \(X \times I^n\) for some \(n\). It is possible to generalize the product theorem of smoothing theory to show that there is a 1–1 correspondence between \(\mathcal{S}_{C_1}(X)\) and \(\mathcal{S}_{C_1}(X \times I)\). (See Levitt [10] for details.) The \(C_1\)-structure
$N \to X \times I^n$ thus determines a unique $C_1$-structure on $X$. It is easily checked that the constructions above are inverses.

The space $BC_1$ is not an $H$-space, but $BO$ acts on it. Let $\mu_{BO} : BO \times BO \to BO$ and $\mu_{BPL} : BPL \times BPL \to BPL$ be the familiar maps characterizing Whitney sum. From the viewpoint of thickenings $\mu_{BO}$ and $\mu_{BPL}$ classify the following constructions. Let $f_1 : X \to M_1$ and $f_2 : X \to M_2$ be two thickenings. Then $f_1 \times f_2 : X \times X \to M_1 \times M_2$ is a thickening. Take the induced thickening over the diagonal $\Delta : X \to X \times X$. The map classifying this thickening is the sum of $f_1$ and $f_2$.

**Lemma 1.3.** There is a natural map $\mu_{BC_1} : BO \times BC_1 \to BC_1$ and the following diagram is homotopy commutative.

\[
\begin{array}{ccc}
BO \times BO & \longrightarrow & BO \\
\downarrow & & \downarrow \\
BO \times BC_1 & \longrightarrow & BC_1 \\
\downarrow & & \downarrow \\
BPL \times BPL & \longrightarrow & BPL \\
\end{array}
\]

**Proof.** Suppose $X$ is a PL-manifold. We construct a natural transformation $[X; BO \times BC_1] \to [X, BC_1]$.

Consider $(\alpha, \beta) : X \to BO \times BC_1$. Its first coordinate defines a smooth thickening $\tilde{\alpha} : X \to M_0$, its second a $C_1$-thickening $\tilde{\beta} : X \to M_{C_1}$. The restriction of the $C_1$-thickening $\tilde{\alpha} \times \tilde{\beta} : X \times X \to M_0 \times M_{C_1}$ to the diagonal determines a $C_1$-thickening, classified by $\gamma : X \to BC_1$. This determines the transformation. It is easy to check that it is natural, and that the diagrams of the lemma commute.

In accordance with usual notation we let $PL/C_1$ denote the homotopy theoretical fibre of the map $BC_1 \to BPL$. In analogy with smoothing theory, one would suspect that $C_1$-structures on a $C_1$-manifold $M$ are classified by homotopy classes in $[M; PL/C_1]$. The lack of a multiplication in $C_1$ prevents us from proving this, but the following restricted version is true.

**Lemma 1.4.** Let $M$ be a smooth manifold. There exists a 1–1 correspondence between $\mathcal{S}_{C_1}(M)$ and $[M, PL/C_1]$.

**Proof.** First we construct a map $[M, PL/C_1] \to \mathcal{S}_{C_1}(M)$ as follows. If $t_0 : M \to BO$ is the tangent bundle, and $\alpha : M \to PL/C_1$ a map, then
\[ M \xrightarrow{(t_0, \gamma)} BO \times PL/C_1 \to BO \times BC_1 \to BC_1 \]

is a lifting of the PL tangent bundle \( t_{PL}: M \to BPL \), so it determines a \( C_1 \)-structure on \( M \).

Next we construct the inverse map \( \mathcal{S}_{C_1}(M) \to [M, PL/C_1] \). Let \( \tilde{\gamma}: M \to \tilde{M} \) be a \( C_1 \)-structure on \( M \), and \( \gamma: M \to BC_1 \) the corresponding lifting of the PL tangent bundle map \( t_{PL} \). We have the homotopy commutative diagram

\[
\begin{array}{c}
M \xrightarrow{(-t_0, \gamma)} BO \times BC_1 \xrightarrow{\mu_{BC_1}} BC_1 \\
\downarrow \quad \downarrow \\
BPL \times BPL \xrightarrow{\mu_{BPL}} BPL
\end{array}
\]

where both \( t_0 \) and \( \gamma \) are liftings of \( t_{PL} \). The composite is trivial, in fact canonically trivialized. This trivialization defines an element of \([M, PL/C_1]\).

Finally, it is easy to see that the two maps are inverses of each other.

On the homogenous space level we get an action of PL/O on PL/C_1 which lifts the action from lemma 1.3. Indeed, let \( M \) be a smooth manifold, and \( \alpha: M \to PL/O, \beta: M \to PL/C_1 \) two maps. They give \( M \times M \) a \( C_1 \)-manifold structure which induces a \( C_1 \)-structure \( \tilde{\gamma} \) on the smooth tangent bundle \( \tau(M) \) of \( M \) via \( \tau(M) \hookrightarrow M \times M \). We get a map \( \gamma: M \hookrightarrow \tau(M) \to PL/C_1 \), and set \( \alpha \cdot \beta = \gamma \). The action is natural and gives rise to a map

\[ \mu_{PL/C_1}: PL/O \times PL/C_1 \to PL/C_1 \]

such that the following diagrams homotopy commute

\[
\begin{array}{c}
PL/O \times PL/O \to PL/O \\
\downarrow \quad \downarrow \\
PL/O \times PL/C_1 \to PL/C_1 \\
\downarrow \quad \downarrow \\
BO \times BC_1 \to BC_1
\end{array}
\]

Given two maps \( \alpha_1, \alpha_2: M \to PL/O \), then \( \alpha_1 \cdot (\alpha_2 \beta) \) and \( (\alpha_1 \cdot \alpha_2) \cdot \beta \) are the pullbacks of \( (\alpha_1, \alpha_2, \beta) \) by

\[ M \xrightarrow{\alpha} M \times M \xrightarrow{id \times \alpha} M \times M \times M \text{ and } M \xrightarrow{\alpha} M \times M \xrightarrow{\alpha \times id} M \times M \times M, \text{ respectively.} \]

These maps are homotopic, so on classifying space level we see that

\[
\begin{array}{c}
PL/O \times PL/O \times PL/C_1 \xrightarrow{\mu_{PL/O} \times id} PL/O \times PL/C_1 \\
\downarrow \quad \downarrow \\
\mu_{PL/C_1}
\end{array}
\]

is homotopy commutative.
2. The map $\chi_1: PL/C_1 \to G/O$.

When we constructed the $C_1$-category we had to decide on a particular set of singularities. We chose the set of singularities that involves cones of spheres in $bP_\ast$. Recall that $bP_{\ast+1}$ can be characterized as the kernel of the map $\chi_0_\ast: \pi_\ast(PL/O) \to \pi_\ast(G/O)$, see for example Sullivan [13]. In this section we shall extend the natural map $\chi_0: PL/O \to G/O$ to a map $\chi_1: PL/C_1 \to G/O$, which will play an analogous role in the definition of the $C_2$-category. With later generalizations in mind, we will give a purely homotopy theoretical definition. First, however, we outline a more conceptual, geometric definition.

