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ON CERTAIN HOMOMORPHISM PROPERTIES OF GRAPHS II

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Abstract.

It is proved that every finite graph with $n \geqq 7$ vertices and at least $\frac{9}{2}n-12$ edges which is not a 4-cockade composed of complete 6-graphs and/or complete 8-graphs with four independent edges deleted is homomorphic to a complete 7-graph with one edge deleted.

Furthermore some estimates of the number of edges required for a graph to be homomorphic to a complete graph and a complete graph with one or two edges deleted are proved.

Introduction.

This paper is a continuation of [3].

For definitions and terminology see [3, section 1], and for an introduction into the problems considered see [3, section 2]. The first section of this paper, section 5, contains some theorems and auxiliary results to be used in sections 6 and 7, where the main results are presented.

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5. Auxiliary results.

Some auxiliary results and basic theorems similar to those of [3 sections 3], will be needed here:

(B) A corollary to an extension of the theorem of Menger ([1]): identical with (B) of section 3 in [3].

(C) If Γ is a finite graph with $n \geqq 6$ vertices and e edges such that $e \geqq \frac{7}{2}n - \frac{15}{2}$ and $\Gamma \notin \mathscr{K}_5^3$, then $\Gamma \succ \langle 6- \rangle$ (Theorem 1 E in [2]; this is Theorem α of [3] for $\nu = 6$).

Furthermore the following lemma will be needed:

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Lemma 3. *Let* $K \in \mathscr{C}^{v-3}_{v-1}$, $v \geqq 7$. *Then*:

A. $e(K) = (v - \frac{5}{2})n(K) - \frac{1}{2}(v-1)(v-3)$.
B. $K \not\succ \langle v - \rangle$.
C. *For* $v=7$: *If* $K \neq \langle 6 \rangle$, *and an edge joining any two vertices not already joined by an edge is added to K and any edge of K deleted, then the resulting graph is homomorphic to a* $\langle 7 - \rangle$.

Proof. A and B: By induction over the number $\varkappa$ of $\langle v-1 \rangle$'s and/or $\langle v+1 \equiv i \rangle$'s of which K is composed. We omit the details.

C: By induction over $\varkappa$.

Let $\varkappa = 1$. $K \neq \langle 6 \rangle$, hence $K = \langle 8 \equiv i \rangle$. Let $k_1, k_2, \ldots, k_8$ be the vertices of K, let (k_1, k_5), (k_2, k_6), (k_3, k_7), and $(k_4, k_8) \notin E(K)$, and assume (k_1, k_5) is added. Without loss of generality it may be assumed that either (k_1, k_2) or (k_2, k_3) is deleted. By contracting $K(k_2, k_8)$ into one vertex the resulting graph is contracted into a $\langle 7 - \rangle$.

Let $\varkappa = 2$. If K is composed of two $\langle 6 \rangle$'s, C is easily verified. If K is composed of a $\langle 8 \equiv i \rangle$ and a $\langle 6 \rangle$ or a $\langle 8 \equiv i \rangle$, then the resulting graph is homomorphic to a $\langle 8 \equiv i \rangle$ with a new edge added and one of its edges missing, and this graph is from case $\varkappa = 1$ homomorphic to a $\langle 7 - \rangle$.

Suppose now that the conclusion holds for cockades composed of fewer than $\varkappa (\varkappa \geqq 3)$ $\langle 6 \rangle$'s and/or $\langle 8 \equiv i \rangle$'s. Let K be composed of the $\varkappa$ $\langle 6 \rangle$'s and/or $\langle 8 \equiv i \rangle$'s $\varphi_1, \varphi_2, \ldots, \varphi_\varkappa$, successively. Let $s, t \in V(K)$ such that $(s,t) \notin E(K)$ and let the vertices of $\varphi_\varkappa$ be $k_1, k_2, \ldots, k_6$, if $\varphi_\varkappa = \langle 6 \rangle$, $k_1, k_2, \ldots, k_8$ if $\varphi_\varkappa = \langle 8 \equiv i \rangle$, the notation being chosen so that

$$V(\varphi_\varkappa \cap K') = \{k_1, k_2, k_3, k_4\} ,$$

where K' is the cockade composed of $\varphi_1, \varphi_2, \ldots, \varphi_{\varkappa-1}$, successively. K' is a member of $\mathscr{C}^4_6$ composed of $\varkappa - 1$ $(\geqq 2)$ $\langle 6 \rangle$'s and/or $\langle 8 \equiv i \rangle$'s hence $K' \neq \langle 6 \rangle$. Two alternative cases are considered:

i) $s, t \in V(K')$. Then (s,t) is added to K'. If any edge is deleted from K' or from $\varphi_\varkappa$, it follows by the induction hypothesis that the resulting graph is homomorphic to a $\langle 7 - \rangle$.

ii) *s or t is one of* k_5, k_6 (k_5, k_6, k_7, k_8 *respectively*). Say $s = k_5$. $t \in \varphi_\varkappa$ implies that $\varphi_\varkappa \neq \langle 6 \rangle$, and the case $\varkappa = 1$ gives the conclusion. Hence assume $t \notin \varphi_\varkappa$. If t is joined by an edge to every vertex of $\{k_1, k_2, k_3, k_4\}$, then (by Remark after Lemma 2) k_1, k_2, k_3, k_4, and t are all vertices of the same φ_i. A $\langle 8 \equiv i \rangle$ does not contain a $\langle 5 \rangle$, hence $\varphi_i = \langle 6 \rangle$. $i \neq \varkappa$, because $t \notin \varphi_\varkappa$. Let ε denote an arbitrary edge of K.

$$\varphi_i \cup \varphi_\varkappa \cup (s,t) - \varepsilon \succ \langle 7 - \rangle$$

by the induction hypothesis because $\varphi_i \cup \varphi_K$ is a member of $\mathscr{C}_6^4$ composed of only 2 $\langle 6 \rangle$'s and/or $\langle 8 \equiv i \rangle$'s. If t is not joined to all of k_1, k_2, k_3, k_4, say $(t, k_1) \notin E(K)$, then if ε again denotes an arbitrary edge of K:

$$K \cup (s,t) - \varepsilon \succ K' \cup (t, k_1) - \varepsilon \succ \langle 7- \rangle$$

by the induction hypothesis.

This completes the proof of Lemma 3.

REMARK. The statement analogous to C for $v > 7$ is not true; e.g. for $v > 7$ a $\langle v+1 \equiv i \rangle$ with an arbitrary edge added and an edge deleted, the latter edge being incident with a vertex of valency v in the $\langle v+1 \equiv i \rangle$ and a vertex incident with the added edge, is not homomorphic to a $\langle v- \rangle$.

Finally, in the proof of Theorem 3 the following result will be needed:

LEMMA 4. *Every regular graph of valency* 4 *with* 8 *vertices is isomorphic to just one of the six graphs* $\Delta_1, \ldots, \Delta_6$ *in Fig.* 1.

PROOF. Obviously $\Delta_1, \ldots, \Delta_6$ are regular graphs of valency 4 with 8 vertices. They are pairwise non-isomorphic, because Δ_4 is the only one containing a $\langle 4 \rangle$, Δ_1 contains 7 $\langle 3 \rangle$'s, Δ_2 8 $\langle 3 \rangle$'s, Δ_3 6, Δ_5 4 and Δ_6 none.

Hence to prove Lemma 4 it is sufficient to show that if Γ is any regular graph of valency 4 with 8 vertices, then Γ is isomorphic to one of $\Delta_1, \ldots, \Delta_6$. Label the vertices of Γ as $x^1, \ldots, x^8$. We will consider three cases (i), (ii) and (iii).

(i) Γ contains 4 independent vertices.

Suppose that $x^1, \ldots, x^4$ are independent. Then each of them is joined to each of $x^5, \ldots, x^8$ and Γ has no other edges. Hence $\Gamma \cong \Delta_6$.

(ii) $\Gamma \supset \langle 4 \rangle$.

Suppose that $\Gamma(x^1, \ldots, x^4) = \langle 4 \rangle$. Then each of $x^1, \ldots, x^4$ is joined to just one of $x^5, \ldots, x^8$. Therefore each of $x^5, \ldots, x^8$ is joined to just one of $x^1, \ldots, x^4$ and $\Gamma(x^5, \ldots, x^8) = \langle 4 \rangle$. Hence $\Gamma \cong \Delta_4$.

(iii) Γ contains a $\langle 4- \rangle$ as a spanned subgraph.

Suppose that $\Gamma(x^1, \ldots, x^4) = \langle 4- \rangle$ and $(x^1, x^4) \notin E(\Gamma)$. The number of edges of Γ between $\{x^1, \ldots, x^4\}$ and $\{x^5, \ldots, x^8\}$ is 6. Hence $\Gamma(x^5, \ldots, x^8) = \langle 4- \rangle$. Suppose that $(x^5, x^8) \notin E(\Gamma)$.

It is sufficient to consider the following typical cases:

(a) $(x^2, x^8), (x^3, x^8) \in E(\Gamma)$.

(b) $(x^2, x^7), (x^3, x^6) \in E(\Gamma)$.

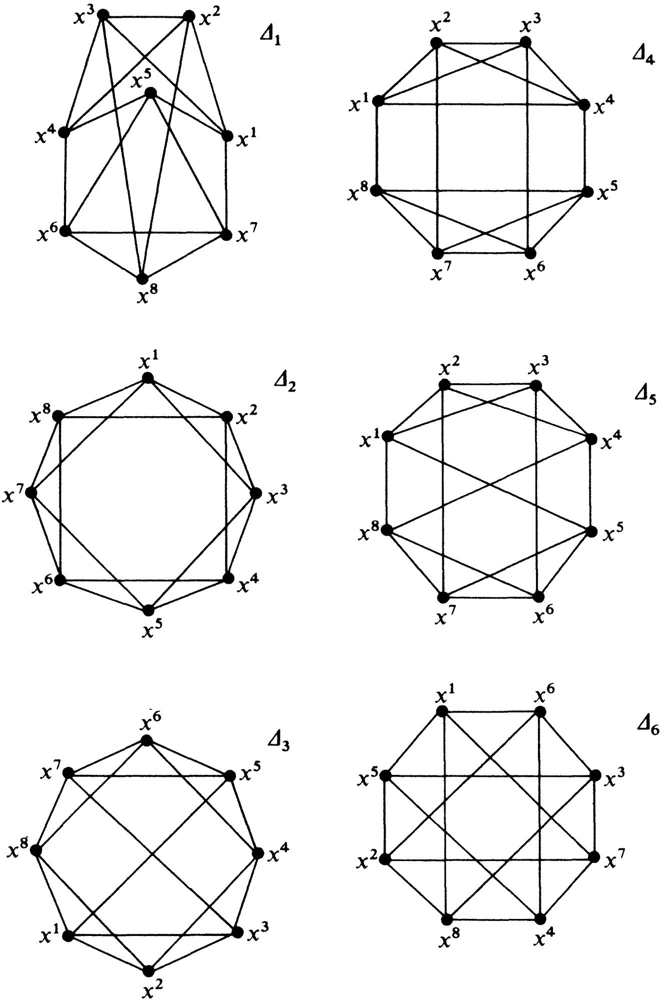

Figure 1

(c) $(x^2, x^8), (x^3, x^7) \in E(\Gamma)$.

(d) $(x^2, x^8), (x^3, x^5), \in E(\Gamma)$.

In case (a) (x^5, x^1), $(x^5, x^4) \in E(\Gamma)$. Now by symmetry we may suppose that the remaining edges are (x^1, x^7) and (x^4, x^6). Hence $\Gamma \cong \Delta_1$.

In case (b) clearly $\Gamma \cong \Delta_5$.

In case (c) (x^5, x^1), $(x^5, x^4) \in E(\Gamma)$, and by the symmetry between x^1 and x^4 we may suppose that the remaining edges are (x^1, x^8) and (x^4, x^7). Hence $\Gamma \cong \Delta_3$.

In case (d) we may suppose that (x^7, x^1), $(x^6, x^4) \in E(\Gamma)$, for otherwise we have (a) or (b) or (c) with a change of notation. Then by symmetry we may suppose that the remaining edges are (x^1, x^8) and (x^4, x^5). Hence $\Gamma \cong \Delta_2$.

(i) or (ii) or (iii) holds. For suppose that (i) does not hold.

Assume first that no three vertices are independent. Then without loss of generality $(x^1, x^8) \notin E(\Gamma)$ and $(x^1, x^i) \in E(\Gamma)$ for $i=2,\ldots,5$ and $(x^8, x^i) \in E(\Gamma)$ for $i=4,\ldots,7$. Then $(x^2, x^3) \in E(\Gamma)$, since $\{x^2, x^3, x^8\}$ is not independent. $\{x^3, x^4, x^5\}$ is not independent, and if $(x^4, x^5) \in E(\Gamma)$, then

$$\Gamma(x^1, x^8, x^4, x^5) = \langle 4- \rangle$$

while if e.g. $(x^3, x^4) \in E(\Gamma)$, then

$$\Gamma(x^1, x^2, x^3, x^4) \supset \langle 4- \rangle .$$

Hence (ii) or (iii) holds.

Assume next that $\{x^6, x^7, x^8\}$ is independent. If some two of x^6, x^7, x^8 have the same four neighbours then, since (i) does not hold, $\Gamma \supset \langle 4- \rangle$. Otherwise the notation can be chosen so that (x^2, x^8), (x^3, x^7), $(x^4, x^6) \notin E(\Gamma)$. Now x^1 is joined to one of $x^2,\ldots,x^5$, say x^j. (Then $(x^1, x^j) \in \langle 4- \rangle \subset \Gamma$. Thus $\Gamma \supset \langle 4- \rangle$, hence (ii) or (iii) holds.

