LINEAR-TOPOLOGICAL CLASSIFICATION
OF MATROID C*-ALGEBRAS

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Abstract.

We classify, up to a linear-topological isomorphism, all matroid C*-algebras
(i.e. direct limits of a sequence of finite dimensional matrix algebras). There are
two isomorphism classes: one is represented by LC(l_2), the C*-algebra of all
compact operators on the Hilbert space l_2, and the other – by the Fermion
algebra F = \bigotimes_{n=1}^{\infty} M_2. In particular, any UHF algebra is isomorphic to F as a
Banach space. We also show that LC(l_2) is isometric to a 1-complemented
subspace of F, but F is not isomorphic to a subspace of a quotient space of
LC(l_2).

1. Introduction.

Let M_n denote the C*-algebra of all complex n \times n-matrices with the usual
algebraic operations and norms. A C*-algebra A is called a matroid C*-algebra
(or, briefly, a matroid) if there exists a sequence \{A_k\}_{k=1}^{\infty} of C*-subalgebras of
A, possibly with different units, so that:

(i) A_k \supseteq A_{k+1}; \quad k = 1, 2, 3, \ldots

(ii) A_k is C*-isomorphic to M_{n(k)} for some positive integer n(k);
    \quad k = 1, 2, 3, \ldots

(iii) \bigcup_{k=1}^{\infty} A_k is dense in A in the norm topology.

If, moreover,

(iv) A has a unit e and e \in A_k, \quad k = 1, 2, \ldots

then A is called a UHF algebra (i.e., uniformly hyper-finite algebra or, a Glimm
algebra, see [6, Chapter 6]). We call the sequence \{A_k\}_{k=1}^{\infty} an admissible
sequence for the matroid A.

The classification of matroids up to a C*-isomorphism is due to Glimm [2]
in the UHF case) and Dixmier [1] (in the general case). Glimm proved that if
\{A_k\}_{k=1}^{\infty} and \{B_k\}_{k=1}^{\infty} are admissible sequences for the UHF algebras A and B

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respectively, with $A_k$ C*-isomorphic to $M_{n(k)}$ and $B_k$ C*-isomorphic to $M_{m(k)}$, then $A$ is C*-isomorphic to $B$ if and only if
\[
\sup \{ j ; \exists k \left( p^j \mid n(k) \right) \} = \sup \{ j ; \exists k \left( p^j \mid m(k) \right) \}
\]
for every prime number $p$. In particular, there exist uncountably many non-C*-isomorphic UHF algebras. Dixmier constructed a "dimension function" $d_A$ on the set $E_A$ of projections of a general matroid $A$, and showed that two matroids $A$ and $B$ are C*-isomorphic if and only if $d_A(E_A) = d_B(E_B)$. He also showed how to compute $d_A(E_A)$ in terms of the dimensions $\{ n(k) \}_{k=1}^{\infty}$ of an arbitrary admissible sequence $\{ A_k \}_{k=1}^{\infty}$ for $A$. Another (easy) remark of Dixmier is that a unital matroid is, in fact, a UHF algebra (see [1, 1.2]).

Using the fact that two C*-algebra are linearly isometric if and only if they are Jordan-*-isomorphic (see [3] and [5]) one obtains easily that the isometric classification of matroids coincides with the Glimm-Dixmier classification as C*-algebras.

We are interested here in the linear-topological classification of matroids, i.e., in the classification up to a Banach-space isomorphism. Our main result is the following theorem, which shows a completely different phenomenon.

**Theorem 1.1.** (a) Every matroid C*-algebra is isomorphic either to LC($l_2$), the C*-algebra of all compact operators on $l_2$, or to the Fermion algebra $F = \bigotimes_{n=1}^{\infty} M_2$.

(b) LC($l_2$) is isometric to a subspace of $F$ which is the range of a contractive projection from $F$.

(c) $F$ is not isomorphic to a subspace of a quotient space of LC($l_2$).

The representation of LC($l_2$) as a matroid is quite obvious. Let $\{ a_k \}_{k=1}^{\infty}$ be an increasing sequence of finite-rank projections on $l_2$ tending strongly to $I$, the identity operator. Let $A_k = a_k \cdot LC(l_2) \cdot a_k$ and $n(k) = \text{rank} (a_k)$. Then $A_k$ is C*-isomorphic to $M_{n(k)}$, $A_k \supseteq A_{k+1}$, and $\bigcup_{k=1}^{\infty} A_k$ is norm-dense in LC($l_2$).

Assuming the notion of infinite tensor product of C*-algebras (see [7, Section 1.23], [8], and section 2 below) the representation of $F = \bigotimes_{n=1}^{\infty} M_2$ as a UHF algebra is also obvious. For $k=1,2,\ldots$ let
\[
A_k = \underbrace{M_2 \otimes M_2 \otimes \ldots \otimes M_2}_{k\text{-factors}}
\]
then $\{ A_k \}_{k=1}^{\infty}$ is (identified with) a strictly increasing sequence of unital C*-subalgebras of $F$, $A_k$ is C*-isomorphic to $M_{2^k}$, and $\bigcup_{k=1}^{\infty} A_k$ is norm-dense in $F$.

Theorem 1.1 answers questions raised by A. Lazar, and may be helpful in the
linear-topological classification of general AF-algebras. We thank Professor Lazar for valuable discussions.

Our methods are elementary and straightforward, and are independent of the delicate analysis of [1] and [2]. After replacing the above definition of matroids by the (equivalent) definition as a direct limit of matrix algebras, we analyze in a greater detail commutative diagrams of the form

$$
\begin{array}{ccc}
M_{n(1)} & \xrightarrow{f} & M_{m(2)} \\
\gamma_1 \downarrow & & \downarrow \gamma_2 \\
M_{m(1)} & \xrightarrow{g} & M_{m(2)}
\end{array}
$$

where $f$ and $g$ are C*-monomorphisms and $\gamma_1$ and $\gamma_2$ are linear isometries of a special kind. This analysis enables us to show that if $A, B$ are matroids with $B \cong LC(l_2)$, then $A$ is isometric to a 1-complemented subspace of $B$. Then we show that every matroid $A$ is isomorphic to $c_0(A)$. These two facts together easily imply parts (a) and (b) of Theorem 1.1. In order to prove part (c) we introduce the notion of the “diagonal” of a matroid which is always a 1-complemented, commutative C*-subalgebra, and show that the diagonal of $F$ is $C(\Delta)$, the algebra of all continuous function on the cantor set $\Delta$. A simple duality argument, together with the fact that $LC(l_2)^* = C_1$ (= the trace class) is separable, imply (c).

A word of caution about our terminology is necessary. Throughout the entire work we shall stay in the category of Banach spaces; so by “operator”, “isomorphism”, “isometry”, “projection”, etc. we shall always mean linear, continuous maps with the specified properties. The prefix “C*” will switch us to the category of C*-algebras, so “C*-homomorphism” “C*-monomorphism”, “C*-isomorphism”, etc. will mean linear, multiplicative, *-preserving, continuous maps. We do not require, however, that a C*-homomorphism from one unital C*-algebra into another preserves the unit element (also, a C*-subalgebra $B$ of a unital C*-algebra $A$ need not have a unit, and if it does — the units of $A$ and $B$ need not be the same). Except for this — our notation and terminology are quite standard, and we refer to [4] for Banach space theory and to [6] and [7] for C*-algebras.