Let $M$ be a smooth manifold, and $\sigma \in [M; PL/C_1]$. Further, let

$$\tilde{\sigma}: M_0 \cup \left( \bigcup_i V_i \times c \Sigma_i \right) \to M$$

be a $C_1$-structure represented by $\sigma$. By the assumption, $\Sigma_i$ is the boundary of some parallelizable manifold $M(\Sigma_i)$. The manifold $\tilde{M} = M_0 \cup \left( \bigcup_i V_i \times M(\Sigma_i) \right)$ is a smooth manifold, and there is a degree one normal map $\tilde{M} \to M$ whose normal cobordism class is classified by a homotopy class $M \to G/O$. We will prove in lemma 2.2 that this defines a natural transformation, inducing a map at classifying spaces $\chi_1: PL/C_1 \to G/O$.

In this construction we have to specify for each given homotopy sphere $\Sigma$ a particular parallelizable manifold $M(\Sigma)$, such that $\partial(M(\Sigma)) = \Sigma$. We will now do that.

Recall that $bP_{4n}$ is a cyclic group of finite order, say $\theta_n$. For $n > 1$ we construct the Milnor manifold $M^{4n}$ of index 8 by plumbing together 8 copies of the tangent disc bundle of $S^{2n}$, see Browder [4] for details. The boundary of $M^{4n}$ is a homotopy sphere, generating $bP_{4n}$.

The group $bP_{4n-2}$ is either $\mathbb{Z}/2$ or 0. The case $bP_{4n-2} = 0$ can, according to Browder [3], only occur if $n$ is a power of 2. We now choose for each $\Sigma \in bP_{2n}$ a framed bounding manifold $M(\Sigma)$. The only restriction we make on the choice is that if $\Sigma$ is the $(4n-1)$-dimensional generator mentioned above, then $M(\Sigma)$ is the Milnor manifold.

We point out that $M(\Sigma)$ defines an extensions of the classifying map of $\Sigma$

$$S^{n-1} \xrightarrow{g_\Sigma} PL/O \quad \downarrow \quad \downarrow$$

$$D^n \xrightarrow{g_D} G/O$$

(2.1)

Now we will give the homotopy theoretical definition. Suppose that $M$ is a smooth manifold, $\sigma \in [M, PL/C_1]$ and that $\tilde{\sigma}: M_0 \cup \left( \bigcup_i V_i \times c \Sigma_i \right) \to M$ is a representative of the class of $C_1$-structures on $M$ defined by $\sigma$. If we consider $M_0, V_n$ and $\Sigma_i$ as pieces of PL-submanifolds of the smooth manifold $M$, they will all inherit a standard smooth structure from the smooth structure on $M$. 

The manifold $\tilde{\sigma}(M_0)$ is a codimension zero submanifold of $M$, so it inherits a smooth structure. Let $\tilde{M}_0$ be $M_0$ with this new structure. In the same way $\tilde{\sigma}(V_i \times c \Sigma_i)$ inherits a smooth structure. The product structure theorem of smoothing theory tells us that $\tilde{\sigma}(V_i \times c \Sigma_i)$ is concordant to $\tilde{V}_i \times D_i$ where $\tilde{V}_i$ is some smooth structure on $V_i$, and $D_i$ is a standard disc. By restricting to the boundary we find that $\partial M_0 = \bigcup_i \tilde{V}_i \times S_i$. Here $S_i$ is a standard sphere.

The differences between the smoothings induced via $\tilde{\sigma}$, and the given smoothing correspond to homotopy classes of maps

$$g_0: M_0 \to \text{PL/O}$$

$$g_{V_i}: V_i \to \text{PL/O}$$

$$g_{\Sigma_i}: \Sigma_i \to \text{PL/O}.$$  

These classes are related by the equations

$$g_0|_{V_i \times \Sigma_i} = \mu_{\text{PL/O}} \cdot (g_{V_i} \times g_{\Sigma_i}).$$

We define the homotopy class $M \to G/O$ as follows

i) On $M_0$ it is the map $M_0 \xrightarrow{g_0} \text{PL/O} \xrightarrow{\chi_0} G/O$.

ii) On $V_i \times c \Sigma_i$ it is

$$V_i \times c \Sigma_i \xrightarrow{g_{V_i} \times g_{\Sigma_i}} \text{PL/O} \times G/O \to G/O \times G/O \to G/O$$

where $g_{\Sigma_i}$ is the extension of $g_{\Sigma_i}$ defined by 2.1.

The map $\chi_0$ is an $H$-map, so the two maps agree up to canonical homotopy on $V_i \times \Sigma_i$. We can glue them together to get a well-defined element $\chi_1(\sigma) \in [M, G/O]$. It is not difficult to check that this agrees with the geometrical definition.

**Lemma 2.2.** $\chi_1: [-; \text{PL/C}_1] \to [-; G/O]$ is a natural transformation.

**Proof.** Let $M, N$ be smooth manifolds, $\sigma: N \to \text{PL/C}_1$ represent a homotopy class, and let $f: M \to N$ be a continuous map. Note that homotopic maps $N \to \text{PL/C}_1$ determines the same element in $[N, G/O]$.

**Case 1.** Suppose $f: M \to N = D(\xi)$ is the inclusion of $M$ as the zero section in a disc bundle. We can represent the homotopy class of $\sigma$ by the homotopic map

$$D(\xi) \xrightarrow{\varepsilon} M \xrightarrow{\sigma} D(\xi) \xrightarrow{\varepsilon} \text{PL/C}_1.$$
The map $\sigma \circ f$ induces a $C_1$-structure on $M$

$$\sigma \circ f : M_0 \cup \left( \bigcup_i (V_i \times c \Sigma_i) \right) \to M$$

$\sigma \circ f$ induces a $C_1$-structure on $D(\xi)$

$$\sigma \circ f \circ \pi : D(\xi|_{M_0}) \cup \left( \bigcup_i (D(\xi|_{V_i}) \times c \Sigma_i) \right) \to D(\xi)$$

whose classifying map is $\sigma \circ f \circ \pi$. Thus $\sigma \circ f \circ \pi$ represents the restriction $f^* (\sigma)$ of $\sigma$ to the zerosection. The maps $\sigma \circ f$ and $\sigma \circ f \circ \pi$ define $g_M \in [M, G/O]$ and $g_{D(\xi)} \in [D(\xi), G/O]$ respectively, and it is easily concluded that $g_{D(\xi)} \circ f = g_M$. This proves lemma 3.1 in this case.