Therefore (i) or (ii) or (iii) holds, and Lemma 4 is proved.

6. Homomorphism theorems for $\langle 7- \rangle$.

THEOREM 3. *Let Γ be a graph containing a vertex x_0 of valency* 8. *Let the vertices joined to x_0 be denoted by $x_1,\ldots,x_8$ and let $\Gamma(x_1,\ldots,x_8)$ be denoted by Γ_8. If Γ satisfies the following conditions*

1) *Γ is 5-fold connected,*
2) *Γ is not separated by a $\langle 5 \rangle$, $\langle 5- \rangle$, or $\langle 5= \rangle$,*
3) *$\forall x \in V(\Gamma): v(x, \Gamma) \geqq 6$ and the neighbour-configuration of every vertex of valency 6 in Γ is a $\langle 6 \equiv i \rangle$,*
4) *$\forall x_k \in V(\Gamma_8): v(x_k, \Gamma_8) \geqq 4$,*

then $\Gamma \succ \langle 7- \rangle$.

PROOF. By 4) $e(\Gamma_8) \geqq 16$. It will first be proved that

(1) If $e(\Gamma_8) = 16$, then $\Gamma \succ \langle 7- \rangle$.

PROOF OF (1). By 4) Γ_8 is a regular graph of valency 4 having 8 vertices. Therefore by Lemma 4, Γ_8 is isomorphic to one of the graphs $\Delta_1,\ldots,\Delta_6$ of Fig. 1. By 3) each vertex of Γ_8 is joined to one or more connected components of $\Gamma - \Gamma_8 - x_0$.

Each of the six cases $\Gamma_8 \cong \Delta_k$, $k=1,2,\ldots,6$ will now be considered in turn.

$\Gamma_8 \cong \Delta_1$.

Let C be a connected component of $\Gamma - \Gamma_8 - x_0$ joined to x_3. Two alternative cases are distinguished.

(a) *Suppose C is not joined to x_2.* Then C is by 1) joined to at least four of $x_1, x_4, x_5, x_6, x_7, x_8$, hence it is joined to at least one pair of diametrically opposite vertices of the 6-circuit $((x_1, x_5, x_4, x_6, x_8, x_7))$; assume without loss of generality that C is joined to x_5 and x_8. Then by contracting each of $C \cup x_5$, $\Gamma(x_6, x_4)$, and $\Gamma(x_1, x_7)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ in case (a).

(b) *Suppose C is joined to x_2.* Then C is joined to both x_1 and x_2. In this case C is by 1) joined to at least three vertices of $x_1, x_5, x_7, x_6, x_8, x_4$, and the case (b) may be divided into two subcases (b1) and (b2):

Subcase (b1): *Suppose C is joined to at least one of x_5, x_6, x_7.* Assume without loss of generality that C is joined to x_5. Then by contracting each of $C \cup x_5$ and $\Gamma(x_6, x_7, x_8)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ in subcase (b1).

Subcase (b2): *Suppose C is not joined to any of x_5, x_6, x_7.* In this case C is by 1) necessarily joined to each of x_1, x_2, x_3, x_4, x_8. Therefore by contracting each of C, $\Gamma(x_4, x_6)$, nad $\Gamma(x_1, x_4)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6 \rangle$ as a subgraph, five vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ also in subcase (b2).

The subcases (b1) and (b2) are alternatives hence $\Gamma \succ \langle 7- \rangle$ in case (b) and consequently $\Gamma \succ \langle 7- \rangle$, if $\Gamma_8 \cong \Delta_1$.

$\Gamma_8 \cong \Delta_2$.

Two alternative cases (a) and (b) are considered.

(a). *Assume that there exists a projection P from $\Gamma - \Gamma_8 - x_0$ onto Γ_8 such that at least one vertex of $P\Gamma_8$ is joined in $P\Gamma_8$ to all the other vertices of Γ_8.* It may without loss of generality be assumed that x_1 is the vertex which in $P\Gamma_8$ is joined to all the other vertices of Γ_8. By contracting $P\Gamma_8(x_2, x_7, x_8)$ into one vertex, $P\Gamma_8$ is then contracted into a $\langle 6- \rangle$, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ in case (a).

(b). *Assume that there does not exist any projection of the kind described in* (a). By 3), $\Gamma - \Gamma_8 - x_0$ contains a connected component, C_1, say. C_1 is by 1) joined to two diametrically opposite vertices of $((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8))$, let the notation be chosen so that C_1 is joined to x_1 and x_5. Then C_1 must be joined to at least one of the vertices x_2, x_4, x_6, x_8, say it is joined to x_4.

C_1 is not joined to x_6 by (6).

By 3) there exists a connected component C_2 of $\Gamma - \Gamma_8 - x_0$ different from C_1 joined to x_6.

If C_2 is joined to x_1 or to x_3, then by contracting each of $C_1 \cup x_1$, $C_2 \cup x_6$, and $\Gamma(x_2, x_7, x_8)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ in this case. *If C_2 is joined to each of x_4, x_7, x_8*, then by contracting each of $C_1 \cup x_4$ and $C_2 \cup x_4$ into one vertex, a projection P is obtained such that x_5 in $P\Gamma_8$ is joined to all the other vertices of Γ_8, contrary to assumption (b). *The remaining possibility is that C_2 is joined to neither x_1 nor x_3, and is not joined to all three of x_4, x_7, x_8*. By 1) C_2 is then joined to exactly two of x_4, x_7, x_8 and to each of x_2, x_5, x_6. Now by contracting each of $C_1 \cup x_1$, $C_2 \cup x_2$, and $\Gamma(x_6, x_7, x_8)$ into one vertex, $\Gamma - x_0$ is contracted into a $\langle 6- \rangle$, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ also in this case, so $\Gamma \succ \langle 7- \rangle$, if (b) holds. Consequently $\Gamma \succ \langle 7- \rangle$ if $\Gamma_8 \cong \Delta_2$.

$\Gamma_8 \cong \Delta_3$.

First it will be proved that

(*) *If there exists a connected component of $\Gamma - \Gamma_8 - x_0$ which is joined either to x_6 and at least two of x_1, x_2, x_3 or to x_2 and at least two of x_5, x_6, x_7, then $\Gamma \succ \langle 7- \rangle$.*

PROOF. Let C denote such a component. Assume without loss of generality that C is joined to x_6 and at least two of x_1, x_2, x_3. By contracting each of $C \cup x_6$, $\Gamma(x_4, x_5)$, and $\Gamma(x_7, x_8)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$, and the assertion is proved.

By 3) there exists a connected component C of $\Gamma - \Gamma_8 - x_0$ joined to x_6. There are now three possibilities left:

(a). *Suppose C is not joined to any of x_1, x_2, x_3*. Then C is by 1) joined to exactly x_4, x_5, x_6, x_7, x_8, and by contracting each of $C \cup x_8$, $\Gamma(x_3, x_7)$, and $\Gamma(x_1, x_2)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ in case (a).

(b). *Suppose C is joined to exactly one of x_1, x_2, x_3*.

If C is joined to x_2, then C must by 1) be joined to at least one of x_5, x_7, hence by (*) $\Gamma \succ \langle 7- \rangle$. If C is not joined to x_2, then assume without loss of generality that C is joined to x_3. There are two alternative subcases:

SUBCASE (b1): *Suppose C is joined to x_5.* Then by contracting each of $C \cup x_3$ and $\Gamma(x_1, x_2, x_8)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ in subcase (b1).

SUBCASE (b2): *Suppose C is not joined to x_5.* Then by 1) C is joined to exactly x_3, x_4, x_6, x_7, x_8. By contracting each of $C \cup x_8$, $\Gamma(x_1, x_5)$ and $\Gamma(x_6, x_7)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ also in subcase (b2).

It has now been proved that $\Gamma \succ \langle 7- \rangle$ in case (b).

(c). *Suppose C is joined to at least two of x_1, x_2, x_3.* Then $\Gamma \succ \langle 7- \rangle$ by (*). Thus $\Gamma \succ \langle 7- \rangle$ if $\Gamma_8 \cong \Delta_3$.

$\Gamma_8 \cong \Delta_4$.

By 3) $\Gamma - \Gamma_8 - x_0$ contains a connected component, C say. C is joined to both vertices in one of the pairs $\{x_1, x_8\}$, $\{x_2, x_7\}$, $\{x_3, x_6\}$, $\{x_4, x_5\}$ by 1); suppose without loss of generality that C is joined to x_4 and x_5. Furthermore, again by 1) it follows that C is joined to at least two vertices in one of the sets $\{x_1, x_2, x_3\}$, $\{x_6, x_7, x_8\}$. By the symmetry it may be assumed that C is joined to x_7, x_8. Then by contracting each of $C \cup x_4$, $\Gamma(x_3, x_6)$, and $\Gamma(x_1, x_2)$ into one vertex $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ if $\Gamma_8 \cong \Delta_4$.

$\Gamma_8 \cong \Delta_5$.

By 3) there exists a connected component C of $\Gamma - \Gamma_8 - x_0$ joined to x_2. Three cases are considered:

(a). *Suppose C is joined to x_5 or x_8.* Without loss of generality suppose that C is joined to x_8. Then by contracting each of $C \cup x_2$, $\Gamma(x_3, x_6)$ and $\Gamma(x_1, x_5)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ in case (a).

(b). *Suppose C is joined to x_6.* By contracting each of $C \cup x_2$, $\Gamma(x_1, x_8)$, and $\Gamma(x_4, x_5)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ in case (b).

(c). *Suppose neither* (a) *nor* (b) *is the case.* Then by 1) C is joined to

x_1, x_2, x_3, x_4, x_7. By contracting each of $C \cup x_7$, $\Gamma(x_1, x_8)$, and $\Gamma(x_5, x_6)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ in case (c) also. Hence $\Gamma \succ \langle 7- \rangle$ if $\Gamma_8 \cong \Delta_5$.

$\Gamma_8 \cong \Delta_6$.

By 1) and 3) there exists a connected component C of $\Gamma - \Gamma_8 - x_0$ joined to at least five vertices of Γ_8. Hence C is joined to two diametrically opposite vertices of $((x_1, x_6, x_3, x_7, x_4, x_8, x_2, x_5))$. Assume without loss of generality C is joined to x_1 and x_4. By contracting each of $C \cup x_1$, $\Gamma(x_2, x_5)$, and $\Gamma(x_3, x_7)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$ if $\Gamma_8 \cong \Delta_6$.

Now (1) is proved.

Next it will be proved that

(2) If $e(\Gamma_8) \geqq 17$, then there exists a projection P from $\Gamma - \Gamma_8 - x_0$ onto Γ_8 such that $e(P\Gamma_8) \geqq 20$.

PROOF OF (2). Suppose (reduction ad absurdum) that (2) is false.

(2.1) $\Gamma - \Gamma_8 - x_0 = \varnothing$.

For Γ_8 contains a vertex of valency 4 in Γ_8, since otherwise $e(\Gamma_8) \geqq 20$. This vertex has valency 5 in $\Gamma_8 \cup x_0$, but valency $\geqq 6$ in Γ by 3).

(2.2) $\Gamma - \Gamma_8 - x_0$ *has at least two connected components.*

PROOF OF (2.2). Suppose that (2.2) is false. Then $\Gamma - \Gamma_8 - x_0$ has exactly one connected component by (2.1), C say.

Each vertex of Γ_8 not joined to C has valency $\geqq 6$ in Γ_8. For suppose a vertex x_k has valency $\leqq 5$ in Γ_8 and is not joined to C; then $v(x_k, \Gamma) = 6$ by the first part of 3). But x_k is joined to x_0, which is joined to all the other vertices adjacent to x_k contrary to the second part of 3). This contradiction proves the assertion.

C is not joined to four vertices of Γ_8 such that one of them is not joined to any of the three others; for otherwise there exists a simple projection P from C onto Γ_8 such that $e(P\Gamma_8) \geqq e(\Gamma_8) + 3$, which is contrary to hypothesis because $e(\Gamma_8) \geqq 17$.

Let j be the number of vertices of Γ_8 to which C is joined; $j \geqq 5$ by 1). Then by the two above assertions

$$e(\Gamma_8) \geqq \tfrac{1}{2}\cdot j\cdot (j-3)+(8-j)6-\tfrac{1}{2}(8-j)(8-j-1) = 20 ,$$

contrary to hypothesis. This contradiction proves (2.2).

(2.3) *Every connected component of* $\Gamma - \Gamma_8 - x_0$ *is joined to three vertices of* Γ_8 *such that one of them is joined to neither of the others.*

Suppose on the contrary that C is a connected component of $\Gamma - \Gamma_8 - x_0$ such that any three vertices of Γ_8 joined to C span a $\langle 3- \rangle$ or a $\langle 3 \rangle$ in Γ_8. Then any five vertices of Γ_8 joined to C span a graph containing a $\langle 5 = i \rangle$ as a subgraph and any six vertices of Γ_8 joined to C span a graph containing a $\langle 6 \equiv i \rangle$ as a subgraph. From this it follows by 2) that it may without loss of generality be assumed that C is joined to $x_1, x_2, \ldots, x_6$ and that $\Gamma_8(x_1,\ldots,x_6) \supseteqq \langle 6 \equiv i \rangle$. The total of edges incident with x_7, and x_8 in Γ_8 is by 4) at least 7. Hence either $e(\Gamma_8) \geqq 22$, or there exists a simple projection P from C onto Γ_8 such that $e(P\Gamma_8) \geqq 20$, contrary to hypothesis. This contradiction proves (2.3).