2. Technical preparation

Let us start with some information on direct (or, inductive) limits of sequences of C*-algebras and infinite tensor products of matrix algebras. Our presentation is a variant of [7, Section 1.23] and [8]. Let $\{A_k\}_{k=1}^\infty$ be a sequence of C*-algebras so that for every $k$ there exists a C*-monomorphism (i.e., an injective C*-homomorphism) $f_k: A_k \to A_{k+1}$. We call $\{A_k, f_k\}_{k=1}^\infty$ a direct sequence. Let $\tilde{A}$ be the *-subalgebra of $\prod_{k=1}^\infty A_k$ consisting of all
\[ a = (a_k)_{k=1}^\infty \] so that \( a_{k+1} = f_k(a_k) \) for all \( k \geq k_0 \), normed by \( \| a \| = \sup_k \| a_k \| \), and let \( A \) be the completion of \( \bar{A} \).\( A \) is called the direct limit of the direct sequence \( \{ A_k, f_k \}_{k=1}^\infty \), in notation \( A = \lim_{\to} (A_k, f_k)_{k=1}^\infty \). For every \( n = 1, 2, \ldots \) the map \( A_n \to \lim_{\to} (A_k, f_k)_{k=1}^\infty \) defined by

\[ a \mapsto (0, 0, \ldots, 0, a, f_n(a), f_{n+1}(f_n(a)), \ldots) \]

is a C*-monomorphism which identifies \( A_n \) with a C*-subalgebra of \( \lim_{\to} (A_k, f_k)_{k=1}^\infty \). For simplicity, we shall regard \( A_n \) itself as being a C*-algebra of \( \lim_{\to} (A_k, f_k)_{k=1}^\infty \).

Suppose now that \( \{ v(j) \}_{j=1}^\infty \) is some sequence of positive integers. Let

\[ A_k = M_{v(1)} \otimes M_{v(2)} \otimes \cdots \otimes M_{v(k)}, \]

with the norm of \( B(l_2^{(1)} \otimes l_2^{(2)} \otimes \cdots \otimes l_2^{(k)}) \). Clearly, \( A_k \) is C*-isomorphic to \( M_{n(k)} \), where \( n(k) = v(1) \cdot v(2) \cdots v(k) \). Let \( f_k : A_k \to A_{k+1} \) be defined by

\[ f_k(x_1 \otimes x_2 \otimes \cdots \otimes x_k) = x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes I_{v(k+1)} \]

Then \( \{ A_k, f_k \}_{k=1}^\infty \) is direct sequence and its direct limit is called the infinite tensor product of \( \{ M_{v(j)} \}_{j=1}^\infty \), in notation

\[ \bigotimes_{j=1}^\infty M_{v(j)} = \lim_{\to} \left\{ \bigotimes_{j=1}^k M_{v(j)}, f_k \right\}_{k=1}^\infty. \]

Next, let us show that the notion of a matroid coincides with the notion of a direct limit of matrix algebras. Let \( \{ A_k \}_{k=1}^\infty \) be an admissible sequence for a matroid \( A \). Let \( \varphi_k : A_k \to M_{n(k)} \) be a C*-isomorphism of \( A_k \) onto \( M_{n(k)} \), and let \( f_k : M_{n(k)} \to M_{n(k+1)} \) be defined by \( f_k = \varphi_{k+1} \circ \varphi_k^{-1} \). Then \( \{ M_{n(k)}, f_k \}_{k=1}^\infty \) is a direct sequence and \( A \) is C*-isomorphic to \( \lim_{\to} \{ M_{n(k)}, f_k \}_{k=1}^\infty \). Using this identification one can easily prove the following.

**Proposition 2.1.** Let \( \{ M_{n(k)}, f_k \}_{k=1}^\infty \) and \( \{ M_{m(k)}, g_k \}_{k=1}^\infty \) be two direct sequences of matrix algebras. Suppose that for every \( k \) there exists an operator \( h_k : M_{n(k)} \to M_{m(k)} \), so that \( g_k \cdot h_k = h_{k+1} \cdot f_k \) for all \( k \), i.e., the following diagram commutes:

\[
\begin{array}{ccc}
M_{n(k)} & \xrightarrow{f_k} & M_{n(k+1)} \\
\downarrow{h_k} & & \downarrow{h_{k+1}} \\
M_{m(k)} & \xrightarrow{g_k} & M_{m(k+1)}
\end{array}
\]

Suppose also that \( \sup_k \| h_k \| < \infty \). Then there exists a unique operator

\[ h : \lim_{\to} \{ M_{n(k)}, f_k \}_{k=1}^\infty \to \lim_{\to} \{ M_{m(k)}, g_k \}_{k=1}^\infty \]

satisfying \( h(M_{n(k)}) = h_k \) for all \( k \) and \( \| h \| = \sup_k \| h_k \| \). Moreover, if all the \( h_k \) are
isometries (or, C*-homomorphisms) then h is an isometry (respectively, a C*-homomorphism).

The operator h whose existence is ensured by Proposition 2.1 is called the direct limit of the sequence \( \{ h_k \}_{k=1}^{\infty} \), and is denoted by \( h = \lim_{\rightarrow} h_k \). One can easily verify the following composition formula:

\[
\lim_{\rightarrow} (h_k \circ h'_k) = (\lim_{\rightarrow} h_k) \circ (\lim_{\rightarrow} h'_k).
\]

As a consequence, we have the following.

**Proposition 2.2.** Let \( \{ M_{n(k)} \}, \{ f_k \}_{k=1}^{\infty} \) and \( \{ M_{m(k)} \}, \{ g_k \}_{k=1}^{\infty} \) be two direct sequences. Suppose that for every k there exists an isometry \( h_k \) of \( M_{n(k)} \) into \( M_{m(k)} \) and a contraction \( p_k \) from \( M_{m(k)} \) into \( M_{n(k)} \) so that

(i) \( g_k \circ h_k = h_{k+1} \circ f_k \)

(ii) \( p_k \circ h_k = id_{M_{n(k)}} \), the identity operator on \( M_{n(k)} \)

and

(iii) \( f_k \circ p_k = p_{k+1} \circ g_k \)

that is, the following diagram commutes

\[
\begin{array}{ccc}
M_{n(k)} & \xrightarrow{id} & M_{n(k)} \\
p_k \downarrow & & \downarrow h_k \\
M_{m(k)} & \xrightarrow{g_k} & M_{m(k+1)} \\
h_{k+1} & & \downarrow p_{k+1} \\
M_{m(k)} & \xrightarrow{id} & M_{m(k+1)} \\
p_k \downarrow & & \downarrow h_{k+1} \\
M_{n(k)} & \xrightarrow{f_k} & M_{n(k+1)}
\end{array}
\]

Then

\[
A = \lim_{\rightarrow} \{ M_{n(k)} \}, \{ f_k \}_{k=1}^{\infty}
\]

is isometric to a subspace \( X \) of

\[
B = \lim_{\rightarrow} \{ M_{m(k)} \}, \{ g_k \}_{k=1}^{\infty},
\]

and there exists a contractive projection from \( B \) onto \( X \).