**Case 2.** Using case 1 we replace $N$ by $N \times D$, where $D$ is a large disc. Thus we can assume that $f : M \to N$ is an embedding. The image $f(M)$ has a normal disc bundle $D(v)$ in $N$. Then the map $f : M \to N$ factors as

$$M \xrightarrow{f} D(v) \xrightarrow{f''} N.$$ 

Case 1 applies to the inclusion $f'$. The map $f''$ is an inclusion of a submanifold of codimension zero. Assume without loss of generality that $f$ is of this form. Let $\sigma : N \to PL/C_1$ induce the $C_1$-structure

$$\tilde{\sigma} : N_0 \cup \left( \bigcup_i (V_i \times c \Sigma_i) \right) \to N.$$ 

The submanifolds $\tilde{\sigma}(V_i \times c \Sigma_i)$ are also of codimension zero. As we already remarked, this structure is of the form $\tilde{V}_i \times D_i$, where $\tilde{V}_i$ is some smoothing of $V_i$. By smooth transversality we can assume that $\tilde{\sigma}$ restricted to $\tilde{\sigma}^{-1}(M)$ is of the form

$$\tilde{\sigma} \circ f : (N_0 \cap \tilde{\sigma}^{-1}(M)) \cup \left( \bigcup_i V'_i \times c \Sigma_i \right) \to M,$$

where $V'_i$ is a smooth submanifold of $V_i$. This induces a $C_1$-structure on $M$. The induced element in $[M, PL/C_1]$ is $\sigma \circ f$. The $C_1$-structures $\tilde{\sigma}$ and $\sigma \circ f$ defines $g_M \in [M; G/O]$ and $g_N \in [N, G/O]$ and $g_N \circ f = g_M$. This proves the lemma in general.

A smooth surgery problem is a degree one normal map $\tilde{M} \to M$. Normal cobordism classes of such maps are classified by elements of $[M, G/O]$. Similarly, PL-surgery classes by elements of $[M, G/PL]$. The homotopy type of $G/PL$ localized at 2 was determined by Sullivan in [13]. Essentially $G/PL_{(2)}$ is a
product of one copy of each of the Eilenberg-MacLane spaces $K(\mathbb{Z}/2, 4n-2)$ and $K(\mathbb{Z}_2, 4n)$ for each $i$. In particular there are indecomposable classes $k^{4n-2} \in H^{4n-2}(G/\text{PL}; \mathbb{Z}/2)$ inducing characteristic classes for surgery problems.

The natural transformation which forgets the smooth structure of a smooth surgery problem induces a map $j: G/O \to G/\text{PL}$. For suitable choice of the $k^{4n-2}$ above (related to the surgery invariant) it was proved in Brumfiel-Madsen-Milgram [8] that $j^*(k^{4n-2}) = 0$ if $n \neq 2r$, but $j^*(k^{2r-2}) \neq 0$.

In dimensions $4n-2 \pm 2r-2$, $bP_{4n-2} = \mathbb{Z}/2$. It is conjectured that $bP_{2r-2}$ is always trivial. This will be the case if $k^{2r-2}$ is spherical. When we study the homotopy of PL/C in section 4, the dimensions where $bP_{4n-2} = 0$ have to be treated separately. We will then need the following lemma.

**Lemma 2.3.** If $bP_{4n-2} = 0$, then $k^{4n-2}$ pulls back to zero under the composite $k: \text{PL}/C_1 \xrightarrow{\chi_1} G/O \xrightarrow{j} G/\text{PL}$.

**Proof.** We use the geometrical interpretation of $\chi_1$. Let $M$ be a smooth manifold, and let $\tilde{\sigma}: M_0 \cup (\bigcup_i V_i \times c\Sigma_i) \to M$ be a $C_1$-structure on $M$. Since $bP_{4n-2} = 0$, none of spheres $\Sigma_i$ has dimension $4n-3$.

Let $\pi: \tilde{M} \to M$ be the surgery problem associated to $\tilde{\sigma}$. It is classified by $g_M \in [M, G/O]$. On the smooth part $\tilde{M}$ the surgery problem $\pi$ a PL-equivalence. In particular, it is a trivial PL-surgery problem. Over the singular part our surgery problem is the disjoint sum of surgery problems of the form

$V_i \times (M(\Sigma_i), \partial M(\Sigma_i)) \xrightarrow{\text{id} \times f_i} V_i \times (c\Sigma_i, \Sigma_i)$,

where the $M(\Sigma_i)$ are parallelizable manifolds, and $\partial M(\Sigma_i)$ PL-spheres. The map $j \circ g_M$ classifying the original surgery problem considered in the PL-category, will factor over a wedge of spheres

$M \to \bigvee_i (V_i \times c\Sigma_i/\Sigma_i) \to \bigvee_i (c\Sigma_i/\Sigma_i)$.

Since no spheres of dimension $4n-2$ occur in the wedge, any cohomology class in $H^{4n-2}(G/\text{PL})$ will pull back to zero over $M$. This proves the lemma.

Next we prove that $\chi_1: \text{PL}/C_1 \to G/O$ preserves the action of PL/O.

**Lemma 2.4.** The following diagram is homotopy commutative

\[
\begin{array}{ccc}
\text{PL/O} \times \text{PL/C}_1 & \xrightarrow{\mu_{\text{PL/C}_1}} & \text{PL/C}_1 \\
\downarrow \chi_0 \times \chi_1 & & \downarrow \chi_1 \\
G/O \times G/O & \xrightarrow{\mu_{G/O}} & G/O
\end{array}
\]
Proof. The spaces PL/O and G/O are loop spaces, in fact infinite loop spaces (see Boardman-Vogt [1]). This means that the multiplication can be chosen to be strictly associative.

The map PL/O → G/O is a loop map, so we can assume that the following diagram is strictly commutative

\[
\begin{array}{ccc}
\text{PL/O} \times \ldots \times \text{PL/O} & \rightarrow & \text{PL/O} \\
\downarrow & & \downarrow \\
\text{G/O} \times \ldots \times \text{G/O} & \rightarrow & \text{G/O}
\end{array}
\]

Now we can test the diagram in the statement of the lemma on a smooth compact manifold M. Suppose \((\alpha, \beta): M \rightarrow (\text{PL/O} \times \text{PL/C}_1)\) is a given map.

We can map M into G/O in two distinct ways. The action of PL/O on PL/C_1 allows us to compose \(\alpha\) and \(\beta\). The product \(\alpha \cdot \beta\) induces a \(C_1\)-structure on M. This defines a class \(\chi_1(\alpha \cdot \beta) \in [M, G/O]\).

But we also know that \(\alpha\) and \(\beta\) defines a smooth structure \(\bar{\alpha}\) and a \(C_1\)-structure \(\bar{\beta}\) on M. These structures define elements \(\chi_1(\alpha)\) and \(\chi_1(\beta) \in [M, G/O]\). Using the multiplicative structure of G/O, we can form their product \(\chi_1(\alpha) \cdot \chi_1(\beta)\). The lemma states that \(\chi_1(\alpha \cdot \beta) = \chi_1(\alpha) \cdot \chi_1(\beta)\). To prove this we factor \((\alpha, \beta)\) over \(M \times M\)

\[(\alpha, \beta): M \rightarrow M \times M \xrightarrow{(\alpha \times \beta)} \text{PL/O} \times \text{PL/C}_1.
\]

By naturality it is enough to calculate the two classes \(\chi_1((\alpha \times \ast) \cdot (\ast \times \beta))\) and \(\chi_1(\alpha \times \ast) \cdot \chi_1(\ast \times \beta)\).