Now let C_1 be a connected component of $\Gamma - \Gamma_8 - x_0$. Assume without loss of generality that C_1 is joined to $x_1, \ldots, x_5$ and by (2.3) that (x_1, x_2), $(x_2, x_3) \notin E(\Gamma_8)$. Let C_2 be another connected component of $\Gamma - \Gamma_8 - x_0$. C_2 is by (2.3) joined to $x_p, x_q, x_r \in V(\Gamma_8)$ such that (x_p, x_q), $(x_q, x_r) \notin E(\Gamma_8)$. Now there always exist simple projections P_1 and P_2 from C_1, C_2 onto Γ_8, respectively, such that

$$e((P_1 \circ P_2)\Gamma_8) \geqq e(\Gamma_8)+3 \geqq 20 ,$$

unless $x_p = x_1$, $x_q = x_2$, $x_r = x_3$ (or analogously, $x_p = x_3$, $x_q = x_2$, $x_r = x_1$) and $\Gamma_8(x_1,\ldots,x_5) = \langle 5 = \rangle$, hence by hypothesis this must be the case. But then by 2), C_1 is joined to at least one more vertex of Γ_8, say to x_6. Again there exist simple projections P_1 and P_2 from C_1, C_2 onto Γ_8, respectively, such that

$$e((P_1 \circ P_2)\Gamma_8) \geqq 20 ,$$

unless x_6 is joined to each of $x_1, \ldots, x_5$. In this case, however, $e(\Gamma_8) \geqq 20$, contrary to hypothesis.

This completes the proof of (2).

(3) If there exists a projection P from $\Gamma - \Gamma_8 - x_0$ onto Γ_8 such that $e(P\Gamma_8) \geqq 21$, then $\Gamma \succ \langle 7- \rangle$.

Proof of (3). $21 > \frac{7}{2}\cdot 8 - \frac{15}{2}$, hence by (C) and Lemma 1, $P\Gamma_8 \succ \langle 6- \rangle$ and consequently $\Gamma \succ \langle 7- \rangle$. This proves (3).

By 4), (1), (2) and (3), in order to prove Theorem 3 it remains only to prove that if $e(\Gamma_8) \geqq 17$ and P is a projection from $\Gamma - \Gamma_8 - x_0$ onto Γ such that $e(P\Gamma_8) = 20$, then $\Gamma \succ \langle 7- \rangle$. This will be done now.

(i) If every vertex of $P\Gamma_8$ has valency 5 in $P\Gamma_8$, then $\Gamma \succ \langle 7- \rangle$.

Proof of (i). $\overline{P\Gamma_8}$ consists of an 8-circuit or of two 4-circuits or of a 3-circuit and a 5-circuit. In the first two cases $P\Gamma_8 \succ \langle 6 \rangle$ and so $\Gamma \succ \langle 7 \rangle$. In the last case $P\Gamma_8 \succ \langle 6- \rangle$ and so $\Gamma \succ \langle 7- \rangle$.

(ii) If there is a vertex of valency 4 in $P\Gamma_8$ whose neighbour-configuration in $P\Gamma_8$ contains a vertex of valency $\leqq 1$ in the neighbour-configuration, then $\Gamma \succ \langle 7- \rangle$.

Proof of (ii). Let the notation be chosen so that x_1 is joined to exactly x_2, x_3, x_4, x_5 in $P\Gamma_8$ and $v(x_2, P\Gamma_8(x_2, \ldots, x_5)) \leqq 1$. By contracting $P\Gamma_8(x_1, x_2)$ into one vertex $P\Gamma_8$ is contracted into a graph Δ such that $n(\Delta) = 7$,

$$e(\Delta) \geqq 20 - 4 + 2 = 18 .$$

By (C) and Lemma 1, $\Delta \succ \langle 6- \rangle$, hence $\Gamma \succ \langle 7- \rangle$. This proves (ii).

Because of 4), the assumption $e(P\Gamma_8) = 20$, and (i) and (ii), in order to complete the proof of Theorem 3, it is sufficient to prove:

(iii) If there exists a vertex of valency 4 in $P\Gamma_8$ and every vertex of valency 4 in $P\Gamma_8$ has one of the three neighbour-configurations $\langle 4 = i \rangle$, $\langle 4- \rangle$, or $\langle 4 \rangle$ in $P\Gamma_8$, then $\Gamma \succ \langle 7- \rangle$.

Proof of (iii). The cases where $P\Gamma_8$ is isomorphic to a graph consisting of a $\langle 5 \rangle$ and a $\langle 6 = \rangle$ with a $\langle 3 \rangle$ in common will be of importance in the sequel. There are six non-isomorphic graphs of this description, three of which have a vertex of valency 3 and consequently by 4) these will not be considered further.

The other three are denoted by $\Lambda_1, \Lambda_2, \Lambda_3$ and may be described as follows: $\Lambda_i(y_1, y_2, y_3, y_4, y_5) = \langle 5 \rangle$ and $\Lambda_i(y_3, y_4, y_5, y_6, y_7, y_8) = \langle 6 = \rangle$ for $i = 1, 2, 3$. In Λ_1 the deleted edges of the $\langle 6 = \rangle$ are (y_8, y_6) and (y_3, y_7). In Λ_2 they are (y_8, y_5), (y_3, y_7). Finally in Λ_3 they are (y_8, y_3), (y_3, y_7). (Fig. 2).

It will now be proved that under the assumption of (iii)

(I) Either $\Gamma \succ \langle 7- \rangle$ or $P\Gamma_8 \cong \Lambda_1, \Lambda_2$ or Λ_3.

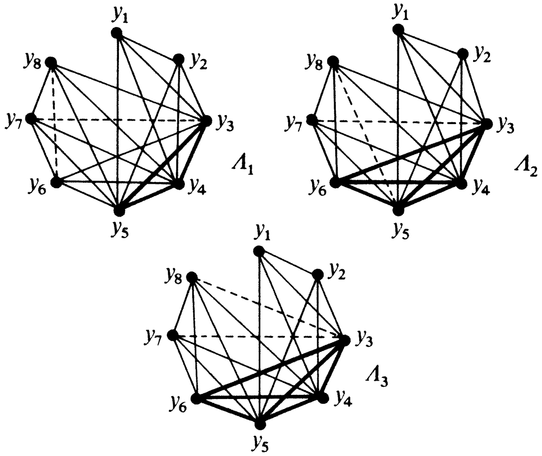

Figure 2

PROOF OF (I). Assume without loss of generality that x_1 is a vertex of valency 4 in $P\Gamma_8$ (by 4) consequently also of valency 4 in Γ_8) joined to x_2, x_3, x_4, x_5. By (iii) exactly three possibilities may be distinguished, namely $P\Gamma_8(x_2,\ldots,x_5) = \langle 4=i \rangle$, $\langle 4- \rangle$, or $\langle 4 \rangle$. These three possibilities will be analysed in turn:

A) $P\Gamma_8(x_2,\ldots,x_5) = \langle 4=i \rangle$.
Assume without loss of generality that $(x_2, x_4), (x_3, x_5) \notin E(P\Gamma_8)$. By contracting $P\Gamma_8(x_1, x_2)$ into one vertex, $P\Gamma_8$ is contracted into a graph Δ such that $n(\Delta)=7$, $e(\Delta)=20-4+1=17$. Suppose firstly that $\Delta \in \mathscr{K}_5^3$; then Δ is a $\langle 7 \rangle$ with the edges of a 4-circuit deleted. This 4-circuit of $\bar{\Delta}$ must include (x_3, x_5) and can without loss of generality be assumed to be $\bar{\Delta}((x_3, x_5, x_8, x_7))$. By contracting $P\Gamma_8(x_1, x_5)$ into one vertex in stead of $P\Gamma_8(x_1, x_2)$, $P\Gamma_8$ is contracted into a graph $\Delta' \notin \mathscr{K}_5^3$ such that $n(\Delta')=7$, $e(\Delta')=17$, hence by (C), $\Delta' \succ \langle 6- \rangle$ and consequently $\Gamma \succ \langle 7- \rangle$. Suppose secondly that $\Delta \notin \mathscr{K}_5^3$. Then by (C), $\Delta \succ \langle 6- \rangle$, hence $\Gamma \succ \langle 7- \rangle$. So if A) holds, then $\Gamma \succ \langle 7- \rangle$.

B) $P\Gamma_8(x_2,\ldots,x_5) = \langle 4- \rangle$.
Assume without loss of generality that $(x_2, x_3) \notin E(P\Gamma_8)$.
By contracting $P\Gamma_8(x_1, x_2)$ into one vertex, $P\Gamma_8$ is contracted into a graph $\Delta \cong (P\Gamma_8 - x_1) \cup (x_2, x_3)$ such that $n(\Delta)=7$, $e(\Delta)=20-4+1=17$.
If $\Delta \notin \mathscr{K}_5^3$, then by (C), $\Delta \succ \langle 6- \rangle$, hence $\Gamma \succ \langle 7- \rangle$ in this case.
Suppose consequently from now on that $\Delta \in \mathscr{K}_5^3$. Then Δ is a $\langle 7 \rangle$ with the edges of a 4-circuit deleted, and $(x_2, x_3) \in E(\Delta)$. This 4-circuit will in the sequel be referred to as *the 4-circuit of* $\bar{\Delta}$. Two alternative cases are considered:

(a): *Suppose the 4-circuit of $\bar{\Delta}$ contains neither x_2 nor x_3*. There are two possibilities: x_4 and x_5 are both contained in the 4-circuit of $\bar{\Delta}$ or just one of x_4, x_5 is contained in it. In both cases one of x_6, x_7, x_8 is joined to both x_2 and x_3 and another one of x_6, x_7, x_8 is joined to all of x_2, x_3, x_4, x_5, hence clearly $P\Gamma_8 \succ \langle 6- \rangle$ and consequently $\Gamma \succ \langle 7- \rangle$ in case (a).

(b): *Suppose the* 4-*circuit of* $\bar{\Delta}$ *contains at least one of* x_2, x_3.

Then three subcases are distinguished:

SUBCASE (b1): *Suppose the* 4-*circuit of* $\bar{\Delta}$ *contains both* x_2 *and* x_3. Then the 4-circuit of $\bar{\Delta}$ may without loss of generality be assumed to be $\bar{\Delta}((x_2, x_6, x_3, x_7))$. $P\Gamma_8$ is then isomorphic to Λ_1, because

$$P\Gamma_8(x_4, x_5, x_6, x_7, x_8) = \langle 5 \rangle,$$

$$P\Gamma_8(x_1, x_2, x_3, x_4, x_5, x_8) = \langle 6= \rangle \ ,$$

and the deleted edges of the $\langle 6= \rangle$ are (x_2, x_3) and (x_1, x_8).

SUBCASE (b2): *The* 4-*circuit of* $\bar{\Delta}$ *contains exactly one of* x_2, x_3 *and neither of* x_4, x_5. Then it may without loss of generality be assumed to be $\bar{\Delta}((x_2, x_6, x_8, x_7))$. $P\Gamma_8$ is then isomorphic to Λ_1, because

$$P\Gamma_8(x_3, x_4, x_5, x_6, x_7) = \langle 5 \rangle \ ,$$

$$P\Gamma_8(x_1, x_2, x_3, x_4, x_5, x_8) = \langle 6= \rangle \ ,$$

and the deleted edges of the $\langle 6= \rangle$ are (x_1, x_8) and (x_2, x_3).

SUBCASE (b3): *The* 4-*circuit of* $\bar{\Delta}$ *contains exactly one of* x_2, x_3 *and exactly one of* x_4, x_5. The 4-circuit of $\bar{\Delta}$ may without loss of generality be assumed to be $\bar{\Delta}((x_2, x_6, x_4, x_7))$. $P\Gamma_8$ is then isomorphic to Λ_2, because

$$P\Gamma_8(x_3, x_5, x_6, x_7, x_8) = \langle 5 \rangle \ ,$$

$$P\Gamma_8(x_1, x_2, x_3, x_4, x_5, x_8) = \langle 6= \rangle \ ,$$

and the deleted edges of the $\langle 6= \rangle$ are (x_1, x_8) and (x_2, x_3).

The subcases (b1)–(b3) clearly exhaust all possibilities of case (b). Hence in case (b), $P\Gamma_8 \cong \Lambda_1$ or Λ_2.

This proves that (I) is true if (B) holds.

C) $P\Gamma_8(x_2, \ldots, x_5) = \langle 4 \rangle$.

Then $P\Gamma_8(x_1, \ldots, x_5) = \langle 5 \rangle$. Two alternative cases will be considered:

(a). *Suppose* $P\Gamma_8(x_6, x_7, x_8)$ *is joined to each of* $x_2, \ldots, x_5$. If $P\Gamma_8(x_6, x_7, x_8)$ is connected, then clearly $P\Gamma_8 \succ \langle 6- \rangle$ and consequently $\Gamma \succ \langle 7- \rangle$. If

$P\Gamma_8(x_6, x_7, x_8)$ is disconnected, then by 4), $P\Gamma_8 \supset \langle 6- \rangle$ and so $\Gamma \succ \langle 7- \rangle$. Hence in case (a), $\Gamma \succ \langle 7- \rangle$.