Indeed, \( h = \lim_{\rightarrow} k_k \) is an isometry from \( A \) into \( B \), \( p = \lim_{\rightarrow} p_k \) is a contraction from \( B \) into \( A \), and

\[
p \circ h = \lim_{\rightarrow} (p_k \circ h_k) = id_A,
\]

so \( h \circ p \) is a contractive projection from \( B \) onto \( X = h(A) \).
Definition 2.3. Let \( n, m, r \) be positive integers with \( rn \leq m \). We define a map \( \varphi_{n,m,r} : M_n \to M_m \) by

\[
(2.1) \quad \varphi_{n,m,r}(a) = \left( \bigoplus_{j=1}^{r} a \right) \oplus 0_l = \begin{bmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0_l \end{bmatrix}
\]

where \( l = m - nr \) and \( 0_l \) denotes the zero matrix of order \( l \times l \).

Clearly, \( \varphi_{n,m,r} \) is a \( C^* \)-monomorphism of \( M_n \) into \( M_m \).

Proposition 2.4. Let \( n \leq m \) and let \( \varphi : M_n \to M_m \) be a \( C^* \)-monomorphism. Then there exists a unitary matrix \( v \in M_m \) so that

\[
\varphi(a) = v^* \cdot (\varphi_{n,m,r}(a)) \cdot v, \quad a \in M_n
\]

where \( r = \text{rank } \varphi(e_{1,1}) \).

This is well-known (any \( C^* \)-monomorphism maps elements with orthogonal ranges (or, orthogonal cokernels) into elements with the same properties. Now apply this to the system of matrix-units \( \{e_{i,j}\}_{i,j=1}^n \) of \( M_n \).

Definition 2.5. Let \( \varphi : M_n \to M_m \) (\( n \leq m \)) be a \( C^* \)-monomorphism. We put

\[
r(\varphi) = \text{rank } \varphi(e_{1,1})
\]

It is clear that \( r(\varphi) = \text{rank } \varphi(e) \) for every rank-one projection \( e \in M_n \). The functional "\( r \)" is multiplicative: if \( M_n \xrightarrow{\varphi} M_m \) and \( M_m \xrightarrow{\psi} M_k \) are \( C^* \)-monomorphisms, then \( r(\psi \circ \varphi) = r(\psi) \cdot r(\varphi) \).

We state without proof the following elementary proposition.

Proposition 2.6. Let \( \{A_{k_j}, f_{k_j}\}_{k=1}^\infty \) be a direct sequence and let \( \{k_j\}_{j=1}^\infty \) be any increasing sequence of positive integers. Let for \( j = 1, 2, \ldots \)

\[
g_j = f_{k_{j+1}} \circ \cdots \circ f_{k_{j+1}} \circ f_{k_j} : A_{k_j} \to A_{k_{j+1}}
\]

Then \( \lim \{A_{k_j}, g_j\}_{j=1}^\infty \) is \( C^* \)-isomorphic to \( \lim \{A_{k_j}, f_{k_j}\}_{j=1}^\infty \).

In particular, if \( \{M_{n(k)}, f_{k_j}\}_{k=1}^\infty \) is any direct sequence of matrix algebras with \( r(f_k) = 1 \) for \( k \geq k_0 \), then \( \lim \{M_{n(k)}, f_{k_j}\}_{j=1}^\infty \) is \( C^* \)-isomorphic to

\[
\text{LC}(l_2) = \lim \{M_{k_j}, \varphi_{k,j+1,1}\}_{j=1}^\infty
\]
DEFINITION 2.7. Let $n \leq m$. We define $p_{m,n} : M_m \rightarrow M_n$ by $(p_{m,n}(a))(i,j) = a(i,j)$, $1 \leq i,j \leq n$, $a \in M_m$.

Clearly, $p_{m,n}$ is a contraction and $p_{m,n} \circ \varphi_{n,m,r} = \text{id}_{M_n}$ for all positive integers $n,m,r$ with $nr \leq m$.

DEFINITION 2.8. Two maps $f,g : M_n \rightarrow M_m$ are said to be equivalent if there exist unitary matrices $u_1,u_2 \in M_n$ and $v_1,V_2 \in M_m$ so that

$$f(a) = v_2(g(u_2au_1))v_1, \quad a \in M_n.$$  

DEFINITION 2.9. For positive integers $n \leq m$ let $\Gamma_{n,m}$ be the set of all linear maps $\gamma : M_n \rightarrow M_m$ that are equivalent to a map $\tilde{\gamma} : M_n \rightarrow M_m$ of the form

$$(2.2) \quad \tilde{\gamma}(a) = a \oplus p_{n,n_1}(a) \oplus p_{n,n_2}(a) \oplus \ldots \oplus p_{n,n_s}(a) \oplus 0_l$$

where $1 \leq n_j < n$, $1 \leq l$, $0 \leq s$, and $n + \sum_{j=1}^{s} n_j + l = m$.

Notice that, up to a permutation, the sequence $\{n_j\}_{j=1}^{s}$ depends only on $\gamma$ (not on $\tilde{\gamma}$). Also, $\tilde{\gamma}(a^*) = \tilde{\gamma}(a)^*$ for all $a \in M_m$, and $\tilde{\gamma}$ is multiplicative if and only if $s = 0$, i.e., $\tilde{\gamma}(a) = a \oplus 0_l$.

PROPOSITION 2.10. Let $n \leq m$. Then every $\gamma \in \Gamma_{n,m}$ is an isometry and there is a contractive (i.e., norm-one) projection from $M_m$ onto $\gamma(M_n)$.

PROOF. It is clearly enough to prove this in the case where $\gamma = \tilde{\gamma}$ is given by (2.2). Now, for any matrices $a,b$

$$\|a \oplus b\| = \max \{\|a\|,\|b\|\}.$$  

So, using the fact that $p_{n,n_j}$ are contractions, we get

$$\|\tilde{\gamma}(a)\| = \max \{\|a\|,\|p_{n,n_1}(a)\|,\ldots,\|p_{n,n_s}(a)\|\} = \|a\|.$$  

Since $p_{m,n} \circ \gamma = \text{id}_{M_m}$, we get that $\gamma \circ p_{m,n}$ is a contractive projection from $M_m$ onto $\gamma(M_n)$.

The following Lemma is the heart of the proof of Theorem 1.1.