Let \(\bar{\beta}\) be realized by

\[\bar{\beta}: M_0 \cup (V_i \times c\Sigma_i) \rightarrow M.\]

The \(C_1\)-structure on \(M \times M\) induced by \(\alpha \times \beta\) is realized by

\[\text{id} \times \bar{\beta}: M_0 \times (\bigcup_i (M_0 \times V_i) \times c\Sigma_i) \rightarrow M \times M.\]

Here \(M_0\) is the smooth structure induced by \(\alpha\).

We can now compute the two maps. On the smooth part the two maps are given by the compositions

\[M_a \times M_0 \rightarrow \text{PL/O} \times \text{PL/O} \rightarrow G/O \times G/O \rightarrow G/O\]

\[M_a \times M_0 \rightarrow \text{PL/O} \times \text{PL/O} \rightarrow \text{PL/O} \rightarrow G/O.\]

On the singular part they are

\[M_a \times V_i \times c\Sigma_i: \text{PL/O} \times \text{PL/O} \times G/O \xrightarrow{\mu_{\text{PL/O} \times \text{id}}} \text{PL/O} \times G/O \rightarrow G/O \times G/O \rightarrow G/O\]
\[ M_z \times V_i \times c\Sigma_i : \text{PL/O} \times \text{PL/O} \times G/O \to G/O \times G/O \times G/O \xrightarrow{id \times \mu_{G,O}} G/O \times G/O \to G/O \]

Because the diagram (*) commutes, the maps agree in both cases. The lemma is proved.

3. **C-manifolds.**

In this section we give an inductive construction of the C-category. In analogy with the method used to construct the C\(_1\)-category, we obtain a general C\(_s\)-manifold by gluing conelike singularities onto a C\(_{s-1}\)-manifold. This process is well-defined once we fixed an allowable set of singularities.

In order to generalize the group

\[ bP_{s+1} = \ker \left\{ \left( \chi_0 \right)_* : \pi_* (\text{PL/O}) \to \pi_* (G/O) \right\} \]

we will define a map \( \chi_s : \text{PL/C}_s \to G/O \) for each category C\(_r\). Then we define the C\(_{s+1}\)-category using cones on the C\(_s\)-spheres corresponding to elements of

\[ \ker \left\{ \left( \chi_s \right)_* : \pi_* (\text{PL/C}_s) \to \pi_* (G/O) \right\}. \]

Thus we must show that all constructions and lemmas of sections 1–2 have counterparts in the C\(_s\)-category.

Suppose inductively that we have defined categories of C\(_r\)-manifolds, 1 ≤ r ≤ s, and corresponding classifying spaces BC\(_r\), with maps

\[ BO \to BC_1 \ldots \to BC_s \to \text{BPL} \]

Furthermore, suppose there are maps

\[ \mu_{BC_r} : BO \times BC_r \to BC_r \]

\[ \mu_{PL/C_r} : \text{PL/O} \times \text{PL/C}_r \to \text{PL/C}_r \]

satisfying the obvious analogues of lemmas 1.3 and 1.5. We will finally assume that there exist maps

\[ \chi_r : \text{PL/C}_r \to G/O \]

such that the diagrams below commute up to homotopy

\[
\begin{array}{c}
\text{PL/C}_{r-1} \xrightarrow{\chi_{r-1}} G/O \\
\downarrow \chi_r \\
\text{PL/C}_r
\end{array}
\]
Then we define $bP^s_{*+1}$ as follows

$$bP^s_{*+1} = \ker \left\{ (\chi_s)_* : \pi_*(PL/C_s) \to \pi_*(G/O) \right\}.$$  

Choose a particular representative for each element.

**Definition 3.1.** A $C_{s+1}$-manifold is a quadruple $(M_0, \{V_i\}, \{\Sigma_i\}, \{f_i\})$ consisting of

(i) A $C_s$-manifold with boundary $M_0$.
(ii) Smooth compact manifolds $V_i = 1 \ldots p$.
    $V_i$s may or may not have boundary.
(iii) $C_s$-spheres $\Sigma_i$, which are among the representatives chosen above.
(iv) Disjoint inclusions $f_i : V_i \times \Sigma_i \to \partial M_0$
    of $V_i \times c\Sigma_i$ in $\partial M_0$ as dimension 0 submanifolds.

In the same way as we did with $C_1$-manifolds, we can glue together $M_0$ and the singular parts of $M$ to get the underlying PL-manifold:

$$M_{PL} = M_0 \cup \left( \bigcup V_i \times c\Sigma_i \right).$$

It is not difficult to see that theorem 1.2 and the lemmas 1.3, 1.4, and 1.5 hold, if they are stated for $C_{s+1}$ instead of $C_1$. Indeed, the proofs are quite similar.

We also have to construct a map

$$\chi_{s+1} : PL/C_{s+1} \to G/O.$$ 

The construction is essentially the same as our homotopy theoretical construction of $\chi_1$. It depends on picking particular extensions to the disc for homotopy trivial maps $S \to G/O$.

If $\Sigma \in bP^s_*$ is classified by a map $g_\Sigma : S \to PL/C_s$ we can choose a map $g_{c\Sigma} : cS \to G/O$ so that the following diagram is commutative

$$
\begin{array}{ccc}
S & \xrightarrow{g_\Sigma} & PL/C_s \\
\downarrow & & \downarrow \chi_s \\
cS & \xrightarrow{g_{c\Sigma}} & G/O
\end{array}
$$
In the next section we discuss the question of to what extent the choices involved will affect the category $C_{s+1}$. It turns out that it is possible to give conditions strong enough to guarantee that the $C_{s+1}$-category is essentially uniquely determined.

It is easy to generalize lemmas 2.2 and 2.4. The generalization of lemma 2.3 will be proved in section 4.

We now have the categories $C_s$ with classifying spaces $BC_s$. A $C_s$-thickening may be considered as a $C_{s+1}$-thickening, so we also have maps

$$BO 	o BC_1 \to \ldots \to BC_s \to \ldots BPL.$$  

We set

$$BC = \lim_{\to} BC.$$  

In the same way we define the $C$-category to be $\lim_{\to} C_s$.

For a given dimension, the direct limit system stabilizes after a finite number of steps. Recall that $bP_n = 0$, if $n \leq 7$, so that any $C_1$-manifold of dimension 7 or less is a smooth manifold. Because $bP^1_n$ consist of $C_1$-manifolds that are not smooth manifolds, it has to be zero if $n \leq 8$. This shows that any $C_2$-manifold of dimension 8 or less is a $C_1$-manifold. In general, all $n$ dimensional $C$-manifolds are $C_{n-7}$ manifolds.

At this point we need the following

**Lemma 3.3.** Let $X$ be a finite CW-complex, $\dim X = m$. Any $C_s$-thickening is stably equivalent to a thickening $X \to N^n$, where $n = \dim N^n \leq 2m$.