(b). *Suppose* $P\Gamma_8(x_6, x_7, x_8)$ *is not joined to all of* $x_2, \ldots, x_5$. It may without loss of generality be assumed that x_2 is not joined to any of x_6, x_7, x_8 in $P\Gamma_8$. Then

$$P\Gamma_8 = P\Gamma_8(x_1, \ldots, x_5) \cup P\Gamma_8(x_3, \ldots, x_8) .$$

The total number of edges incident with x_6, x_7, x_8 is 10 because $e(P\Gamma_8) = 20$. Hence $P\Gamma_8(x_3, \ldots, x_8) = \langle 6= \rangle$.

$$P\Gamma_8(x_1, \ldots, x_5) \cap P\Gamma_8(x_3, \ldots, x_8) = \langle 3 \rangle .$$

Therefore by 4), $P\Gamma_8$ is isomorphic to either Λ_1, Λ_2 or Λ_3 in case (b).

Thus (I) is true if C holds.

Now (I) is proved.

Finally it will be proved that under the assumptions of (iii):

(II) If $P\Gamma_8 \cong \Lambda_1, \Lambda_2$, or Λ_3, then $\Gamma \succ \langle 7- \rangle$.

Proof of (II). By 4) no edge incident with a vertex of valency 4 in $P\Gamma_8$ can have been provided by P. Hence in Λ_1 at most the edges (y_3, y_4), (y_3, y_5), (y_4, y_5) can have been provided for Γ_8 by P, and in Λ_2 and Λ_3 at most the edges (y_3, y_4), (y_3, y_5), (y_4, y_5), (y_3, y_6), (y_4, y_6), (y_5, y_6) can have been provided for Γ_8 by P. (These edges are heavily drawn in Fig. 2.)

By 3) there exists a connected component C_k of $\Gamma - \Gamma_8 - x_0$ joined to y_k, $k = 1, 2$. If there is a simple projection from C_k onto Γ_8 which is part of P, let it be denoted by P_k, $k = 1, 2$. If not, let P_k denote the identical mapping on $V(\Gamma_8)$ (then $P - P_k = P$), $k = 1, 2$. C_1 and C_2 and hence P_1 and P_2 need not be different.

$P\Gamma_8 \cong \Lambda_1$ or Λ_2 and $P\Gamma_8 \cong \Lambda_3$ will be considered separately.

$P\Gamma_8 \cong \Lambda_1$ or Λ_2.

Two alternatives are considered:

(a). *Suppose either* C_1 *or* C_2 *is not joined to* y_6. By the symmetry assume without loss of generality that C_1 is not joined to y_6. Then none of (y_3, y_6), (y_4, y_6), (y_5, y_6) can have been provided for Γ_8 by P_1, hence they are all contained in $(P - P_1)\Gamma_8$. If C_1 is joined to y_7 or y_8, let P' denote the simple projection from C_1 onto Γ_8 obtained by contracting $C_1 \cup y_7$ or $C_1 \cup y_8$, respectively, into one vertex, and let $P^* = P' \cdot (P - P_1)$. By contracting each of $P^*\Gamma_8(y_3, y_8)$ and $P^*\Gamma_8(y_5, y_6)$ or each of $P^*\Gamma_8(y_5, y_7)$ and $P^*\Gamma_8(y_3, y_6)$, respectively, into one vertex, $P^*\Gamma_8$ is contracted into a graph containing a $\langle 6-\rangle$ as a subgraph, hence $\Gamma \succ \langle 7-\rangle$ in this case. If C_1 is joined to neither y_7 nor y_8, then C_1 is by 1) joined to each $y_1, \ldots, y_5$, and then by contracting each of $(P-P_1)\Gamma_8(y_3, y_6)$, $(P-P_1)\Gamma_8(y_5, y_7)$, and C_1 into one vertex $\Gamma - x_0$ is contracted into a graph containing a $\langle 6\rangle$ as a subgraph, five vertices of which are joined to x_0, hence $\Gamma \succ \langle 7-\rangle$. Consequently $\Gamma \succ \langle 7-\rangle$ in case (a).

b). *Suppose both C_1 and C_2 are joined to y_6.* Using only the edges of Λ_1, which are thinly drawn in Fig. 2, if $P\Gamma_8 \cong \Lambda_1$, and only the edges of Λ_2, which are thinly drawn in Fig. 2, if $P\Gamma_8 \cong \Lambda_2$—in each case all these edges are in $E(\Gamma_8)$ by 4)—it is seen that by contracting each of $C_1 \cup C_2 \cup y_6$ (possibly $C_1 = C_2$), $\Gamma(y_3, y_8)$ and $\Gamma(y_5, y_7)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6-\rangle$ as a subgraph, all the vertices of which are joined to x_0. Hence $\Gamma \succ \langle 7-\rangle$ also in case (b).

(a) and (b) are alternatives, hence it is proved that $\Gamma \succ \langle 7-\rangle$ if $P\Gamma_8 \cong \Lambda_1$ or Λ_2.

$P\Gamma_8 \cong \Lambda_3$.
Two alternatives are considered:

(a). *Suppose either C_1 or C_2 is not joined to y_6.* By the symmetry assume without loss of generality that C_1 is not joined to y_6. The none of (y_3, y_6), (y_4, y_6), (y_5, y_6) can have been provided for Γ_8 by P_1, hence they are all contained in $(P-P_1)\Gamma_8$. If C_1 is joined to y_7 or y_8, then from the symmetry between y_7 and y_8 assume without loss of generality that C_1 is joined to y_8. Then let P' denote the simple projection from C_1 onto Γ_8 obtained by contracting $C_1 \cup y_8$ into one vertex, and let $P^* = P' \cdot (P - P_1)$. By contracting each of $P^*\Gamma_8(y_5, y_7)$ and $P^*\Gamma_8(y_3, y_6)$ into one vertex, $P^*\Gamma_8$ is contracted into a graph containing a $\langle 6-\rangle$ as a subgraph, hence $\Gamma \succ \langle 7-\rangle$ in this case. If C_1 is joined to neither y_7 nor y_8, then it is proved that $\Gamma \succ \langle 7-\rangle$ just as in case (a) above, when $P\Gamma_8 \cong \Lambda_1$ or Λ_2. Consequently $\Gamma \succ \langle 7-\rangle$ in case (a).

(b). *Suppose both C_1 and C_2 are joined to y_6.* Let the graph which consists of the vertices $y_1, \ldots, y_8$ and the thinly drawn edges of Λ_3 in Fig. 2 be denoted by Φ. By 4), $\Gamma_8 \supset \Phi$ and in addition $E(\Gamma_8)$ contains at least two of the edges (y_3, y_4), (y_3, y_5), (y_3, y_6). Which ever two of these three edges is in $E(\Gamma_8)$, and also if all three of them are in $E(\Gamma_8)$, by contracting each of $C_1 \cup C_2 \cup y_6$ (possibly $C_1 = C_2$), $\Gamma(y_6, y_7)$ and $\Gamma(y_5, y_8)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6-\rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7-\rangle$ also in case (b).

(a) and (b) are alternatives, hence it is proved that $\Gamma \succ \langle 7-\rangle$ if $P\Gamma_8 \cong \Lambda_3$. This completes the proof of (II).

From (I) and (II) it follows that (iii) holds. Hence the proof of Theorem 3 is completed.

THEOREM 4. *Let Γ be a finite graph with n vertices and e edges. If $n \geqq 7$, $e \geqq \frac{9}{2}n - 12$ and $\Gamma \notin \mathscr{C}_6^4$, then $\Gamma \succ \langle 7- \rangle$.*

PROOF. By induction over n. The theorem is trivially true for $n = 7$.

INDUCTION HYPOTHESIS: Assume that the theorem is true for all graphs with m vertices satisfying the conditions, where $7 \leqq m \leqq n-1$.

Let Γ be a graph with n vertices satisfying the conditions of the theorem.

It is sufficient to consider the case $e = \{\frac{9}{2}n - 12\}$ in the rest of the proof.

For assume that $e > \{\frac{9}{2}n - 12\}$ and that the theorem holds for all graphs having exactly $\{\frac{9}{2} - 12\}$ edges. By deleting edges from Γ, a graph Γ^* may be obtained such that $e(\Gamma^*) = \{\frac{9}{2}n - 12\}$. If $\Gamma^* \notin \mathscr{C}_6^4$, then $\Gamma^* \succ \langle 7- \rangle$ by the last assumption, and therefore $\Gamma \succ \langle 7- \rangle$. If $\Gamma^* \in \mathscr{C}_6^4$ then, because $n \geqq 7$ and by Lemma 3.C again $\Gamma \succ \langle 7- \rangle$. This proves the assertion.

Assume then in the sequel that $e = \{\frac{9}{2}n - 12\}$.

(1) If $\exists\, x \in V(\Gamma):\ v(x, \Gamma) \leqq 4$, then $\Gamma \succ \langle 7- \rangle$.

PROOF OF (1). $n(\Gamma - x) = n - 1 \geqq 7$,

$$e(\Gamma - x) \geqq e - 4 \geqq \tfrac{9}{2}n - 12 - 4 = \tfrac{9}{2}(n-1) - \tfrac{23}{2} > \tfrac{9}{2}(n-1) - 12\,.$$

By Lemma 3.A and the induction hypothesis $\Gamma - x \succ \langle 7- \rangle$.

(2) Let Γ' be a graph with $n' < n$ vertices and e' edges. If $n' \geqq 6$ and $e' \geqq \frac{9}{2}n' - 12$, then either $\Gamma' \succ \langle 7- \rangle$ or $\Gamma' \in \mathscr{C}_6^4$.

PROOF OF (2). If $n' = 6$, $e' \geqq 15$ that is $\Gamma' = \langle 6 \rangle \in \mathscr{C}_6^4$. If $n' \geqq 7$ then by the induction hypothesis $\Gamma' \succ \langle 7- \rangle$ or $\Gamma' \in \mathscr{C}_6^4$.

(3) If Γ is disconnected or has a cut-set S such that $|S| \leqq 4$ or such that $|S| = 5$ and $\Gamma(S) = \langle 5 \rangle$, $\langle 5- \rangle$ or $\langle 5= \rangle$, then $\Gamma \succ \langle 7- \rangle$.

PROOF OF (3). If Γ has a cut-set, then it has a minimal cut-set. In the sequel let S denote a minimal cut-set of Γ if Γ has a cut-set, and $\varnothing$ if Γ is disconnected.

Let now $\Gamma_1, \Gamma_2, \sigma, s_1, \ldots, s_\sigma$, p, $n_1, n_2, e_1, e_2, \Gamma_1', \Gamma_2'$, P_1, and P_2 be defined just as in the beginning of the proof of (3) in Theorem 2.

If for $i=1$ or 2, $n_i \leqq 5$, then every vertex of $\Gamma_i - S$ has valency $\leqq 4$ in Γ, therefore by (1), $\Gamma \succ \langle 7-\rangle$ in this case. Hence it may be assumed from now on that

$$n_i \geqq 6, \quad i=1,2\,. \tag{3.1}$$

Now

$$\begin{aligned} e_1+e_2 &= e+p \geqq \tfrac{9}{2}n-12+p \\ &= \tfrac{9}{2}(n_1+n_2-\sigma)-12+p = \tfrac{9}{2}(n_1+n_2)-(\tfrac{9}{2}+12-p)\,. \end{aligned}$$

By the symmetry between Γ_1 and Γ_2 it may be assumed that

$$e_1 \geqq \tfrac{9}{2}n_1 - \tfrac{1}{2}(\tfrac{9}{2}\sigma+12-p)\,.$$

i) $\sigma \leqq 2$.

Then $p=0$ or 1. $e_1 \geqq \frac{9}{2}n_1 - \frac{21}{2} > \frac{9}{2}n_1 - 12$. By (3.1), Lemma 3.A, and (2), $\Gamma_1 \succ \langle 7-\rangle$.

ii) $\sigma = 3$.

1) $p \leqq 1$. Then $e_1 \geqq \frac{9}{2}n_1 - \frac{51}{4}$. Assume without loss of generality $(s_1,s_2), (s_1,s_3) \notin E(\Gamma)$. Consider

$$P_2\Gamma_1 = \Gamma_1 \cup (s_1,s_2) \cup (s_1,s_3)\,.$$

$n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1),

$$e(P_2\Gamma_1) = e_1+2 \geqq \tfrac{9}{2}n_1 - \tfrac{43}{4} > \tfrac{9}{2}n_1 - 12\,.$$

By (2) and Lemma 3.A, $P_2\Gamma_1 \succ \langle 7-\rangle$.

2) $p \geqq 2$. Then $e_1 \geqq \frac{9}{2}n_1 - \frac{47}{4}$. By (3.1), Lemma 3.A and (2) $\Gamma_1 \succ \langle 7-\rangle$.

It may now be assumed that Γ has no cut-sets with 3 vertices i.e. that Γ is 4-fold connected.

iii) $\sigma = 4$.