LEMMA 2.11. Let $f : M_{m(1)} \rightarrow M_{m(2)}$ and $g : M_{m(1)} \rightarrow M_{m(2)}$ be C*-monomorphism. Let $\nu = r(f)$, $\mu = r(g)$ and $\sigma = n(2) - n(1)\nu$, and suppose that $\mu = \prod_{j=1}^{\nu+1} \mu(j)$, where $\mu(j)$ are positive integers satisfying $\mu(1) \geq \nu$ and $\mu(j) \geq 3$ for all $j$. Assume also that $m(1) \geq n(1) + 1$ and let $\gamma_1 \in \Gamma_{n(1),m(1)}$. Then there exists a $\gamma_2 \in \Gamma_{n(2),m(2)}$ so that $\gamma_2 \circ f = g \circ \gamma_1$, i.e., the following diagram commutes.
\[
M_{n(1)} \xrightarrow{f} M_{n(2)} \\
\gamma_1 \downarrow \quad \downarrow \gamma_2 \\
M_{m(1)} \xrightarrow{g} M_{m(2)}
\]

**Proof.** By Proposition 2.4, there is no loss of generality in assuming that

\[ f = \varphi_{n(1), n(2), \nu}, \quad g = \varphi_{m(1), m(2), \mu} \]

and that for all \( a \in M_{n(1)} \nu \),

\[ \gamma_1(a) = a \oplus p_{n(1), k(1)}(a) \oplus \ldots \oplus p_{n(1), k(s)}(a) \oplus 0 \],

where \( 1 \leq k(j) < n(1) \), \( 0 \leq s \), and \( 1 \leq l \). Next, let us factor \( f \) as

\[ f = f_0 \circ \ldots \circ f_1 \circ f_0, \]

where

\[ f_0 = \varphi_{n(1), n(1) \nu, \nu} \]

and

\[ f_j = \varphi_{n(1) \nu + j - 1, n(1) \nu + j, 1}, \quad 1 \leq j \leq \sigma. \]

By our assumption on \( \mu = r(g) \) there is also a factorization \( g = g_{\sigma + 1} \circ \ldots \circ g_1 \circ g_0 \), where for \( 0 \leq j \leq \sigma \),

\[ g_j = \varphi_{m(1) \mu(j)!, \mu(1) \mu(j + 1)!, \mu(j + 1)} \]

(here \( \mu(j)! = \prod_{i=1}^{j} \mu(i) \), with the understanding that \( \mu(0)! = 1 \)) and

\[ g_{\sigma + 1} = \varphi_{m(1) \mu, m(2), 1}. \]

It is therefore enough to prove the existence of maps

\[ \gamma^{(j)} \in \Gamma_{n(1) \nu + j, m(1) \mu(j + 1)!}, \quad j = 0, 1, \ldots, \sigma \]

satisfying

\[ g_j \circ \gamma^{(j-1)} = \gamma^{(j)} \circ f_j, \quad j = 0, 1, \ldots, \sigma, \]

(where \( \gamma_1 = \gamma^{(-1)} \)). Indeed, using these \( \gamma^{(j)} \) we define \( \gamma_2 = g_{\sigma + 1} \circ \gamma^{(\sigma)} \). It is clear that \( \gamma_2 \in \Gamma_{n(2), m(2)} \) and that \( \gamma_2 \circ f = g \circ \gamma_1 \).

The following commutative diagram describes the factorizations of \( f \) and \( g \) and the maps \( \gamma^{(j)} \) (the broken lines describes the maps to be constructed):

\[
\begin{array}{cccccccc}
M_{n(1)} & \xrightarrow{f_0} & M_{n(1) \nu} & \xrightarrow{f_1} & M_{n(1) \nu + 1} & \xrightarrow{f_2} & M_{n(1) \nu + 2} & \xrightarrow{f_3} & \ldots & \xrightarrow{f_{\sigma}} & M_{n(2)} \\
\gamma_1 & \downarrow & \gamma^{(0)} & \downarrow & \gamma^{(1)} & \downarrow & \gamma^{(2)} & \downarrow & \gamma^{(\sigma)} & \Downarrow & \gamma_2 \\
M_{m(1)} & \xrightarrow{g_0} & M_{m(1) \mu(1)} & \xrightarrow{g_1} & M_{m(1) \mu(2)} & \xrightarrow{g_2} & M_{m(1) \mu(3)} & \xrightarrow{g_3} & \ldots & \xrightarrow{g_{\sigma + 1}} & M_{m(2)}
\end{array}
\]
Thus, it is enough to prove the lemma in the following special cases:

**CASE 1.** \( n(2) = n(1) \nu, \quad m(2) = m(1) \mu, \quad \nu \leq \mu, \) and
\[
f = \varphi_{n(1), n(1) \nu, \nu}, \quad g = \varphi_{m(1), m(1) \mu, \mu}.
\]

**CASE 2.** \( n(2) = n(1) + 1, \quad m(2) = m(1) \mu, \quad \mu \geq 3, \) and
\[
f = \varphi_{n(1), n(1) + 1, 1}, \quad g = \varphi_{m(1), m(1) \mu, \mu}.
\]

**PROOF OF THE LEMMA IN CASE 1.** Write (2.3) as
\[
(2.4) \quad \gamma_1(a) = a \oplus p(a) \oplus 0_l = \begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0_l \end{bmatrix}, \quad a \in M_{n(1)}.
\]

By our assumption,
\[
f(a) = \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & \cdots & a \end{bmatrix} = \underbrace{a \oplus a \oplus \cdots \oplus a}_{\nu \text{-terms}}, \quad a \in M_{n(1)}
\]

and
\[
g(b) = \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & \cdots & b \end{bmatrix} = \underbrace{b \oplus b \oplus \cdots \oplus b}_{\mu \text{-terms}}, \quad b \in M_{m(1)}.
\]

Define \( \gamma_2 : M_{n(2)} \to M_{m(2)} \) by
\[
\gamma_2 \begin{bmatrix} a_{1,1} & \cdots & a_{1, \nu} \\ \vdots & \ddots & \vdots \\ a_{v,1} & \cdots & a_{v, \nu} \end{bmatrix} =
\]
where the large blocks belong to $M_{m(1)}$, $a_{i,j} \in M_{n(1)}$ and $p(a_{1,1}) \in M_{m(1) - n(1) - 1}$.

It is clear that $\gamma_2$ is unitarily equivalent to the map $\tilde{\gamma} : M_{m(2)} \to M_{m(2)}$ defined by

$$
\gamma_2 \begin{bmatrix}
    a_{1,1} & \cdots & a_{1,v} \\
    \vdots & & \vdots \\
    a_{v,1} & \cdots & a_{v,v}
\end{bmatrix} = \begin{bmatrix}
    \bar{a}_{1,1} & \cdots & a_{1,v} \\
    \vdots & & \vdots \\
    a_{v,1} & \cdots & a_{v,v}
\end{bmatrix}
$$

$$
= \sum_{\mu \text{-terms}} \left( a_{1,1} \oplus \cdots \oplus a_{1,1} \right) \oplus \left( p(a_{1,1}) \oplus \cdots \oplus p(a_{1,1}) \right)
$$

So $\gamma_2 \in \Gamma_{m(2),m(2)}$. Also, for $a \in M_{m(1)}$
\[ \gamma_2(f(a)) = \gamma_2 \begin{bmatrix} a & 0 \\ a & \cdot \\ 0 & \cdot \\ a & \end{bmatrix} \]

\[ \text{\mu-terms} \]

\[ = \begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ = g(\gamma_1(a)). \]

This completes the proof of the lemma in Case 1.