**Proof.** In [14] Wall proves a similar result for PL-thickenings. Suppose that $X \to N'$ is any $C_s$-thickening. It is stably PL-equivalent to some PL-thickening $X \to N$ with $\dim N \leq 2m$. We must show that we can give $N$ a $C_s$-structure so that $N$ and $N'$ are stably $C_s$-equivalent thickenings.

The concordance $W$ between $N \times I'$ and $N'$ is PL-equivalent to $N' \times I$. We can use this PL-equivalence to give $W$ a $C_s$-structure which is a product of $N'$ and $I$. By the $C_s$ product structure theorem, this is concordant to the product of a $C_s$-structure on $N$ and the standard structure on $I^{*+1}$. The $C_s$-structure on $N$ allows us to consider $X \to N$ as a $C_s$-thickening, and $W$ is a stable $C_s$-equivalence between this thickening and $X \to N'$.

**Lemma 3.3** shows that for a finite complex $X$ the sequence

$$[X,BO] \to [X,BC_1] \to \ldots \to [X,BC_s] \to \ldots$$
will stabilize. This shows that the space $BC$ is the classifying space for $C$-thickenings.

Since a $C$-thickening is a PL-thickening we have a map of classifying spaces $BC \to BPL$ whose fibre is denoted $PL/C$. Then $PL/C = \lim_{\to} PL/C_s$ and the maps $\chi_s$ induce a mapping $\chi : PL/C \to G/O$. By the construction,

$$\chi_* : \pi_*(PL/C) \to \pi_*(G/O)$$

is injective.

4. How to choose trivializations.

In the last section the category $C_{s+1}$ was defined using the map $\chi_s : PL/C_s \to G/O$. The construction of the map $\chi_s$ depended on choices. In this section, we first limit the element of choice, and then prove that the remaining choices are essentially equivalent.

The map $(\chi_0)_* : \pi_*(PL/O) \to \pi_*(G/O)$ has been studied by Kervaire-Milnor [9], Sullivan [13], Brumfiel [5], [6], [7] and others. The image of $(\chi_0)_*$ is a complicated (unknown) subgroup of $\pi_*(G/O)$, but the cokernel is much simpler.

There is a fibration

$$PL/O \xrightarrow{\chi_0} G/O \xrightarrow{j} G/PL.$$ 

The homotopy groups of $G/PL$ are wellknown. They are periodic with period 4 as follows

$$\begin{cases} 
\pi_{4n}(G/PL) = \mathbb{Z} & \pi_{4n+1}(G/PL) = 0 \\
\pi_{4n+2}(G/PL) = \mathbb{Z}/2 & \pi_{4n+3}(G/PL) = 0.
\end{cases}$$

Since $\text{coker} (\chi_0)_* = \text{im} (j_*)$, this group has to be cyclic. In dimension $4n$ we can describe a generator of $\text{coker} (\chi_0)_* = \pi_*(G/O)$ as follows.

Recall that $G/O$ is the space classifying normal cobordism classes of degree one normal maps. Also recall the Milnor manifold $M^{4n}$ with index 8 mentioned in section 2. The boundary $\partial M$ is a homotopy sphere that generates $bP_{4n}$. Consider the manifold $N$ which is a connected sum along the boundary of $bP_{4n}$ copies of $M$. Then $N$ is framed, and its boundary $\partial N$ is a standard smooth sphere $S$. The union with a standard disc $\tilde{N} = N \cup_{\partial N} D$ has a natural smooth structure. There is a natural degree one normal map $\tilde{f} : \tilde{N} \to S$ which is classified by $f : S \to G/O$. The group $\text{coker} (\chi_0)$ is generated by $f$.

In dimensions $4n - 2$ the long exact sequence of the fibration $PL/O \to G/O \to G/PL$ specializes to

$$0 \to (\text{coker} \chi_0)_{4n-2} \to \mathbb{Z}/2 \to bP_{4n-2} \to 0.$$
If \( n \neq 2^r \), \( bP_{4n-2} = \mathbb{Z}/2 \), so \( \text{coker} \chi_0 \) \( 4n-2 = 0 \). If \( n = 2^r \) it is conjectured that \( bP_{4n-2} = 0 \). Let \( A \subset \mathbb{Z} \) be the set of dimensions \( 4n-2 \) for which \( \text{coker} \chi_0 \) \( 4n-2 = \mathbb{Z}/2 \).

Now we are able to compute \( \text{coker} \chi_1 \).

The image of
\[
\chi_1 \star : \pi_{4n} (\text{PL}/C_1) \to \pi_{4n} (G/O)
\]
contains the image of \( \chi_0 \) in this dimension, but also the element \( f: S \to G/O \) generating \( \text{coker} \chi_0 \) for the following reason:

Take \( \{bP_4\} \) copies of the boundary \( \partial N^{4n} \) of the Milnor manifold. The connected sum along the boundary of the cones of these homotopy spheres is a \( C_1 \)-manifold \( \tilde{D} \). The boundary of \( \tilde{D} \) is a standard sphere, so we can take the union with a standard disc \( \tilde{D} \cup S \tilde{D} \). This defines a \( C_1 \)-structure \( S_x \) on a PL \( 4n \)-sphere. From the geometrical definition of \( \chi_1 \) it is clear that

\[
S_x \to \text{PL}/C_1 \xrightarrow{f} G/O
\]
is the element \( f \). This is where we use that the trivialization of the generator of \( bP_1 \) is chosen in a particular way.

From lemma 2.3 follows now that \( \text{coker} \chi_1 \) \( n = \mathbb{Z}/2 \) if \( n \in A \), \( \text{coker} \chi_1 \) \( n = 0 \) else.

In 2 we discussed classes \( k \in H^{2^r-2} (G/\text{PL}, \mathbb{Z}/2) \) which are mapped to non-zero elements in \( H^{2^r-2} (G/O) \). Consider the map

\[
p: G/O \rightarrow \prod_{n \in A} K(\mathbb{Z}/2, n).
\]

It is proved in Madsen-Milgram [11] that each map \( k: G/\text{PL} \to K(\mathbb{Z}/2, 2^r-2) \) is twice deloopable, so \( p \) is also twice deloopable.

**Definition 4.1.** The homotopy fibre of \( p \) is \( \Phi_0 \).

\( \Phi_0 \) is a loop space. Because \( \text{PL}/O \to G/O \to G/\text{PL} \) is trivial we can lift \( \text{PL}/O \to G/O \) to a loop map \( \text{PL}/O \to \Phi_0 \). Let \( \varphi: \Phi_0 \to G/O \) be the inclusion.

**Lemma 4.2.** We can choose \( \chi_s: \text{PL}/C_s \to G/O \) so that it factors \( \text{PL}/C_s \xrightarrow{\tilde{\chi}_s} \Phi_0 \xrightarrow{\varphi} G/O \).