1) $p=0$. Then $e_1 \geqq \frac{9}{2}n_1 - 15$. Consider

$$P_2\Gamma_1 = \Gamma_1 \cup (s_1,s_2) \cup (s_1,s_3) \cup (s_1,s_4)\,.$$

$n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1). $e(P_2\Gamma_1) = e_1+3 \geqq \frac{9}{2}n_1 - 12$. By (2), $P_2\Gamma_1 \succ \langle 7-\rangle$ except when $P_2\Gamma_1 \in \mathscr{C}_6^4$. Assume then that $P_2\Gamma_1 \in \mathscr{C}_6^4$:

$\Gamma(s_1,s_4)$ is a $\langle 2\rangle$ contained in a $\langle 6\rangle$ or a $\langle 8 \equiv i\rangle$ of $P_2\Gamma_1$; let this $\langle 6\rangle$ or $\langle 8 \equiv i\rangle$ be denoted by X. Two alternatives are considered:

A. *Suppose s_2 or $s_3 \in V(X)$.*

Assume without loss of generality $s_2 \in V(X)$; then $s_3 \notin V(X)$ because

(s_3, s_4), $(s_3, s_2) \notin E(P_2\Gamma_1)$. $(s_2, s_4) \notin E(P_2\Gamma_1)$, hence $X \neq \langle 6 \rangle$, therefore $X = \langle 8 \equiv i \rangle$. Let P' denote the projection from Γ_2' onto Γ_1 obtained by contracting $\Gamma_2' \cup s_2$ into one vertex.

$$P'\Gamma_1 \supseteqq X \cup (s_2, s_4) - (s_1, s_4)$$

and $(s_2, s_4) \notin E(X)$, hence by Lemma 3.C, $P'\Gamma_1 \succ \langle 7- \rangle$ in case A.

B. *Suppose* $s_2, s_3 \notin V(X)$.

Γ is assumed to be 4-fold connected hence by (B), $\Gamma - s_1 - s_4$ contains 2 disjoint paths Π_1 and Π_2 from $\{s_2, s_3\}$ to $X - s_1 - s_4$; clearly

$$\Pi_i \cap (\Gamma_2 - s_2 - s_3) = \emptyset, \quad i = 1, 2 .$$

The notation may be chosen so that Π_1 has s_2 and t_2 as end-vertices and Π_2 has s_3 and t_3. If t_2 and t_3 are joined to s_3 and s_4 then, since $X - s_1 - s_4$ contains a (t_2, t_3)-path Π, by contracting each of Π_1 and Π_2 and $\Pi - t_3$ into one vertex, Γ can be contracted into a graph containing

$$\Gamma_2 \cup (s_1, s_2) \cup (s_2, s_3) \cup (s_3, s_1) \cup (s_2, s_4) \cup (s_3, s_4)$$

as a subgraph. Let Γ' denote this graph. If e.g. $(s_1, t_2) \in E(X)$, then by contracting each of Π_1 and $\Gamma_2' \cup s_1$ into one vertex, Γ is contracted into a graph containing $X \cup (s_1, t_2)$, hence by Lemma 3.C, $\Gamma \succ \langle 7- \rangle$.

Now $P_2\Gamma_1 \in \mathscr{C}_6^4$ implies that $e_1 = \frac{9}{2}n_1 - 15$ and this implies that $e_2 \geqq \frac{9}{2}n_2 - 15$. By (3.1), $n_2 \geqq 6$, and

$$e(\Gamma') = e_2 + 5 \geqq \tfrac{9}{2}n_2 - 10 .$$

Therefore by (2) and Lemma 3.A $\Gamma' \succ \langle 7- \rangle$. Hence in case (B) $\Gamma \succ \langle 7- \rangle$.

Thus we have proved that if $\sigma = 4$ and $p = 0$, then $\Gamma' \succ \langle 7- \rangle$.

2) $p = 1$. Then $e_1 \geqq \frac{9}{2}n_1 - \frac{29}{2}$. Assume without loss of generality (s_3, s_4) to be the only edge of $\Gamma(S)$. Consider

$$P_2\Gamma_1 = \Gamma_1 \cup (s_1, s_2) \cup (s_1, s_3) \cup (s_1, s_4) .$$

$n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1),

$$e(P_2\Gamma_1) = e_1 + 3 \geqq \tfrac{9}{2}n_1 - \tfrac{23}{2} > \tfrac{9}{2}n_1 - 12 .$$

By (2) and Lemma 3.A, $P_2\Gamma_1 \succ \langle 7- \rangle$.

3) $p = 2$. Then $e_1 \geqq \frac{9}{2}n_1 - 14$. Assume without loss of generality (s_1, s_2), $(s_1, s_3) \notin E(\Gamma)$, $(s_1, s_4) \in E(\Gamma)$ and s_2 is incident with a third missing edge denoted by ε. Consider

$$P_2\Gamma_1 = \Gamma_1 \cup (s_1, s_2) \cup (s_1, s_3) .$$

$n(P_2\Gamma_1)=n_1\geqq 6$ by (3.1) and $e(P_2\Gamma_1)=e_1+2\geqq\frac{9}{2}n_1-12$. By (2), $P_2\Gamma_1\succ\langle 7-\rangle$ except when $P_2\Gamma_1\in\mathscr{C}_6^4$. Assume then that $P_2\Gamma_1\in\mathscr{C}_6^4$. $P_2\Gamma_1\nsupseteq\langle 6\rangle$ because $\varepsilon\notin E(P_2\Gamma_1)$. Let P' denote the projection from Γ_2' onto Γ_1 obtained by contracting $\Gamma_2'\cup s_2$ into one vertex.

$$P'\Gamma_1 \supseteqq P_2\Gamma_1\cup\varepsilon-(s_1,s_3)\,.$$

$\varepsilon\notin E(P_2\Gamma_1)$, hence by Lemma 3.C, $P'\Gamma_1\succ\langle 7-\rangle$.

4) $p=3$. Then $e_1\geqq\frac{9}{2}n_1-\frac{27}{2}$. $\Gamma(S)$ contains at least one vertex of valency $\leqq 1$ in $\Gamma(S)$. Assume without loss of generality s_1 is such a vertex and $(s_1,s_2),(s_1,s_3)\notin E(\Gamma)$. Consider

$$P_2\Gamma_1 = \Gamma_1\cup(s_1,s_2)\cup(s_1,s_3)\,.$$

$n(P_2\Gamma_1)=n_1\geqq 6$ by (3.1),

$$e(P_2\Gamma_1) = e_1+2 \geqq \tfrac{9}{2}n_1-\tfrac{23}{2} > \tfrac{9}{2}n_1-12\,.$$

By (2) and Lemma 3.A, $P_2\Gamma_1\succ\langle 7-\rangle$.

5) $p=4$. Then $e_1\geqq\frac{9}{2}n_1-13$. Assume without loss of generality that $(s_1,s_2)\notin E(\Gamma)$, and s_3 is incident with the other missing edge of $\Gamma(S)$, denoted by ε, and s_1 is not incident with ε. Consider

$$P_2\Gamma_1 = \Gamma_1\cup(s_1,s_2)\,.$$

$n(P_2\Gamma_1)=n_1\geqq 6$ by (3.1),

$$e(P_2\Gamma_1) = e_1+1 \geqq \tfrac{9}{2}n_1-12\,.$$

By (2) therefore $P_2\Gamma_1\succ\langle 7-\rangle$ except when $P_2\Gamma_1\in\mathscr{C}_6^4$; assume then that $P_2\Gamma_1\in\mathscr{C}_6^4$. $P_2\Gamma_1\nsupseteq\langle 6\rangle$ because $\varepsilon\notin E(P_2\Gamma_1)$. P' denotes the projection from Γ_2' onto Γ_1 obtained by contracting $\Gamma_2'\cup s_3$ into one vertex.

$$P'\Gamma_1 = P_2\Gamma_1\cup\varepsilon-(s_1,s_2);$$

$\varepsilon\notin E(P_2\Gamma_1)$, hence by Lemma 3.C, $P'\Gamma_1\succ\langle 7-\rangle$.

6) $p=5$. Then $e_1\geqq\frac{9}{2}n_1-\frac{25}{2}$. Assume without loss of generality $(s_1,s_2)\notin E(\Gamma)$. Consider

$$P_2\Gamma_1 = \Gamma_1\cup(s_1,s_2)\,.$$

$n(P_2\Gamma_1)=n_1\geqq 6$ by (3.1),

$$e(P_2\Gamma_1) = e_1+1 \geqq \tfrac{9}{2}n_1-\tfrac{23}{2} > \tfrac{9}{2}n_1-12\,.$$

By (2) and Lemma 3.A, $P_2\Gamma_1\succ\langle 7-\rangle$.

7) $p=6$. Then $e_1\geqq\frac{9}{2}n_1-12$. By (3.1) and (2), $\Gamma_1\succ\langle 7-\rangle$ except when

$\Gamma_1 \in \mathscr{C}_6^4$. In this case $e_1 = \frac{9}{2}n_1 - 12$ by Lemma 3.A, hence $e_2 \geqq \frac{9}{2}n_2 - 12$ and by (3.1) and (2), $\Gamma_2 \succ \langle 7- \rangle$ except when $\Gamma_2 \in \mathscr{C}_6^4$. But if $\Gamma_1 \in \mathscr{C}_6^4$ and $\Gamma_2 \in \mathscr{C}_6^4$, then $\Gamma \in \mathscr{C}_6^4$, contrary to hypothesis.

iv) $\sigma = 5$.

1) $\Gamma(S) = \langle 5 \rangle$. Then $p = 10$ and $e_1 \geqq \frac{9}{2}n_1 - \frac{49}{4}$ that is

$$2e_1 \geqq 9n_1 - \tfrac{49}{2} = 9n_1 - 24 - \tfrac{1}{2},$$

hence $2e_1 \geqq 9n_1 - 24$ or $e_1 \geqq \frac{9}{2}n_1 - 12$. By (3.1) and (2), $\Gamma_1 \succ \langle 7- \rangle$ except when $\Gamma_1 \in \mathscr{C}_6^4$. In this case $\Gamma(S)$ is contained in a $\langle 6 \rangle \subseteqq \Gamma_1$ (it is not contained in a $\langle 8 \equiv i \rangle$ because a $\langle 8 \equiv i \rangle$ does not contain a $\langle 5 \rangle$). By contracting Γ_2' into one vertex, Γ is contracted into a graph containing a $\langle 7- \rangle$ as a subgraph.

2) $\Gamma(S) = \langle 5- \rangle$. Then $p = 9$ and $e_1 \geqq \frac{9}{2}n_1 - \frac{51}{4}$. Assume without loss of generality $(s_1, s_2) \notin E(\Gamma)$. Consider

$$P_2\Gamma_1 = \Gamma_1 \cup (s_1, s_2).$$

$n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1),

$$e(P_2\Gamma_1) = e_1 + 1 \geqq \tfrac{9}{2}n_1 - \tfrac{47}{4} > \tfrac{9}{2}n_1 - 12.$$

By (2) and Lemma 3.A, $P_2\Gamma_1 \succ \langle 7- \rangle$.

3) $\Gamma(S) = \langle 5= \rangle$. Then $p = 8$ and $e_1 \geqq \frac{9}{2}n_1 - \frac{53}{4}$. Assume without loss of generality that $(s_1, s_2) \notin E(\Gamma)$ and s_3 is incident with the other missing edge, denoted by ε, and s_1 is not incident with ε. Consider

$$P_2\Gamma_1 = \Gamma_1 \cup (s_1, s_2).$$

$n(P_2\Gamma_1) = n_1 \geqq 6$ by (3.1),

$$e(P_2\Gamma_1) = e_1 + 1 \geqq \tfrac{9}{2}n_1 - \tfrac{49}{4}$$

and by an argument as under iv) 1),

$$e(P_2\Gamma_1) \geqq \tfrac{9}{2}n_1 - 12.$$

By (2), $P_2\Gamma_1 \succ \langle 7- \rangle$ except when $P_2\Gamma_1 \in \mathscr{C}_6^4$. Assume then that $P_2\Gamma_1 \in \mathscr{C}_6^4$. $P_2\Gamma_1 \neq \langle 6 \rangle$ because $\varepsilon \notin E(P_2\Gamma_1)$. P' denotes the projection from Γ_2' onto Γ_1 obtained by contracting $\Gamma_2' \cup s_3$ into one vertex.

$$P'\Gamma_1 = P_2\Gamma_1 \cup \varepsilon - (s_1, s_2).$$

$\varepsilon \notin E(P_2\Gamma_1)$, hence by Lemma 3.C, $P'\Gamma_1 \succ \langle 7- \rangle$.

This completes the proof of (3).

(4) If Γ is 5-fold connected and is not separated by a $\langle 5\rangle$, $\langle 5-\rangle$, or $\langle 5=\rangle$, then $\Gamma \succ \langle 7-\rangle$.

Proof of (4). Assume that Γ has the properties stated in (4).

(4.1) If $\Gamma \supseteqq \langle 6\rangle$, then $\Gamma \succ \langle 7-\rangle$.

Proof of (4.1). Let Λ be a $\langle 6\rangle \subseteqq \Gamma$. $\Gamma-\Lambda \neq \emptyset$ because $n>7$. Let C be a connected component of $\Gamma-\Lambda$. Γ is 5-fold connected, hence C is joined to at least 5 of the vertices of Λ and by contracting C into one vertex, Γ is contracted into a graph containing a $\langle 7-\rangle$ as a subgraph. This proves (4.1).

Every vertex of Γ has valency $\geqq 5$, because Γ is 5-fold connected. If

$$\forall x \in V(\Gamma):\ v(x,\Gamma) \geqq 9\,,$$

then $e \geqq \frac{9}{2}n > \{\frac{9}{2}-12\}$, contrary to hypothesis. Hence it may be assumed that

$$\forall x \in V(\Gamma):\ v(x,\Gamma) \geqq 5$$

and

$$\exists x' \in V(\Gamma):\ 5 \leqq v(x',\Gamma) \leqq 8\,.$$

Let x_0 be an arbitrary vertex of Γ such that $v(x_0,\Gamma)=j$, $5 \leqq j \leqq 8$. Let the vertices joined to x_0 be denoted by $x_1, x_2, \dots, x_j$. $\Gamma(x_1, x_2, \dots, x_j)$ is denoted by Γ_j.