Proof of the lemma in Case 2. In this case \( f: M_{n(1)} \rightarrow M_{n(1)+1} = M_{n(2)} \) is given by

\[ f \begin{bmatrix} a_{1,1} & \ldots & a_{1,n(1)} \\ \vdots & \ddots & \vdots \\ a_{n(1),1} & \ldots & a_{n(1),n(1)} \end{bmatrix} = \begin{bmatrix} a_{1,1} & \ldots & a_{1,n(1)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n(1),1} & \ldots & a_{n(1),n(1)} & 0 \\ 0 & \ldots & 0 & 0 \end{bmatrix} \]

while \( \gamma_1 \) and \( g \) are given by (2.4) and (2.5) respectively. Let every \( x = (x(i,j))_{i,j=1}^{n(2)} \in M_{n(2)} \) be written as

\[ x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

where

\[ a = \begin{bmatrix} x(1,1) & \ldots & x(1,n(1)) \\ \vdots & \ddots & \vdots \\ x(n(1),1) & \ldots & x(n(1),n(1)) \end{bmatrix} = P_{n(2),n(1)}(x) \in M_{n(1)} \]

\[ b = \begin{bmatrix} x(1,n(2)) \\ \vdots \\ x(n(1),n(2)) \end{bmatrix} \in M_{n(1),1} \]

\[ c = (x(n(2),1), \ldots, x(n(2),n(1))) \in M_{1,n(1)} \]

and

\[ d = x(n(2),n(2)) \in M_{1,1} \]
Using the fact that $\mu \geq 3$ we define a map $\gamma_2 : M_{n(2)} \to M_{m(2)}$ by

$$\gamma_2(x) = \gamma_2 \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) =$$

\[
\begin{array}{cccc}
\sigma & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
\end{array}
\]

$(\mu \times \mu$ block matrices from $M_{m(1)})$. Clearly, $\gamma_2 \in \Gamma_{n(2), m(2)}$. Also, for $a \in M_{n(1)}$ we have

$$\gamma_2(f(a)) = \gamma_2 \left( \begin{array}{c} a \\ 0 \\ 0 \end{array} \right)$$

$$= \left[ \begin{array}{ccc} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{array} \right] \oplus \cdots \oplus \left[ \begin{array}{ccc} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$(\mu$ blocks)

$$= \gamma_1(a) \oplus \cdots \oplus \gamma_1(a)$$

$(\mu$ blocks)

$$= g(\gamma_1(a))$$
This completes the proof of the Lemma in Case 2.

**LEMMA 2.12.** Let \( n \leq m \) and let \( \gamma = \tilde{\gamma} \) be given by (2.2). Let \( \beta : M_m \to M_n \) be a contraction satisfying \( \beta \circ \gamma = \text{id}_{M_n} \). Then \( \beta = p_{m,n} \).

**Proof.** Let \( \{e_{i,j}\}_{i,j=1}^m \) denote the matrix units of \( M_m \), and let \( k = \max_{1 \leq j \leq s} n_j \), where \( s \) and \( n_j \) appear in (2.2). Let us write for short
\[
\gamma(a) = a \oplus p(a), \quad a \in M_n
\]
instead of (2.2). Then for \( (i,j) \) with \( k < \max \{i,j\} \leq n \) we have \( p(e_{i,j}) = 0 \) and so
\[
e_{i,j} = \beta(\gamma(e_{i,j})) = \beta(e_{i,j}).
\]
If \( \max \{i,j\} \leq k \) and \( a = \beta(e_{i,j}) \) then for any \( (i_1,j_1) \neq (i,j) \) with \( \max \{i_1,j_1\} \leq n \) we have
\[
\|e_{i_1,j_1} + \lambda e_{i,j}\| \leq (1 + |\lambda|^2)^{\frac{1}{2}}, \quad |\lambda| \leq 1
\]
So, for all \( |\lambda| \leq 1 \),
\[
|1 + \lambda a(i_1,j_1)| \leq \|e_{i_1,j_1} + \lambda a\|
\]
\[
= \|\beta(\gamma(e_{i_1,j_1}) + \lambda e_{i,j})\|
\]
\[
\leq \|\gamma(e_{i_1,j_1}) + \lambda e_{i,j}\|
\]
\[
= \|e_{i_1,j_1} + \lambda e_{i,j}\| \leq (1 + |\lambda|^2)^{\frac{1}{2}}
\]
and thus \( a(i_1,j_1) = 0 \). It follows that \( \beta(e_{i,j}) = \lambda e_{i,j} \). Now
\[
\|\beta(e_{i,n} \oplus p(e_{i,j}))\| \leq \|e_{i,n} \oplus p(e_{i,j})\| = 1.
\]
But also
\[
\|\beta(e_{i,n} \oplus p(e_{i,j}))\| = \|e_{i,n} + (1 - \lambda_i,j) e_{i,j}\| = (1 + |1 - \lambda_i,j|^2)^{\frac{1}{2}}.
\]
So \( \lambda_i,j = 1 \) and \( \beta(e_{i,j}) = e_{i,j} \).

Finally, let \( \max \{i,j\} > n \), let \( a = \beta(e_{i,j}) \) and let \( \max \{i_1,j_1\} \leq n \). Then \( \|e_{i_1,j_1} + \lambda e_{i,j}\| \leq (1 + |\lambda|^2)^{\frac{1}{2}} \), but
\[
\|\beta(e_{i_1,j_1} + \lambda e_{i,j})\| = \|e_{i_1,j_1} + \lambda a\| \geq |1 + \lambda a(i_1,j_1)|
\]
for all \( \lambda \). This implies that \( a(i_1,j_1) = 0 \) for all \( i_1,j_1 \leq n \). So \( a = 0 \). This proves that \( \beta(e_{i,j}) = e_{i,j} \) if \( \max \{i,j\} \leq n \) and \( \beta(e_{i,j}) = 0 \) if \( \max \{i,j\} > n \). So \( \beta = p_{m,n} \).