**Proof.** We know from lemma 2.3 that \( p\chi_1 \sim 0 \), so we can assume that \( \chi_1 = \varphi \tilde{\chi}_1 \). In the construction of \( \chi_s \) we only used that \( \text{PL}/O \to G/O \) is a loop map. Suppose inductively that \( \chi_s = \varphi \tilde{\chi}_s \). If

\[
\alpha \in \ker [\chi_s : \pi_* \text{PL}/C_s \to \pi_* G/O],
\]

then
\[ \alpha \in \ker [\tilde{\chi}_{s+1} : \pi_* (\text{PL}/C_s) \to \pi_* \varphi_0], \]

since \( \varphi \) is injective on homotopy groups. We can construct a map

\[ \tilde{\chi}_{s+1} : \text{PL}/C_{s+1} \to \Phi_0 \]

in the same way as we constructed \( \chi_{s+1} \) in section 3. Then put \( \chi_{s+1} = \varphi \tilde{\chi}_{s+1} \).

If we let \( s \) increase, in the limit we obtain a map \( \tilde{\chi} : \text{PL}/C \to \Phi_0 \). This map induces an isomorphism on all homotopy groups, so it is a homotopy equivalence. We can formulate this as follows:

**Theorem 4.3.** Let \( A \subset \mathbb{Z} \) be the set of numbers of form \( n = 2^r - 2 \) such that there exists a framed manifold of dimension \( n \) with Kervaire invariant 1. Let \( C \) be the manifold category constructed above. Then there is a canonical fibration.

\[ \text{PL}/C \to G/O \to \prod_{n \in A} K(\mathbb{Z}/2, n) \]

In particular it follows that \( \text{PL}/C \) is independent of the choices we made in the construction of the \( C \)-category. But we can prove more. Assume that by making different choices, we have arrived at two different categories \( C \) and \( C' \). We will construct a natural equivalence \( E : S_C(-) \to S_{C'}(-) \). The map \( E \) is constructed inductively. We can assume that \( C_s = C'_s \), but that \( \tilde{\chi}_s, \tilde{\chi}'_s : \text{PL}/C_s \to \Phi \) might be different maps.

Let \( \Sigma \) be any \( C_s \)-structure on \( S^{n-1} \) which is in \( bP^n \). The two different trivializations of the classifying map \( \Sigma \to \text{PL}/C_s \) give us a map \( f : S^n \to \Phi \).

Since \( \pi_* (\text{PL}/C_s) \to \pi_* (G/O) \) is an epimorphism we can lift \( f \) to \( f' : S^n \to \text{PL}/C_s \).

This corresponds to a \( C_s \)-structure \( \tilde{f} \) on \( S^n \).

Let \( \Sigma \times I \) be a trivial concordance. Take the connected sum of this with \( \tilde{f} \). We obtain a \( C_s \)-concordance \( W(\Sigma) \) from \( \Sigma \) to itself.

Now we can define the map \( E \). If \( M \) has a \( C_s \) structure \( M_s \):

\[ \bar{\alpha} : M_0 \cup \left( \bigcup_i V_i \times c\Sigma_i \right) \to M \]

we define \( M_{E(\alpha)} \) to be

\[ \overline{E(\alpha)} : M_0 \cup \left( \bigcup_i V_i \times W(\Sigma_i) \right) \cup \left( \bigcup_i V_i \times c\Sigma_i \right) \to M_0 \cup \left( \bigcup_i V_i \times c\Sigma_i \right) \xrightarrow{\bar{\alpha}} M, \]

where

\[ (V_i \times W(\Sigma_i)) \cup (V_i \times c\Sigma_i) \to (V_i \times c\Sigma_i) \]

is the obvious PL-homeomorphism.
The following diagram is commutative

\[
\begin{array}{ccccc}
M_{E(a)} & \longrightarrow & \text{PL}/C_s & \xrightarrow{\chi} & \\
\downarrow & & \downarrow \phi & & \\
M_\alpha & \longrightarrow & \text{PL}/C_s & \xrightarrow{\chi} &
\end{array}
\]

Interchanging the roles of \( \chi \) and \( \chi' \) we get a map \( E':\mathcal{SF}_{C_s}(-) \rightarrow \mathcal{SF}_{C_s}(-) \) which is the inverse of \( E \). We can extend \( E \) inductively to get an isomorphism \( E: \mathcal{SF}(M) \rightarrow \mathcal{SF}(M) \).

**Theorem 4.4.** Let \( C \) and \( C' \) be two categories constructed using the inductive procedure above. Then there exist \( S \) a natural equivalence

\[
e: S_C(-) \rightarrow S_{C'}(-)
\]

**5. The homotopy type of BC.**

In this section we determine the homotopy type of BC. We also remark that the machine of sections 1–3 can be generalized. We consider the special case when \( G/O \) is replaced with the localization of coker \( J \) at odd primes.

For any \( C \)-manifold \( M \) we will define a map \( \eta_M: M \rightarrow G/\text{PL} \times BO \). This will give us a map on classifying space level

\[
\eta = (\eta_1 \times \eta_2): BC \rightarrow G/\text{PL} \times BO
\]

which splits \( BC \) as a product.

If \( M \) is a smooth manifold, let \( (\eta_1)_M \) be a trivial map, and \( (\eta_2)_M \) the smooth tangent bundle.

Let \( N \) be any \( C_s \)-manifold with trivial PL-tangent bundle. Then \( N \) has some framed structure, and the \( C_s \)-structure of \( N \) is given by a map \( f: N \rightarrow \text{PL}/C_s \).

Suppose we have defined \( \eta \) on \( C_s \)-manifolds so that for all \( N \) of the type above, \( \eta_N \) is equal to the composite

\[
N \xrightarrow{f} \text{PL}/C_s \xrightarrow{\chi} G/O \xrightarrow{(j,p)} G/\text{PL} \times BO,
\]

where \( j \) and \( p \) are the natural maps. It is easy to see that this is valid for \( s=0 \), that is \( \text{PL}/C_s = \text{PL}/O \), if \( \eta \) is defined as above.

Let \( \Sigma \) be a sphere with a \( C_s \)-structure. It has a trivial PL-tangent bundle. If \( \chi(\Sigma) \sim 0 \) can we extend \( \eta_\Sigma \) from \( \Sigma \) to \( c\Sigma \) as the composite

\[
c\Sigma \xrightarrow{\chi} G/O \xrightarrow{(j,p)} G/\text{PL} \times BO.
\]

Then we can define \( \eta \) in the obvious way on all \( C_{s+1} \)-manifold. In particular, on \( V \times c\Sigma \) it is the composite

\[
V \times c\Sigma \xrightarrow{\chi_0 \times \chi_{s+1}} \text{PL}/O \times G/O \rightarrow G/O \rightarrow G/\text{PL} \times BO.
\]
To complete the induction, let \( N \) be a framed manifold. It has a \( C_\ast \)-structure induced by \( f: N \to PL/C_\ast \). As usual we can decompose \( N \)

\[
N = N_0 \cup \left( \bigcup_i V_i \times c\Sigma_i \right).
\]