$\Gamma_j \supseteqq \langle 5\rangle$ implies $\Gamma \supseteqq \langle 6\rangle$, therefore $\Gamma \succ \langle 7-\rangle$ by (4.1) in this case. Hence it may be assumed that:

(4.2) $$\Gamma_j \not\supseteqq \langle 5\rangle\,.$$

(4.3) If $\exists x_i \in V(\Gamma_j):\ v(x_i,\Gamma_j) \leqq 3$, then $\Gamma \succ \langle 7-\rangle$.

Proof of (4.3) Assume without loss of generality that $i=1$ and x_1 is joined to at most x_2, x_3, x_4. By contracting $\Gamma(x_0,x_1)$ into one vertex, Γ is contracted into

$$(\Gamma - x_0) \cup \bigcup_{k=5}^{j} (x_1, x_k) = \Gamma'\,.$$

$n(\Gamma') = n-1 \geqq 7$;

$$e(\Gamma') \geqq e-j+j-4 \geqq \tfrac{9}{2}n-16 = \tfrac{9}{2}(n-1)-\tfrac{23}{2} > \tfrac{9}{2}(n-1)-12\,.$$

By (2) and Lemma 3.A, $\Gamma' \succ \langle 7-\rangle$. This proves (4.3).

It may then be assumed that:

(4.4) $$\forall x_k \in V(\Gamma_j) : v(x_k, \Gamma_j) \geqq 4 .$$

By (4.2) and (4.4) it may from now on be assumed that

(4.5) $$\forall x \in V(\Gamma) : v(x, \Gamma) \geqq 6 .$$

If $\Gamma - \Gamma_j - x_0 \neq \emptyset$, then every connected component of $\Gamma - \Gamma_j - x_0$ is joined to at least 5 vertices of Γ_j, because Γ is 5-fold connected.

$v_i^{(j)}$ denotes the number of vertices of Γ_j that have valency i in Γ_j.

I. *Suppose that* $j=6$.

If $v_5^{(6)} \geqq 3$, then by (4.4), $\Gamma_6 \supseteqq \langle 6- \rangle$, and then $\Gamma \supseteqq \langle 7- \rangle$. Hence assume $v_5^{(6)} \leqq 2$; the number of vertices of odd valency cannot be odd, hence by (4.4), $v_5^{(6)} = 2$ or 0. There are exactly 2 possibilities:

1) $v_5^{(6)} = 2$ *and* $v_4^{(6)} = 4$. Then $\Gamma_6 = \langle 6 = \rangle$. $\Gamma - \Gamma_6 - x_0 \neq \emptyset$ because $n > 7$. Let C be a connected component of $\Gamma - \Gamma_6 - x_0$. Γ is 5-fold connected, hence C is joined to at least 5 vertices of Γ_6, and Γ is not separated by a $\langle 5 \rangle$, hence C is joined to $x_q, x_r \in V(\Gamma_6)$ such that $(x_q, x_r) \notin E(\Gamma)$. By contracting $C \cup x_q$ into one vertex $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$.

2) $v_5^{(6)} = 0$ *and* $v_4^{(6)} = 6$. Then $\Gamma_6 = \langle 6 \equiv i \rangle$. Hence it may from now on in addition to (4.5) be assumed that:

(4.6) The neighbour-configuration of every vertex of valency 6 in Γ is a $\langle 6 \equiv i \rangle$ (the so-called octahedron-graph).

II. *Suppose that* $j=7$.

By (4.4), $e(\Gamma_7) \geqq 14$.

(4.7) If $e(\Gamma_7) \geqq 15$, then there exists a projection P from $\Gamma - \Gamma_7 - x_0$ onto Γ_7 (possibly P is the identical mapping on $V(\Gamma_7)$) such that $e(P\Gamma_7) \geqq 17$.

Proof of (4.7). Suppose om the contrary that there does not exist any projection P from $\Gamma - \Gamma_7 - x_0$ onto Γ_7 such that $e(P\Gamma_7) \geqq 17$.

$v_4^{(7)} \geqq 1$ otherwise $e(\Gamma_7) > 17$ contrary to hypothesis. Then by (4.5), $\Gamma - \Gamma_7 - x_0 \neq \emptyset$. Let C be a connected component of $\Gamma - \Gamma_7 - x_0$. C is joined to at least five vertices of Γ_7. C is not joined to any three vertices such that one of them is not joined to either of the others, because otherwise a contradiction to hypothesis would occur. Hence C is joined to five vertices spanning a graph containing a $\langle 5=i\rangle$ as a subgraph. But a $\langle 5=\rangle$, $\langle 5-\rangle$, or a $\langle 5\rangle$ does not separate Γ, hence C is joined to at least six vertices of Γ_7, assume without loss of generality to $x_1, x_2, \ldots, x_6$. By the hypothesis $x_1, x_2, \ldots, x_6$ span a graph containing a $\langle 6\equiv i\rangle$ as a subgraph. $v(x_7, \Gamma_7) \geqq 4$ by (4.4). Let P be the projection from $\Gamma-\Gamma_7-x_0$ onto Γ_7 obtained by contracting $C \cup x_1$ into one vertex; then $e(P\Gamma_7) \geqq 17$, contrary to hypothesis.

Hence (4.7) has been proved by reductio ad absurdum.

Consider now the two cases $e(\Gamma_7) \geqq 15$ and $e(\Gamma_7)=14$.

1) $e(\Gamma_7) \geqq 15$. By (4.7) there exists a projection P from $\Gamma-\Gamma_7-x_0$ onto Γ_7 such that $e(P\Gamma_7) \geqq 17$.

$17=\frac{7}{2}\cdot 7-\frac{15}{2}$. If $P\Gamma_7 \notin \mathscr{K}_5^3$ then by (C), $P\Gamma_7 \succ \langle 6-\rangle$, hence $\Gamma \succ \langle 7-\rangle$. Assume then that $P\Gamma_7 \in \mathscr{K}_5^3$, composed of two $\langle 5\rangle-s$. Assume without loss of generality that

$$P\Gamma_7(x_1, x_2, x_3, x_4, x_5) = \langle 5\rangle ,$$

$$P\Gamma_7(x_3, x_4, x_5, x_6, x_7) = \langle 5\rangle .$$

$e(P\Gamma_7)=17$ i.e. at most two new edges have been provided for Γ_7 by P.

The edges of $P\Gamma_7$ incident with x_1, x_2, x_6, x_7 cannot have been provided by P because of (4.4). Therefore the set of edges provided by P is a proper subset of $\{(x_3, x_4), (x_4, x_5), (x_5, x_3)\}$; assume without loss of generality that at most (x_3, x_4), (x_3, x_5), have been provided by P. Thus $\Gamma_7(x_1, \ldots x_5) \supseteqq \langle 5=\rangle$.

By (4.5), $v(x_1, \Gamma) \geqq 6$, hence x_1 is joined to a connected component C of $\Gamma-\Gamma_7-x_0$. Γ is 5-fold connected, and is not separated by a $\langle 5\rangle$, $\langle 5-\rangle$, or $\langle 5=\rangle$, therefore C is joined to one of x_6, x_7, say to x_7. By contracting each of $\Gamma(x_6, x_3)$ and $C \cup x_7$ into one vertex, $\Gamma-x_0$ is contracted into a graph containing a $\langle 6-\rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7-\rangle$.

2) $e(\Gamma_7)=14$. Γ_7 is then a regular graph of valency 4, i.e. $\bar{\Gamma}_7$ is a regular graph of valency 2. There are consequently two possibilities:

i) $\bar{\Gamma}_7$ consists of a 7-circuit.
ii) $\bar{\Gamma}_7$ consists of a 3-circuit and a 4-circuit disjoint from each other.

By (4.5), $\Gamma-\Gamma_7-x_0 \neq \varnothing$. Let C be a connected component of $\Gamma-\Gamma_7-x_0$. C is joined to at least five vertices of Γ_7.

i) *Suppose $\bar{\Gamma}_7$ consists of a 7-circuit.*

Assume without loss of generality the 7-circuit to be $\bar{\Gamma}((x_1, \ldots, x_7))$. C is necessarily joined to three consecutive vertices of the 7-circuit of $\bar{\Gamma}_7$, say to x_2, x_3, x_4. By contracting each of $C \cup x_3$ and $\Gamma(x_1, x_5)$ onto one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$.

ii) *$\bar{\Gamma}_7$ consists of a 3-circuit and a 4-circuit disjoint from each other.*

Assume without loss of generality $\bar{\Gamma}_7((x_1, x_2, x_3, x_4))$ to be the 4-circuit of $\bar{\Gamma}_7$ and $\bar{\Gamma}_7((x_5, x_6, x_7))$ to be the 3-circuit of $\bar{\Gamma}_7$.

If C is joined to three consecutive vertices of the 4-circuit of $\bar{\Gamma}_7$, say to x_3, x_4, x_1, then by contracting each of $C \cup x_4$ and $\Gamma(x_2, x_7)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$.

Assume that this is not the case; then C is joined to x_5, x_6, x_7 and exactly two vertices of the 4-circuit of $\bar{\Gamma}_7$. There ar two alternatives, a) and b):

a) *Suppose C is joined to 2 non-consecutive vertices of $\bar{\Gamma}_7((x_1, x_2, x_3, x_4))$.*

Assume without loss of generality that C is joined to x_2, x_4. By contracting each of C, $\Gamma(x_3, x_7)$, and $\Gamma(x_1, x_6)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6 \rangle$ as a subgraph, five vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$.

b) *Suppose C is joined to 2 consecutive vertices of $\bar{\Gamma}_7((x_1, x_2, x_3, x_4))$.*

Assume without loss of generality that C is joined to x_3, x_4. Then by (4.5), x_2 is joined to another connected component C' of $\Gamma - \Gamma_7 - x_0$, because $v(x_2, \Gamma_7) = 4$. As above $\Gamma \succ \langle 7- \rangle$ except perhaps when C' is joined to all of x_5, x_6, x_7. Assume then that this is the case. By contracting each of $C \cup x_3$, $C' \cup x_5$, and $\Gamma(x_1, x_7)$ into one vertex, $\Gamma - x_0$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x_0, hence $\Gamma \succ \langle 7- \rangle$.

III. *Suppose that $j = 8$.*

In Theorem 3 the conditions 1) and 2) are the assumptions of (4), the condition 3) is (4.5) and (4.6) and 4) is (4.4). Hence all the conditions of Theorem 3 are satisfied and consequently $\Gamma \succ \langle 7- \rangle$.

It has now been proved that $\Gamma \succ \langle 7- \rangle$, if Γ has a vertex x_0 of valency 7 or 8 in Γ.

x_0 was an arbitrary vertex of valency $\leqq 8$ of Γ, hence by (4.5), (4.6) and the above it may now be assumed that:

(4.8) Each vertex of Γ has valency either 6 or $\geqq 9$, and if it has valency 6, then its neighbour-configuration is a $\langle 6 \equiv i \rangle$.

It follows, since $e = \{\frac{9}{2}n - 12\}$, that the number of vertices of valency 6 in Γ is at least 8 for otherwise

$$e(\Gamma) \geqq \tfrac{1}{2}(7\cdot 6+(n-7)9) = \tfrac{9}{2}n-\tfrac{21}{2} > \{\tfrac{9}{2}n-12\}\,.$$

It these vertices span a complete graph, then $\Gamma \geqq \langle 8\rangle$, hence it may be assumed that there exist two vertices, x and y, of valency 6 in Γ, not joined by an edge.

Let $x_1, x_2, \ldots, x_6$ denote the vertices joined to x and let Γ_6 denote $\Gamma(x_1, \ldots, x_6)$. By (4.6), $\Gamma_6 = \langle 6 \equiv i\rangle$; assume without loss of generality, $(x_1, x_4), (x_2, x_5), (x_3, x_6) \notin E(\Gamma)$.

Let $y_1, y_2, \ldots, y_6$ denote the vertices joined to y and let Γ'_6 denote $\Gamma(y_1, \ldots, y_6)$. By (4.6), $\Gamma'_6 = \langle 6 \equiv i\rangle$; assume without loss of generality $(y_1, y_4), (y_2, y_5), (y_3, y_6) \notin E(\Gamma)$.

(4.9) If y is joined to two non-adjacent vertices of Γ_6, then $\Gamma \succ \langle 7-\rangle$.

PROOF OF (4.9). Assume without loss of generality that y is joined to x_1 and x_4.

$\Gamma - \Gamma_6 - x - y \neq \emptyset$, for otherwise y is joined to all vertices of Γ_6, and then $\Gamma = \langle 8 \equiv i\rangle$ contrary to the assumption that $\Gamma \notin \mathscr{C}_6^4$.