**COROLLARY 2.13.** Any \( \gamma \in \Gamma_{n,m} \) \( (n \leq m) \) has a unique contractive left inverse, denoted \( \gamma^{(+)}. \) So
\[\gamma^{(+)} : M_m \to M_n, \quad \|\gamma^{(+)}\| = 1, \quad \gamma^{(+)} \circ \gamma = \text{id}_{M_n}.\]

**Corollary 2.14.** Under the assumptions of Lemma 2.11 we have \(f \circ \gamma_1^{(+)} = \gamma_2^{(+)} \circ g\), i.e., the following diagram commutes:

\[
\begin{array}{c}
M_{n(1)} \xleftarrow{\text{id}} M_{n(1)} \xrightarrow{f} M_{n(2)} \xleftarrow{\text{id}} M_{n(2)} \\
\gamma_1^{(+)} \downarrow \quad \gamma_2 \downarrow \quad \gamma_2^{(+)} \\
M_{m(1)} \xrightarrow{g} M_{m(2)}
\end{array}
\]

**Proof.** We present the proof in Case 1 of the Proof of Lemma 2.11; the proof in Case 2 is essentially the same, but the (obvious) formula for \(\gamma_2^{(+)}\) happens to be more complicated. We have

\[\gamma_1^{(+)}(b) = p_{m(1), n(1)}(b), \quad b \in M_{m(1)},\]

and for \(b = (b_{i,j})_{i,j=1}^{\mu} \in M_{m(2)}\) with \(b_{i,j} \in M_{m(1)}\),

\[\gamma_2^{(+)}(b) = \gamma_2^{(+)}((b_{i,j})_{i,j=1}^{\mu}) = (a_{i,j})_{i,j=1}^{\nu} \in M_{n(2)},\]

where

\[a_{i,j} = p_{m(1), n(1)}(b_{i,j}) \in M_{m(1)}.\]

So, if \(b \in M_{m(1)}\) and \(a = p_{m(1), n(1)}(b)\), then

\[\gamma_2^{(+)}(g(b)) = \gamma_2^{(+)} \begin{bmatrix} b & 0 \\ b & \ddots \\ 0 & & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ a & \ddots \\ 0 & \cdots & a \end{bmatrix} = f(a) = f(\gamma_1^{(+)}(b)).\]

3. The main results.

Let us start with the following

**Lemma 3.1.** Let \(\{M_{m(k)}, g_k\}_{k=1}^{\infty}\) be a direct sequence, and let the positive integers \(r(g_k), k = 1, 2, \ldots\), be defined by Definition 2.5. Assume that \(\lim_{k \to \infty} r(g_k) \geq 2\), and let \(\{M_{n(k)}, f_k\}_{k=1}^{\infty}\) be any other direct sequence. Then \(A = \lim_{k \to \infty} \{M_{m(k)}, f_k\}_{k=1}^{\infty}\) is isometric to a 1-complemented subspace of \(B = \lim_{k \to \infty} \{M_{m(k)}, g_k\}_{k=1}^{\infty}\).
PROOF. By the multiplicativity of the function $r$, we get

$$r(g_{k,j+1-1} \circ \ldots \circ g_{k,j+1} \circ g_{k,j}) = \prod_{k=k_j}^{k_{j+1}-1} r(g_k),$$

for any increasing sequence $\{k_j\}$ of positive integers. Using Proposition 2.6 we can assume without loss of generality that $m(k)$ and $r(g_k)$ are arbitrarily large. Precisely, if $v_k = r(f_k)$, $\mu_k = r(g_k)$, $\sigma_k = n(k+1) - n(k)v_k$, then we assume that

$$\mu_k = \prod_{j=1}^{q_k+1} \mu_k(j),$$

where $\mu_k(j)$ are positive integers satisfying $\mu_k(1) \geq v_k$ and $\mu_k(j) \geq 3$ for all $j$. We also assume that $m(1) > n(1)$.

We define $\gamma_1 \in \Gamma(n(1), m(1), 1)$ by $\gamma_1 = \varphi_{m(1)}\varphi_{m(1)}^{-1}$. Using Lemma 2.11 we construct inductively a sequence $\gamma_k \in \Gamma(n(k), m(k))$ so that $g_k \circ \gamma_k = \gamma_{k+1} \circ f_k$, $k = 1, 2, 3, \ldots$. Let

$$\gamma_k^{(+)} : M_{m(k)} \to M_{n(k)}$$

be the (unique) contractive left inverse of $\gamma_k$ (see Lemma 2.12 and Corollary 2.13), i.e., $\|\gamma_k^{(+)}\| = 1$ and

$$\gamma_k^{(+)} \circ \gamma_k = \text{id}_{M_{n(k)}}, \quad k = 1, 2, \ldots .$$

By Corollary 2.13 we have also $f_k \circ \gamma_k^{(+)} = \gamma_{k+1}^{(+)} \circ g_k$ for all $k$. Let

$$\gamma = \lim_{\gamma_k} : A \to B \quad \text{and} \quad \gamma^{(+)} = \lim_{\gamma_k^{(+)}} : B \to A.$$

By Proposition 2.2 $\gamma$ is an isometry of $A$ onto a subspace, say $X$, of $B$, $\|\gamma^{(+)}\| = 1$ and $\gamma^{(+)} \circ \gamma = \text{id}_A$. So $p = \gamma \circ \gamma^{(+)}$ is a contractive projection from $B$ onto $X = \gamma(A)$.

Let us concentrate now on the Fermion algebra

$$F = \bigotimes_{n=1}^{\infty} M_2^{(n)} = \lim_{\gamma} \{M_2^{(k)}, \varphi_{2^k}, 2^{k+1}, 2^{k+1}\}_{k=1}^{\infty},$$

where $M_2^{(n)}$ denotes the $n$th factor $M_2$ (for basic information see [5]). The canonical, normalized trace of $F$ is given by

$$\tau = \lim_{\gamma} \{2^{-k}, \text{trace}|_{M_k}\}_{k=1}^{\infty},$$

(see Proposition 2.1 and the discussion preceding it). The action of $\tau$ on an elementary tensor is

$$\tau \left( \bigotimes_{n=1}^{m} x_n \right) = \prod_{n=1}^{m} 2^{-1} \cdot (\text{trace} x_n), \quad x_n \in M_2^{(n)}.$$
Let also \( \{ e^{(n)}_{i,j} \}_{i,j=0}^1 \) denote the standard matrix units of \( M_2^{(n)} \).

**Proposition 3.2.** Let \( v_m = \bigotimes_{n=1}^m e^{(n)}_{1,1} \). Then for all \( x \in F \)

\[
\delta_{(1,1)}(x) = \lim_{m \to \infty} \tau(2^m v_m x)
\]

exists. \( \delta_{(1,1)} \) a norm-one linear functional on \( F \).

**Proof.** Let \( x \) be an elementary tensor from \( \bigotimes_{n=1}^k M_2^{(n)} \),

\[
x = \bigotimes_{n=1}^k x_n, \quad x_n \in M_2^{(n)}.
\]

Then for \( m \geq k \),

\[
\begin{align*}
\tau(2^m v_m x) &= \tau(2^m v_m x) \\
&= \prod_{n=1}^k 2^{m-1} \tau(e^{(n)}_{1,1} x_n) \\
&= 2^k \prod_{n=1}^k 2^{m-1} \tau(e^{(n)}_{1,1} x_n) = \tau(2^k v_k x).
\end{align*}
\]

This clearly implies that \( \delta_{(1,1)}(x) = \lim_{m \to \infty} \tau(2^m v_m x) \) exists for every \( x \) in the dense \(*\)-subalgebra \( \bigcup_{k=1}^\infty \bigotimes_{n=1}^k M_2^{(n)} \) of \( F \). Since each functional \( x \mapsto \tau(2^m v_m x) \) has norm one,

\[
\delta_{(1,1)}(x) = \lim_{m \to \infty} \tau(2^m v_m x)
\]

exists for every \( x \in F \), and \( \| \delta_{(1,1)} \| \leq 1 \). Finally, if \( 1 \) denotes the unit of \( F \), then

\[
\delta_{(1,1)}(1) = \lim_{m \to \infty} \tau(2^m v_m 1) = \lim_{m \to \infty} \tau(2^m v_m) = 1.
\]

So \( \| \delta_{(1,1)} \| = 1 \).