Recall from section 2 that the smooth structure on \( N \) induces a smooth structure on \( N_0 \) and \( V_i \). This smooth structure is framed, since the smooth structure on \( N \) is. We must prove that \( \eta_N \) is equal to the composite

\[
N \xrightarrow{f} PL/C_\ast \xrightarrow{x} G/O \to G/PL \times BO.
\]

By the induction hypothesis this is true on \( N_0 \). On \( V_i \times c\Sigma_i \) it is also true, since the composite \( V_i \to PL/O \to G/O \to BO \) is just the tangent bundle of \( V_i \), and since we have a commutative diagram

\[
\begin{array}{ccc}
V_i \times c\Sigma_i & \to & PL/O \times PL/C_{s+1} \\
\downarrow & & \downarrow \\
PL/O \times G/O & \to & G/O \to G/PL \times BO
\end{array}
\]

This concludes the construction of \( \eta_M \). We want to construct a natural transformation \([-,-BC]\) \to \([-,-G/PL \times BO]\). Assume that \( X \) is a smooth manifold with tangent bundle \( \tau_X \). Let \( \sigma: X \to M \) be a \( C \)-thickening of \( X \). Then put

\[
\eta_1 = \sigma(\eta_M)_1: X \to G/PL,
\]

\[
\eta_2 = (\sigma(\eta_M)_2 - \tau_X): X \to BO
\]

and finally

\[
\eta = (\eta_1, \eta_2): X \to G/PL \times BO.
\]

In particular, let \( M \) be the disc bundle \( D(\xi) \) of some vector bundle \( \xi \) on \( X \). Then \( \eta_1 \) is trivial, and \( \eta_2 \) represents the vector bundle \( (\tau_X \oplus \xi) - (\tau_X) = \xi \).

It is easy to see that the homotopy class of \( \eta \circ \sigma \) only depends on the stable class of the thickening \( M \). This defines a map

\[
\eta: [X, BC] \to [X, G/PL \times BO].
\]

**Lemma 5.1.** The map \([-,-BC]\) \to \([-,-G/PL \times BO]\) is a natural transformation in the category of smooth manifolds and homotopy classes of maps.

**Proof.** Let \( X, Y \) be smooth manifolds, \( \sigma: Y \to BC \) represent a homotopy class, and let \( f: X \to Y \) be a continuous map. As in the proof of lemma 2.1, we consider two cases.
CASE 1: $f: X \rightarrow Y = D(\xi)$ is the inclusion of $X$ as the zero section in a disc bundle. Let $\bar{\sigma}: Y \rightarrow M$ be a thickening classified by $\sigma$. Then $\bar{\sigma}$ is a simple homotopy equivalence. Choose a vector bundle $\theta$ over $M$ such that $D(\theta \oplus \bar{\sigma}^{-1}(\xi)) = M \times D^n$. Then there is a pullback of thickenings

$$D(\theta) \longrightarrow M \times D^n$$
$$\uparrow \bar{\sigma} \quad \uparrow \bar{\sigma}$$
$$X \longrightarrow D(\xi) = Y$$

It is clear that $\bar{\sigma}f$ is a thickening representing $\sigma f$. From the definition of $\eta$ we see that

$$(\eta_{D(\theta)}(\bar{\sigma}f)) \sim (\eta_M \bar{\sigma})f.$$  

CASE 2. Now we can assume that $X$ is a submanifold of codimension zero in $Y$. Any stable thickening of $Y$ can be represented by a $C$-structure on a PL-disc bundle $\xi$ on $Y$. Take the restriction of $\xi$ to $X$. The total space of this bundle is a manifold, which has an induced $C$-structure. The inclusion $i_X: X \rightarrow D(\xi_X)$ is a thickening of $X$. It is clear that $$(\eta_{D(\xi_X)}i_X \sim (\eta_{D(\xi)}i_Y)f.$$  

The natural transformation of lemma 5.1 defines a map of classifying spaces

$$\eta: BC \rightarrow G/PL \times BO.$$  

If we approximate PL/C with a framed manifold $M$, and use that $\eta_M: M \rightarrow G/PL \times BO$ is equal to the composite $M \rightarrow PL/C \rightarrow G/O \rightarrow G/PL \times BO$ we get a commutative diagram

$$
\begin{array}{ccc}
\text{PL/C} & \xrightarrow{\chi} & G/O \\
\downarrow & & \downarrow (j,p) \\
BC & \xrightarrow{\eta} & G/PL \times BO \\
\end{array}
$$

Recall the classes $k^{2-2} \in H^{2-2}(G/PL; \mathbb{Z}/2)$. Let $\Phi_{PL}$ be the fibre of the map

$$G/PL \rightarrow \prod_{n \in A} K(\mathbb{Z}/2, n),$$

where $A$ is the set of dimensions congruent to 2 modulo 4, for which $bP_n = 0$. If we replace $\chi$ in the argument above with the map $\tilde{\chi}: PL/C \rightarrow \Phi_0$ defined in section 4, we get a corresponding map $\tilde{\eta}: BC \rightarrow \Phi_{PL} \times BO$.

Consider the homotopy commutative diagram

$$
\begin{array}{ccc}
\text{PL} & \longrightarrow & \text{PL} \\
\downarrow & & \downarrow * \\
\Phi_0 & \longrightarrow & G/O \\
\downarrow & & \downarrow \prod_{n \in A} K(\mathbb{Z}/2, n) \\
\Phi_{PL} \times BO & \rightarrow & G/PL \times BO \\
\downarrow & & \downarrow \prod_{n \in A} K(\mathbb{Z}/2, n) \\
\end{array}
$$
Since all other columns and rows are fibrations, the left column is a fibration. Recall that $PL \to G/O$ factors as $PL \to PL/O \to PL/C \to G/O$. This shows that the following diagram is homotopy commutative

$$
\begin{array}{ccc}
PL & \to & PL/C \\
\downarrow{id} & & \downarrow{\tilde{\chi}} \\
PL & \to & \Phi_0 \\
\downarrow{\epsilon} & & \downarrow{\tilde{\eta}} \\
\Phi_0 & \to & \Phi_{PL} \times BO
\end{array}
$$

Since both rows are fibrations, and the maps $id$ and $\tilde{\chi}$ are homotopy equivalences, the map $\tilde{\eta}$ is also an equivalence. We have proved

**Theorem 5.3.** There is a fibration

$$BC \to G/PL \times BO \to \prod_{n \in A} K(\mathbb{Z}/2, n) ,$$

where $k$ is given by the classes $k^n \in H^n(G/PL; \mathbb{Z}/2)$, and $A$ is the set of dimensions of form $4m-2$ for which $bP_{4m-2}=0$.

If $X$ is a loop space, and $PL/O \to X$ is a loop map we can construct other categories of manifolds, similar to $C$.