Any connected component of $\Gamma - \Gamma_6 - x - y$ is joined to at least four vertices of Γ_6 because $\Gamma - y$ is 4-fold connected. Two alternative cases are considered:

a) *Suppose there exists a connected component of $\Gamma - \Gamma_6 - x - y$ which is joined to two non-adjacent vertices of Γ_6 other than x_1, x_4.* Let C be a connected component of $\Gamma - \Gamma_6 - x - y$ and assume without loss of generality that C is joined to x_2 and x_5; then by contracting each of $C \cup x_2$ and $\Gamma(y, x_1)$ into one vertex, $\Gamma - x$ is contracted into a graph containing a $\langle 6-\rangle$ as a subgraph, all the vertices of which are joined to x, hence $\Gamma \succ \langle 7-\rangle$ in case a).

b) *Suppose* a) *does not hold.* The each connected component of $\Gamma - \Gamma_6 - x - y$ is joined to exactly four vertices of Γ_6 including x_1 and x_4, and to y as well because Γ is 5-fold connected.

Let C be a connected component of $\Gamma - \Gamma_6 - x - y$. C is then joined to y and to exactly four vertices of Γ_6 including x_1 and x_4, say to x_1, x_2, x_3, x_4. Then y cannot be joined to both of x_5 and x_6, because $\Gamma'_6 = \langle 6 \equiv i\rangle$. Hence by (4.5) there exists a connected component C' of $\Gamma - \Gamma_6 - x - y$ different from C. By the above, C' is joined to y and to exactly four vertices of Γ_6 including x_1 and x_4. (Fig. 3.)

Then y is joined to exactly one vertex in each of C and C', because $\Gamma'_6 = \langle 6 \equiv i\rangle$. For the same reason y cannot be joined to any vertex of $\Gamma - \Gamma_6 - x - y - C - C'$, because such a vertex is not joined to either of C, C'. Hence, again because $\Gamma'_6 = \langle 6 \equiv i\rangle$, y is joined to two non-adjacent vertices of Γ_6 different from x_1 and x_4, say x_2 and x_5. By contracting each of $C \cup x_1$ and $\Gamma(y, x_2)$ into one vertex, $\Gamma - x$ is contracted into a graph containing a $\langle 6-\rangle$ as subgraph, all the vertices of which are joined to x. Hence $\Gamma \succ \langle 7-\rangle$ also in case b).

This proves (4.9).

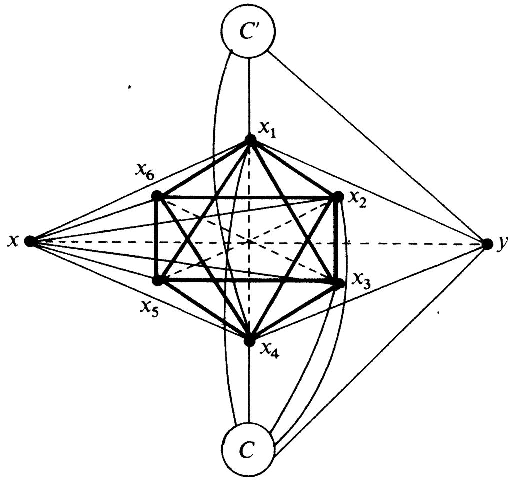

Figure 3

In what follows it may because of (4.9) be assumed that

y is not joined to any pair of non-adjacent vertices of Γ_6.

Let $|V(\Gamma_6 \cap \Gamma'_6)|$ be denoted by α. From our assumption $\alpha \leqq 3$. It may without loss of generality be assumed that $x_\beta = y_\beta$ for $1 \leqq \beta \leqq \alpha$.

i) *Suppose* $\alpha = 3$. $\Gamma - x_1 - x_2 - x_3$ is 2-fold connected, hence by (B) it contains 2 disjoint $(\{x_4, x_5, x_6\}, \{y_4, y_5, y_6\})$-paths, say Π_1 and Π_2. Obviously $x, y \notin \Pi_1 \cup \Pi_2$. Let y_q, $q = 4$, 5 or 6 be one end-vertex of Π_1. By contracting each of $\Pi_1 \cup y$ and $\Pi_2 \cup (\Gamma(y_4, y_5, y_6) - y_q)$ into one vertex, $\Gamma - x$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x, hence $\Gamma \succ \langle 7- \rangle$.

ii) *Suppose* $\alpha = 2$. $\Gamma - x_1 - x_2$ is 3-fold connected, hence by (B) it contains 3 disjoint $(\{x_3, x_4, x_5, x_6\}, \{y_3, y_4, y_5, y_6\})$-paths Π_1, Π_2, and Π_3, not containing x and y. It may without loss of generality be assumed that Π_1 has x_4 as end-vertex, and Π_2 has x_3 as end-vertex. Then Π_3 has either x_5 or x_6 as end-vertex. Let y_q, $q = 3$, 4, 5 or 6 be the other end-vertex of Π_1. By contracting $\Pi_1 \cup y$ and either (if Π_3 and x_5 as end-vertex)

$$\Pi_3 \cup (\Gamma(y_3, y_4, y_5, y_6) - y_q)$$

or (if Π_3 has x_6 as end-vertex)

$$\Pi_2 \cup (\Pi_3 - x_6) \cup (\Gamma(y_3, y_4, y_5, y_6) - y_q)$$

into one vertex, $\Gamma - x$ is contracted into a graph containing a $\langle 6- \rangle$ as a subgraph, all the vertices of which are joined to x, hence $\Gamma \succ \langle 7- \rangle$.

iii) *suppose* $\alpha=1$. $\Gamma-x_1$ is 4-fold connected, hence by (B) it contains 4 disjoint $(\Gamma(x_2,x_3,\ldots,x_6))$, $(\Gamma(y_2,y_3,\ldots,y_6))$-paths Π_1, Π_2, Π_3, Π_4 not containing x and y. It may without loss of generality be assumed that Π_1 has x_3 as end-vertex, Π_2 has x_6 as end-vertex and Π_3 has x_5 as end-vertex. Then Π_4 has either x_4 or x_2 as end-vertex. Let y_q, y_r, be the other end-vertex of Π_1 and Π_2, respectively. By contracting $(\Pi_1\cup\Pi_2\cup y)-x_6$ and either (if Π_4 has x_4 as end-vertex)

$$\Pi_4 \cup (\Gamma(y_2,\ldots,y_6)-y_q-y_r)$$

or (if Π_4 has x_2 as end-vertex)

$$\Pi_4 \cup (\Gamma(y_2,\ldots,y_6)-y_q-y_r) \cup (\Pi_3-x_5)$$

into one vertex, $\Gamma-x$ is contracted into a graph containing a $\langle 6-\rangle$ as a subgraph, all the vertices of which are joined to x, hence $\Gamma\succ\langle 7-\rangle$.

iv) *suppose* $\alpha=0$. Γ is 5-fold connected, hence by (B) contains 5 disjoint $(\Gamma_6)(\Gamma_6')$-paths Π_1, Π_2, Π_3, Π_4, Π_5, not containing x and y. It may be assumed that Π_i has x_i as end-vertex for $i=1,\ldots 5$. Let y_q, y_r be the other end-vertices of Π_1 and Π_4, respectively. By contracting each of $(\Pi_1\cup\Pi_4\cup y)-x_4$ and

$$\Pi_2 \cup (\Gamma(y_1,y_2,\ldots,y_6)-y_q-y_r) \cup (\Pi_5-x_5)$$

into one vertex, $\Gamma-x$ is contracted into a graph containing a $\langle 6-\rangle$ as a subgraph, all the vertices of which are joined to x, hence $\Gamma\succ\langle 7-\rangle$.

This completes the proof of the fact that also under the assumption (4.8), $\Gamma\succ\langle 7-\rangle$. All possibilities have now been exhausted and (4) is proved.

(3) and (4) together imply that $\Gamma\succ\langle 7-\rangle$.

This completes the proof of Theorem 4.

7. General problems and results.

In this section we will try to estimate the conditions on the number of edges required for a graph to be homomorphic to a complete or almost complete graph in a way similar to but not identical with that in [4].

Let n and v be integers such that $n\geqq v\geqq 4$. For $i=0,1,2$, $f_i(v,n)$ is defined as the least natural number such that every graph with n vertices and $\geqq f_i(v,n)$ edges is homomorphic to a complete v-graph with exactly i edges deleted.

Clearly $f_i(v,n)$ exists for all v and n, $n\geqq v\geqq 4$, because

$$f_2(v,n) \leqq f_1(v,n) \leqq f_0(v,n) \leqq \tfrac{1}{2}n(n-1)\,.$$

Having proved the theorems stated as theorem α and β in section 2 of I, Dirac ([2]) asked two general questions which can be re-formulated like this:

QUESTION 1. Is it true that for $n \geqq v \geqq 4$,

$$f_1(v,n) \leqq (v-\tfrac{5}{2})n-\tfrac{1}{2}(v-1)(v-3)+1$$

and the only graphs with exactly $(v-\frac{5}{2})n-\frac{1}{2}(v-1)(v-3)$ edges not being homomorphic to a $\langle v-\rangle$ are the members of $\mathscr{K}_{v-1}^{v-3}$?

QUESTION 2. Is it true that for $n \geqq v \geqq 4$,

$$f_2(v,n) \leqq (v-3)n-\tfrac{1}{2}(v-1)(v-4)+1$$

and the only graphs with exactly $(v-3)n-\frac{1}{2}(v-1)(v-4)$ edges not being homomorphic to a $\langle v=\rangle$ are the members of $\mathscr{K}_{v-1}^{v-4}$?

Theorems α and β and a remark about $v=4$ just after β, show that the answers to questions 1 and 2 are *yes* for $4 \leqq v \leqq 6$.

A similar question may be asked for $f_0(v,n)$. Theorem γ and the remark just following it show that

$$f_0(v,n) = (v-2)n-\tfrac{1}{2}v(v-3) \quad \text{for } 3 \leqq v \leqq 7 .$$

However, Mader ([4]) pointed out something that is equivalent to the fact that

$$f_0(v,n) \geqq (v-2)n-\tfrac{1}{2}v(v-3)+1$$

for all $v \geqq 8$ and at least one value of $n \geqq v$. Theorem 5 will provide further information about this.

The purpose of this section is to show that for all $v \geqq 9$ and infinitely many values of n for each v, the answers to the two questions are in the negative and to establish some lower bounds for $f_i(v,n)$.

Let p be a fixed natural number.

For $p \geqq 3$ and $v \geqq 3p-1$, $X_0^{p,v}$ denotes a $\langle v+p-1\rangle$ with $2p-1$ independent edges deleted.

For $p \geqq 2$ and $v \geqq 3p+1$, $X_1^{p,v}$ denotes a $\langle v+p-1\rangle$ with $2p$ independent edges deleted.

For $p \geqq 2$ and $v \geqq 3p+3$, $X_2^{p,v}$ denotes a $\langle v+p-1\rangle$ with $2p+1$ independent edges deleted.

Observe that $X_0^{p,v} = X_2^{p-1,v+1}$. The complete graph with the greatest possible number of vertices contained in a $X_0^{p,v}$ as a subgraph is a $\langle v-p\rangle$, in a $X_1^{p,v}$ a $\langle v-p-1\rangle$, in a $X_2^{p,v}$ a $\langle v-p-2\rangle$.

Consider now for fixed p and v, $\mathscr{K}^{v-p-1}(X_1^{p,v})$ and $\mathscr{K}^{v-p-2}(X_2^{p,v})$.

LEMMA 5. *Let v and p be fixed natural numbers.*

A. $p \geqq 2$, $v \geqq 3p+1$. *Let $K \in \mathscr{K}^{v-p-1}(X_1^{p,v})$ and let $\varkappa$ be the number of $X_1^{p,v}$'s of which K is composed. Then:*

i) $n(K) = (\nu - p - 1) + 2p\varkappa$,
ii) $e(K) = (\nu - \frac{5}{2})n(K) - \frac{1}{2}(\nu - 1)(\nu - 3) + \frac{1}{2}p(p-2)$,
iii) $K \not\succ \langle \nu - \rangle$.

B. $p \geqq 2$, $\nu \geqq 3p+3$. *Let* $K \in \mathscr{K}^{\nu-p-2}(X_2^{p,\nu})$ *and let* $\varkappa$ *be the number of* $X_2^{p,\nu}$*'s of which* K *is composed. Then*:

i) $n(K) = (\nu - p - 2) + (2p+1)\varkappa$,
ii) $e(K) = (\nu - 3)n(K) - \frac{1}{2}(\nu - 1)(\nu - 4) + \frac{1}{2}p(p-1) - 1$,
iii) $K \not\succ \langle \nu = \rangle$.

PROOF. By induction over $\varkappa$.
$\varkappa = 1$: i) is trivially true in both A and B. In case A,

$$c(K) = \tfrac{1}{2}(\nu+p-1)(\nu+p-2) - 2p$$
$$= (\nu - \tfrac{5}{2})(\nu+p-1) - \tfrac{1}{2}(\nu-1)(\nu-3) + \tfrac{1}{2}p(p-2)\,,$$

in case B,

$$e(K) = \tfrac{1}{2}(\nu+p-1)(\nu+p-2) - 2p - 1$$
$$= (\nu-3)(\nu+p-1) - \tfrac{1}{2}(\nu-1)(\nu-4) + \tfrac{1}{2}p(p-1) - 1$$

hence ii) is true as well. To show that $K \not\succ \langle \nu - \rangle$ in case A, and $K \not\succ \langle \nu = \rangle$ in case B, observe that each contraction of an edge decreases the number of vertices by 1, hence K is contracted into a graph containing a $\langle \nu - \rangle$ through at most $p-1$ successive contractions of edges. After an arbitrary number $r \leqq p-1$ of succesive contractions of edges the resulting graph has $\geqq \nu$ vertices but has at least $2p - 2r \geqq 2$ edges missing in case A and at least $2p+1-2r \geqq 3$ missing in case B. Hence $K \not\succ \langle \nu - \rangle$ in case A, and $K \not\succ \langle \nu = \rangle$ in case B.