**Remark.** \( \delta_{(1,1)} \) correspond to "point-evaluation at \((1,1)\)". If \( 0 \leq s, t \leq 1 \) are given by

\[
s = \sum_{i=1}^\infty s_i 2^{-i} \quad \text{and} \quad t = \sum_{i=1}^\infty t_i 2^{-i}
\]

(where \( s_i, t_i \in \{0, 1\} \) and \( \sum_{i=1}^\infty s_i = \sum_{i=1}^\infty t_i = \infty \), we define

\[
\delta_{(s,t)}^{(m)}(x) = \tau \left[ 2^m \left( \bigotimes_{i=1}^m e^{(i)}_{s_i,t_i} \right) x \right].
\]

Then

\[
\delta_{(s,t)}(x) = \lim_{m \to \infty} \delta_{(s,t)}^{(m)}(x)
\]

. 
exists for all \( x \in F \). \( \delta_{(n,i)} \) is a norm-one functional which corresponds to "point-evaluation at \((s,t)\)". This exhibits \( F \) as a space of functions on the unit square \([0,1] \times [0,1]\) (which, however, is very different from the classical function spaces).

For any Banach space \( X \) we denote by \( c_0(X) \) the space of all sequences \( x = (x_1, x_2, \ldots) \) with \( x_j \in X \) and \( \|x_j\| \to 0 \), normed by \( \|x\| = \sup \|x_j\| \). If \( X \) is a C*-algebra, then \( c_0(X) \) is also C*-algebra.

**Lemma 3.3** The Fermion algebra \( F = \bigotimes_{n=1}^{\infty} M_2^{(n)} \) has a C*-subalgebra \( A \) which is C*-isomorphic to \( c_0(F) \), and there is a projection \( P \) from \( F \) onto \( A \) with \( \|P\| \leq 2 \).

**Proof.** Let \( \delta_{(1,1)} \) and \( v_m \) have the same meaning as in Proposition 3.2. Let \( F_0 = \ker \delta_{(1,1)} \) and let \( Q : F \to F_0 \) be given by \( Qx = x - \delta_{(1,1)}(x)1 \). Then \( Q \) is a projection of norm 2. Define for \( j = 1, 2, \ldots \)

\[
a_j = \left( \bigotimes_{i=1}^{j-1} e_{i,1}^{(i)} \right) \otimes e_{0,0}^{(j)}.
\]

Then \( \{a_j\} \) are mutually orthogonal projections. Also

\[
a_j F a_j = a_j F_0 a_j = a_j \bigotimes_{n=j+1}^{\infty} M_2^{(n)}, \quad j = 1, 2, \ldots .
\]

So \( a_j F a_j \) is C*-isomorphic in the natural way to \( F \). Let \( A = \overline{\text{span} \{a_j F a_j\}_{j=1}^{\infty}} \). Then \( A \) is a C*-subalgebra of \( F \) which is C*-isomorphic to \( c_0(F) \). We now claim that \( A \subset F_0 \) and that

\[
\tilde{P}(x) = \sum_{j=1}^{\infty} a_j x a_j
\]

defines a contractive projection from \( F_0 \) onto \( A \). Proving this, we complete the proof of the lemma by letting \( P = \tilde{P} Q \).

Indeed, for all \( j \) and \( m \)

\[
(3.1) \quad v_m a_j = a_j v_m = \begin{cases} 0 & ; \ j \leq m \\ a_j & ; \ j > m \end{cases}.
\]

This implies that for all \( x \in F \),

\[
\delta_{(1,1)}(a_j x a_j) = \lim_{m \to \infty} \tau(2^m v_m a_j x a_j) = 0
\]

and so \( A \subset F_0 = \ker \delta_{(1,1)} \). Next, let us define
\[ P_m(x) = \sum_{j=1}^{m} a_j x a_j + \tau(2^m v_m x) v_m, \quad x \in F, \quad m = 1, 2, \ldots. \]

By (3.1), \( P_m^2 = P_m \) and \( \|P_m\| = 1 \). If \( x \in \bigotimes_{k=1}^{k} M_2^{(n)} \), then for all \( m \geq k \),

\[ a_m x a_m = \tau(2^k v_k x) a_m, \quad \tau(2^m v_m x) = \tau(2^k v_k x) \]

and also

\[ \sum_{j=k+1}^{m} a_j + v_m = v_k \]

(the last formula follows by an easy induction on \( m \geq k \)). So

\[
P_m(x) = \sum_{j=1}^{m} a_j x a_j + \tau(2^m v_m x) v_m \\
= \sum_{j=1}^{k} a_j x a_j + \sum_{j=k+1}^{m} \tau(2^k v_k x) a_j + \tau(2^k v_k x) v_m \\
= \sum_{j=1}^{k} a_j x a_j + \tau(2^k v_k x) v_k \\
= P_k(x).
\]

This clearly implies that \( \bar{P}(x) = \lim_{m \to \infty} P_m(x) \) exists for all \( x \in F \), and that \( \bar{P} \) is a contractive projection. If \( x \in F_0 \) then \( \tau(2^m v_m x) \to \delta_{(1,1)}(x) = 0 \), and so

\[ \bar{P}(x) = \lim_{m \to \infty} \sum_{j=1}^{m} a_j x a_j = \sum_{j=1}^{\infty} a_j x a_j, \quad x \in F_0. \]

So \( \bar{P}(F_0) \subset A \). Finally, \( P_m(a_j x a_j) = a_j x a_j \) for \( m \geq j \), so \( \bar{P} |_A = \text{id}_A \).

For Banach spaces \( X, Y \) let \( X \cong Y \) (respectively, \( X \hookrightarrow Y \)) denotes that \( X \) is isomorphic to \( Y \) (respectively, to a complemented subspace of \( Y \)).

**Lemma 3.4.** Let \( A \) be any matroid C*-algebra. Then \( A \cong c_0(A) \).

**Proof.** It is enough to prove that \( c_0(A) \) is isometric to a complemented subspace of \( A \). Indeed, proving this, we get for some Banach space \( X \) that

\[ A \cong c_0(A) \oplus X \cong c_0(A) \oplus c_0(A) \oplus X \cong c_0(A) \oplus A \cong c_0(A). \]

If \( A = \text{LC}(l_2) \), let \( \{a_{jj}\}_{j=1}^{\infty} \) be a sequence of mutually orthogonal infinite-rank
projections on $l_2$. Then $P x = \sum_{j=1}^{\infty} a_j x a_j$ defines a contractive projection in $A$ and $P(A)$ is isometric to $c_0(A)$, since $a_j A a_j$ is C*-isomorphic to $A$.