Assume for example that $X$ is the localisation of $\text{Coker} J$ at odd primes. Recall that the localisation of $G/O$ at odd primes splits as a product of loop spaces

$$G/O[\frac{1}{2}] \to (\text{Coker} J)[\frac{1}{2}] \times BSO_0[\frac{1}{2}] .$$

Let

$$\lambda_0 : PL/O \to (\text{Coker} J)[\frac{1}{2}]$$

be the map

$$PL/O \to G/O \to G/O[\frac{1}{2}] \to (\text{Coker} J)[\frac{1}{2}] .$$

Since $(\text{Coker} \chi_0) \in \pi_* (G/O)$ contains no odd torsion, and $\pi_* (\text{Coker} J)$ only contains torsion we know that

$$\lambda_0 : \pi_* PL/O \to \pi_* (\text{Coker} J)[\frac{1}{2}]$$

is an epimorphism on homotopy groups. We define the $\tilde{C}_1$-category as the category with singularities $c\Sigma$, where $\Sigma$ are representatives of the concordance classes satisfying $\lambda_0 (\Sigma) = 0$.

Following the procedure in section 2 we can construct a map

$$\lambda_1 : PL/\tilde{C}_1 \to (\text{Coker} J)[\frac{1}{2}]$$

given by a natural transformation
\[ \lambda_1 : [-, \text{PL}/\mathcal{C}_1] \to [-, \text{G}/\mathcal{O}] \, . \]

More precisely, let \( M \) be a smooth manifold, and let \( \sigma \in [M, \text{PL}/\mathcal{C}_1] \) be represented by a \( \mathcal{C} \)-structure

\[ \tilde{\sigma} : M_0 \cup \bigcup_i (V_i \times c\Sigma_i) \to M \, . \]

We get a set of classifying maps

\[ g_0 : M_0 \to \text{PL}/\text{O} \]
\[ g_{V_i} : V_i \to \text{PL}/\text{O} \]
\[ g_{\Sigma_i} : \Sigma_i \to \text{PL}/\text{O} \]

related by

\[ g_0|_{V_i \times \Sigma_i} = \mu_{\text{PL}/\text{O}}(g_{V_i} \times g_{\Sigma_i}) \, . \]

For each \( g_{\Sigma_i} : \Sigma_i \to \text{PL}/\text{O} \) satisfying that the composite

\[ \Sigma_i \to \text{PL}/\text{O} \to (\text{Coker } J)[\frac{1}{2}] \]

is nullhomotopic, we can choose a particular nullhomotopy

\[ g_{c\Sigma_i} : c\Sigma_i \to (\text{Coker } J)[\frac{1}{2}] \, . \]

We can now define \( \lambda_i(\sigma) \in [M, (\text{Coker } J)[\frac{1}{2}]] \) by the following formulas:

i) On \( M_0 \) it is the composite \( M_0 \xrightarrow{g_0} \text{PL}/\text{O} \xrightarrow{\tilde{\sigma}_0} \text{G}/\text{O} \).

ii) On \( V_i \times c\Sigma_i \) it is given by the composite

\[ V_i \times c\Sigma_i \xrightarrow{g_{V_i} \times g_{c\Sigma_i}} \text{PL}/\text{O} \times (\text{Coker } J)[\frac{1}{2}] \to (\text{Coker } J)[\frac{1}{2}] \times (\text{Coker } J)[\frac{1}{2}] \xrightarrow{\mu} (\text{Coker } J)[\frac{1}{2}] \, . \]

Following section 2 we can now prove that \( \sigma|_\mathcal{C} \to \lambda_1(\sigma) \) is a natural transformation. In particular it does define a map

\[ \lambda_1 : \text{PL}/\mathcal{C}_1 \to (\text{Coker } J)[\frac{1}{2}] \, . \]

By induction we, as in section 3-4 obtain a category of manifolds with singularities we denote the \( \mathcal{C} \)-category. We also obtain a classifying space \( \mathcal{B}\mathcal{C} \).

Let \( \text{PL}/\mathcal{C} \) denote the fibre of the natural map \( \mathcal{B}\mathcal{C} \to \text{BPL} \).

The arguments in sections 3-4 generalize to yield

**Theorem 5.4.** There is a category \( \mathcal{C} \) of manifolds with singularities. This category depends on choices, but if \( \mathcal{C}' \) is another category, constructed in the same way, there is a natural equivalence

\[ E : S_{\mathcal{C}}(-) \to S_{\mathcal{C}'}(-) \, . \]
Furthermore, there is a homotopy equivalence

$$\hat{\lambda} : \text{PL}/\tilde{C} \to (\text{Coker } J)[\frac{1}{2}] \ .$$

Recall that $\text{BSPL}[\frac{1}{2}]$ splits as a product

$$B(\text{Coker } J)[\frac{1}{2}] \times \text{BSO}[\frac{1}{2}] \xrightarrow{j \times \gamma} \text{BSPL}[\frac{1}{2}] \ .$$

The right inverse of $\gamma$ is a map

$$\varrho : \text{BSPL}[\frac{1}{2}] \longrightarrow \text{BSO}[\frac{1}{2}] \ .$$

The natural map $\text{SPL} \to \text{Coker } J$ factors as $\text{SPL} \to \text{PL/O} \to G/O \to \text{Coker } J$. Also,

$$\text{SPL} \to \text{PL/O} \to \text{PL}/\tilde{C} \to (\text{Coker } J)[\frac{1}{2}]$$

is the natural map, since $\hat{\lambda}$ is an extension of $\hat{\lambda}_0$. We conclude that the following diagram is commutative

$$\begin{array}{ccc}
\pi_* (\text{SPL}) & & \pi_* (\text{PL}/\tilde{C}) \\
\downarrow \approx & & \downarrow \\
\pi_* (\text{Coker } J)[\frac{1}{2}] & & \\
\end{array}$$

Consider the map $\tilde{\varrho} : \text{BSC} \to \text{BSPL} \xrightarrow{\varrho} \text{BSO}$.

There is a commutative diagram with exact rows:

$$0 \to \pi_* (\text{BSCL}) \to \pi_* (\text{BSPL}) \to \pi_{*-1} (\text{PL}/\tilde{C}) \to 0$$

$$\downarrow \tilde{\varrho}_* \quad \downarrow \text{id} \quad \downarrow \hat{\lambda}_*$$

$$0 \to \pi_* (\text{BSO}) \xrightarrow{\gamma} \pi_* (\text{BSPL}) \to \pi_{*-1} (\text{Coker } J)[\frac{1}{2}] \to 0 \ .$$

**Theorem 5.5.** There is a homotopy equivalence

$$\tilde{\varrho} : \text{BSC} \to \text{BSO}[\frac{1}{2}] \ .$$

Finally we note that we could have defined the $C$-manifolds with the additional restriction that the singular manifolds must be framed. This is the main case considered in Levitt [10]. We can still construct the map $\chi$, and we get a classifying space $\text{BC}$ homotopy equivalent to the one considered in this paper. The proofs carry over with minimal changes. In this case there is no obvious way to construct the action $\text{BO} \times \text{BC} \to \text{BC}$, and the statement $\mathcal{S}_C(M) \approx [M, \text{PL}/C]$ is not obvious if $M$ is not framed.
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