Assume then that the conclusion holds for cockades composed of fewer than $\varkappa$ $X_i^{p,\nu}$'s, $\varkappa \geqq 2$, $i=1$ in case A, $i=2$ in case B. Let K be composed of the $\varkappa$ $X_i^{p,\nu}$'s, $\varphi_1, \varphi_2, \ldots, \varphi_\varkappa$, sucessively.

In case A the cockade K' composed of $\varphi_1, \ldots, \varphi_{\varkappa-1}$, successively, is a member of $\mathscr{K}^{\nu-p-1}(X_1^{p,\nu})$. By the induction hypothesis

$$n(K') = (\nu - p - 1) + 2p(\varkappa - 1)\,,$$
$$e(K') = (\nu - \tfrac{5}{2})n(K') - \tfrac{1}{2}(\nu-1)(\nu-3) + \tfrac{1}{2}p(p-2)\,,$$

and $K' \not\succ \langle \nu - \rangle$. Hence:

$$n(K) = n(K') + 2p = (\nu - p - 1) + 2p\varkappa\,.$$
$$e(K) = e(K') + 2p(\nu - p - 1) + \tfrac{1}{2}\cdot 2p(2p-1) - 2p$$

$$\begin{aligned} &= (\nu-\tfrac{5}{2})(n(K')+2p)-\tfrac{1}{2}(\nu-1)(\nu-3)+\tfrac{1}{2}p(p-2) \\ &\quad -(\nu-\tfrac{5}{2})2p+2p(\nu-p-1)+2p(p-\tfrac{1}{2})-2p \\ &= (\nu-\tfrac{5}{2})n(K)-\tfrac{1}{2}(\nu-1)(\nu-3)+\tfrac{1}{2}p(p-2)\,. \end{aligned}$$

$K' \not\succ \langle \nu - \rangle$, $\varphi_\varkappa \not\succ \langle \nu - \rangle$, $K' \cap \varphi_\varkappa = \langle \nu-p-1 \rangle$, and $p \geqq 2$, hence clearly $K \not\succ \langle \nu - \rangle$.

In case B the cockade K' composed of $\varphi_1, \varphi_2, \ldots, \varphi_{\varkappa-1}$, successively, is a member of $\mathscr{K}^{\nu-p-2}(X_2^{p,\nu})$. By the induction hypothesis

$$\begin{aligned} n(K') &= (\nu-p-2)+(2p+1)(\varkappa-1)\,, \\ e(K') &= (\nu-3)n(K')-\tfrac{1}{2}(\nu-1)(\nu-4)+\tfrac{1}{2}p(p-1)-1 \end{aligned}$$

and $K' \not\succ \langle \nu = \rangle$. Hence:

$$\begin{aligned} n(K) &= n(K')+2p+1 = (\nu-p-2)+(2p+1)\varkappa\,. \\ e(K) &= e(K')+(2p+1)(\nu-p-2)+\tfrac{1}{2}(2p+1)\cdot 2p-(2p+1) \\ &= (\nu-3)(n(K')+2p+1)-\tfrac{1}{2}(\nu-1)(\nu-4)+\tfrac{1}{2}p(p-1)-1 \\ &\quad -(\nu-3)(2p+1)+(2p+1)(\nu-p-2)+(2p+1)p-(2p+1) \\ &= (\nu-3)n(K)-\tfrac{1}{2}(\nu-1)(\nu-4)+\tfrac{1}{2}p(p-1)-1\,. \end{aligned}$$

$K' \not\succ \langle \nu = \rangle$, $\varphi_\varkappa \not\succ \langle \nu = \rangle$, $K' \cap \varphi_\varkappa = \langle \nu-p-2 \rangle$, and $p \geqq 2$, hence clearly $K \not\succ \langle \nu = \rangle$.

Therefore the Lemma is true for cockades composed of $\varkappa$ $X_i^{p,\nu}$'s $(i=1,2)$ if it is true for cockades composed of $\varkappa-1$ $X_i^{p,\nu}$'s. It is true for cockades composed of a single $X_i^{p,\nu}$, therefore it is true generally.

From the fact that

$$\mathscr{K}^{\nu-p}(X_0^{p,\nu}) = \mathscr{K}^{(\nu+1)-(p-1)-2}(X_2^{p-1,\nu+1})$$

—as noted above $X_0^{p,\nu} = X_2^{p-1,\nu+1}$—and from Lemma 5 B, it follows immediately that i) and ii) of the following lemma is true:

LEMMA 6. *Let ν and p be fixed natural numbers such that $p \geqq 3$, $\nu \geqq 3p-1$. Let $K \in \mathscr{K}^{\nu-p}(X_0^{p,\nu})$ and let $\varkappa$ be the number of $X_0^{p,\nu}$'s of which K is composed. Then*:

i) $n(K) = \nu-p+(2p-1)\varkappa$,
ii) $e(K) = (\nu-2)n(K)-\tfrac{1}{2}\nu(\nu-3)+\tfrac{1}{2}(p-1)(p-2)-1$,
iii) $K \not\succ \langle \nu \rangle$.

PROOF OF iii). Assume first that $\varkappa=1$ and $K \succ \langle \nu \rangle$. Each contraction of an edge decreases the number of vertices by 1, hence K is contracted into a graph containing a $\langle v \rangle$ through at most $p-1$ successive contractions of edges. After an arbitrary number $r \leqq p-1$ of successive contractions of edges the resulting graph has $\geqq v$ vertices but has at least $2p-1-2r \geqq 1$ edges missing. Hence $K \not> \langle 5 \rangle$. By induction over $\varkappa$, iii) follows just as in the analogous cases of Lemma 5.

THEOREM 5. *Let $p, v, \varkappa$ be natural numbers.*

A. $f_0(v,n) \geqq (v-2)n - \frac{1}{2}v(v-3) + \frac{1}{2}(p-1)(p-2)$
for $p \geqq 3, \quad v \geqq 3p-1, \quad n = v-p+(2p-1)\varkappa, \quad \varkappa \geqq 1.$

B. $f_1(v,n) > (v-\frac{5}{2})n - \frac{1}{2}(v-1)(v-3) + \frac{1}{2}p(p-2)$
for $p \geqq 2, \quad v \geqq 3p+1, \quad n = v-p-1+2p\cdot\varkappa, \quad \varkappa \geqq 1.$

C. $f_2(v,n) \geqq (v-3)n - \frac{1}{2}(v-1)(v-4) + \frac{1}{2}p(p-1)$
for $p \geqq 2, \quad v \geqq 3p+3, \quad n = v-p-2+(2p+1)\varkappa, \quad \varkappa \geqq 1.$

PROOF. Follows from Lemma 5 and Lemma 6 and the facts that $f_i(v,n)$ is a natural number for each i and that for a natural number k, $\frac{1}{2}k(k-1)$ and $\frac{1}{2}k(k-3)$ are both integers.

Specific lower bounds for $f_i(v,n)$, $i=0,1,2$, may be obtained for each v and infinitely many n from Theorem 5, by choosing p

$$\text{for } v \equiv 0 \bmod 3 \text{ as } \frac{v}{3} \text{ in } A, \quad \frac{v-3}{3} \text{ in } B, \quad \frac{v-3}{3} \text{ in } C,$$

$$\text{for } v \equiv 1 \bmod 3 \text{ as } \frac{v-1}{3} \text{ in } A, \quad \frac{v-1}{3} \text{ in } B, \quad \frac{v-4}{3} \text{ in } C,$$

$$\text{for } v \equiv 2 \bmod 3 \text{ as } \frac{v+1}{3} \text{ in } A, \quad \frac{v-2}{3} \text{ in } B, \quad \frac{v-5}{3} \text{ in } C.$$

However I think it is unlikely that these lower bounds are best possible.

The consequences of Theorem 5 for Questions 1 and 2 will now be considered and the possibilities for establishing analogues of Theorems α and β for higher but still small values of v will be discussed. Part of the answer has, of course, already been provided by Theorems 2 and 4.

QUESTION 1. From Theorem 5.B it follows that for $p \geqq 3$ and corresponding to each $v \geqq 10$ for infinitely many n:

$$f_1(v,n) > (v-\tfrac{5}{2})n - \tfrac{1}{2}(v-1)(v-3) + \tfrac{3}{2},$$

hence the answer is *no* for $v \geqq 10$.

As stated above the answer is *yes* for $4 \leqq v \leqq 6$.

It remains to consider $v=7$, 8, and 9.

By Lemma 5.A for $p=2$ and $v \geqq 7$ a member K of $\mathscr{K}^{v-3}(X_1^{2,v})$ is not homomorphic to a $\langle v- \rangle$ and has $(v-\frac{5}{2})n(K)-\frac{1}{2}(v-1)(v-3)$ edges;

$$\mathscr{K}^{v-3}(X_1^{2,v}) \cap \mathscr{K}_{v-1}^{v-3} = \varnothing ,$$

because a member of $\mathscr{K}_{v-1}^{v-3}$ always has vertices of valency $v-2$ while each vertex of a member of $\mathscr{K}^{v-3}(X_1^{2,v})$ has valency $\geqq v-1$. Hence K is not a member of $\mathscr{K}_{v-1}^{v-3}$ and consequently for $v=7$, 8 and 9 the answer to question 1 is *no*.

For $v=7$, however, it follows from Theorem 4 that the answer changes to *yes*, if $\mathscr{K}_{v-1}^{v-3}$ is replaced by $\mathscr{C}_{v-1}^{v-3}$, $\mathscr{C}_{v-1}^{v-3}$ being an extension of both $\mathscr{K}^{v-3}(X_1^{2,v})$ and $\mathscr{K}_{v-1}^{v-3}$, because

$$\mathscr{C}_{v-1}^{v-3} = \mathscr{K}^{v-3}(\langle v-1 \rangle, X_1^{2,v}) .$$

The case $v=8$ is not dealt with in this paper but it might be conjectured that, analogously to the case $v=7$, the answer is yes, if $\mathscr{K}_{v-1}^{v-3}$ is replaced by $\mathscr{C}_{v-1}^{v-3}$. If true, the proof is likely to present a good deal of difficulty as indicated e.g. by the fact that Lemma 3.C ceases to be true for $v=8$.

In the case $v=9$ not even a conjecture analogous to that of case $v=8$ can hold. Consider for example the graph Λ_1, where $\bar{\Lambda}_1$ is the disjoint union of a $\langle 3 \rangle$ and 4 $\langle 2 \rangle$'s.

$$n(\Lambda_1) = 11 ,$$

$$e(\Lambda_1) = \tfrac{1}{2}\cdot 11 \cdot 10 - 7 = 48 > \tfrac{95}{2} = (9-\tfrac{5}{2})\cdot 11 - \tfrac{1}{2}(9-1)(9-3) .$$

$$\Lambda_1 \not\succ \langle 9- \rangle \quad \text{and} \quad \Lambda_1 \notin \mathscr{C}_{v-1}^{v-3} .$$

QUESTION 2. From Theorem 5.C it follows that for $p \geqq 3$ and $v \geqq 12$ and infinitely many n for each v:

$$f_2(v,n) \geqq (v-3)n - \tfrac{1}{2}(v-1)(v-4)+3 ,$$

hence the answer is *no* for $v \geqq 12$.

As stated above the answer is *yes* for $4 \leqq v \leqq 6$.

It remains to consider the cases $v=7$, 8, 9, 10, 11.

By Theorem 2 the answer is *yes* for $v=7$ and 8.

By Lemma 4.B for $p=2$ and $v \geqq 9$ a member K of $\mathscr{K}^{v-4}(X_2^{2,v})$ is not homomorphic to a $\langle v= \rangle$ and has $(v-3)n(K)-\frac{1}{2}(v-1)(v-4)$ edges;

$$\mathscr{K}^{\nu-4}(X_2^{2,\nu}) \cap \mathscr{K}_{\nu-1}^{\nu-4} = \emptyset ,$$

because a member of $\mathscr{K}_{\nu-1}^{\nu-4}$ always has vertices of valency $\nu-2$ while each vertex of a member of $\mathscr{K}^{\nu-4}(X_2^{2,\nu})$ has valency $\geqq \nu-1$. Hence $K \notin \mathscr{K}_{\nu-1}^{\nu-4}$ and consequently for $\nu=9$, 10, 11 the answer to question 2 is *no*.

For the cases $\nu=9$ and 10 it might, however, be conjectured that analogously to the case $\nu=7$ for Question 1, the answer changes to yes, if $\mathscr{K}_{\nu-1}^{\nu-4}$ is replaced by $\mathscr{K}^{\nu-4}(\langle \nu-1\rangle, X_2^{2,\nu})$.

In the case $\nu=11$ such a conjecture does not hold. Consider for example the graph Λ_2, where $\bar{\Lambda}_2$ is the disjoint union of a $\langle 3\rangle$ and 5 $\langle 2\rangle$'s.

$$n(\Lambda_2) = 13 ,$$

$$e(\Lambda_2) = \tfrac{1}{2}\cdot 13\cdot 12-8 = 70 > 69 = (11-3)\cdot 13-\tfrac{1}{2}(11-1)(11-4) .$$

$$\Lambda_2 \not\succ \langle 11= \rangle \quad \text{and} \quad \Lambda_2 \notin \mathscr{K}^{\nu-4}(\langle \nu-1\rangle, X_2^{2,\nu}) .$$

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