If $A \neq LC(l_2)$ and $A = \varinjlim \{ M_{m(k), n(k)} g_k \}_{k=1}^{\infty}$ is some representation of $A$ as a direct limit of matrix algebras, then by Proposition 2.6, $\limsup_{k \to \infty} r(g_k) \geq 2$. So, by Lemma 3.1, $F \hookrightarrow A$. It follows by Lemma 3.3 that

$$c_0(A) \hookrightarrow c_0(F) \hookrightarrow F \hookrightarrow A,$$

where the isomorphisms are actually isometries. It follows that $c_0(A)$ is isometric to a 2-complemented subspace of $A$.

**Proof of part (a) of Theorem 1.1.** Let $A$ be any matroid C*-algebra, and let $A = \varinjlim \{ M_{m(k), n(k)} f_k \}_{k=1}^{\infty}$ be any representation of $A$ as a direct limit of matrix algebras. If $\limsup_{k \to \infty} r(f_k) = 1$, then by Proposition 2.6, $A$ is C*-isomorphic (and therefore linearly isometric) to $LC(l_2)$. If $\limsup_{k \to \infty} r(f_k) \geq 2$, then by Lemma 3.1, $A \cong F \oplus X$ and $F \cong A \oplus Y$ for some Banach spaces $X$ and $Y$. Also, by Lemma 3.4, $A \cong A \oplus A$ and $F \cong F \oplus F$. Thus

$$A \cong F \oplus X \cong F \oplus F \oplus X \cong F \oplus A$$

and similarly $F \cong A \oplus F$. So $A \cong F$.

This proof shows, in fact, that the isomorphism type of a matroid C*-algebra can be decided by the asymptotic behavior of the numbers $\{ r(f_k) \}_{k=1}^{\infty}$ (see Definition 2.5) in the representations $A = \varinjlim \{ M_{m(k), n(k)} f_k \}_{k=1}^{\infty}$. Precisely, we have the following:

**Corollary 3.5.** Let $A$ be a matroid C*-algebra,

(i) *If $\limsup_{k \to \infty} r(f_k) = 1$ for some representation $A = \varinjlim \{ M_{m(k), n(k)} f_k \}_{k=1}^{\infty}$, then this is the case for all other representations;*

(ii) *$A \cong LC(l_2)$ if and only if $\limsup_{k \to \infty} r(f_k) = 1$;

(iii) *$A \cong F$ if and only if $\limsup_{k \to \infty} r(f_k) \geq 2$.*

**Proof of part (b) of Theorem 1.1.** Let us apply Lemma 3.1 in the special case where $m(k) = 2^k$, $n(k) = k$, $g_k = \varphi_{2^k, 2^{k+1}, 2}$, and $f_k = \varphi_{k, k+1, 1}$. We have

$$A = \varinjlim \{ M_{k, \varphi_{k, k+1, 1}} \} = LC(l_2)$$

and

$$B = \varinjlim \{ M_{2^k, \varphi_{2^k, 2^{k+1}, 2}} \}_{k=1}^{\infty} = F.$$

So $LC(l_2)$ is isometric to a 1-complemented subspace of $F$. 
For the proof of part (c) of Theorem 1.1 we need the following Lemma. Here \( \Delta = \{0, 1\}^{\mathbb{N}_0} \) is the Cantor set and \( C(\Delta) \) denotes the C*-algebra of all complex-valued continuous functions on \( \Delta \) with the supremum norm.

**Lemma 3.6.** \( C(\Delta) \) is C*-isomorphic to a 1-complemented C*-subalgebra of the Fermion algebra \( F \).

**Proof.** Write \( F = \lim \{ M_{2^n}, f_k \}_{k=1}^\infty \) with \( f_k = \varphi_{2^{2k}, 2^{2k+1}, 2} \). Let \( D_k \) denote the diagonal projection in \( M_{2^n} \) (i.e., \( (D_k a)(i, j) = \delta_{i,j} a(i, i) \)). Then \( f_k \circ D_k = D_{k+1} \circ f_k \). So \( D = \lim D_k \) exists and is a contractive projection from \( F \) onto its C*-subalgebra \( A = \lim \{ D_k M_{2^n}, f_k \}_{k=1}^\infty \). Let

\[
\psi_k : C(\{0, 1\}^k) \to C(\{0, 1\}^{k+1})
\]

be the natural map:

\[
(\psi_k g)(t_1, \ldots, t_{k+1}) = g(t_1, \ldots, t_k); \quad t_j \in \{0, 1\}.
\]

Then there exist C*-isomorphisms \( h_k \) from \( C(\{0, 1\}^k) \) onto \( D_k M_{2^n}, k = 1, 2, \ldots \), so that \( f_k \circ h_k = h_{k+1} \circ \psi_k \) for all \( k \). It follows by Proposition 2.2 that \( h = \lim h_k \) is a C*-isomorphism from

\[
B = \lim \{ C(\{0, 1\}^k), \psi_k \}_{k=1}^\infty
\]

onto \( A \). Finally, \( B \) is C*-isomorphic to \( C(\Delta) \). Indeed, if \( u = (u_k), u_k \in C(\{0, 1\}^k) \), is so that \( \psi_k(u_k) = u_{k+1} \) for \( k > k_u \), let \( \varphi(u) = v \) be defined on \( \Delta \) by

\[
v(t_1, \ldots, t_j, \ldots) = u_k(t_1, \ldots, t_k); \quad t_j \in \{0, 1\},
\]

where \( k \geq k_u \). Clearly, \( v \) is well defined and \( \varphi \) extends to a unital C*-isomorphism of \( B \) onto \( C(\Delta) \).

**Proof of part (c) of Theorem 1.1.** Assume the converse, i.e., that \( F \) is isomorphic to a subquotient (that is, a subspace of a quotient space) of \( LC(l_2) \). By Lemma 3.6, \( C(\Delta) \) is also isomorphic to a subquotient of \( LC(l_2) \). By standard arguments, this implies that \( C(\Delta)^* \) is isomorphic to a subquotient of \( LC(l_2)^* = C_1 \) (= the trace class), which is separable. This contradicts the well-known fact that \( C(\Delta)^* \) is not separable.

The construction in Lemma 3.6 can be generalized to an arbitrary matroid C*-algebra \( A \), as follows. Let

\[
A = \lim \{ M_{n(j)}, f_j \}_{j=1}^\infty, \quad f_j = \varphi_{n(j), n(j+1), r(j)}, \quad (r(j) = r(f_j))
\]

be any representation of \( A \) as a direct limit of matrix algebras. Then
\[ B = \lim_{\rightarrow} \{ D_{n(j)} M_{n(j)} : f_j | D_{n(j)} M_{n(j)} \}_{j=1}^{\infty} \]

is a commutative C*-subalgebra of \( A \), and \( D = \lim_{\rightarrow} D_{n(j)} \) is a contractive projection from \( A \) onto \( B \). We call \( B \) "the" diagonal of \( A \) and denote it by \( DA \).

We now establish the properties of \( DA \), and in particular its independence of the particular representation \( A = \lim_{\rightarrow} \{ M_{n(j)} : f_j \}_{j=1}^{\infty} \). Let \( K_j = \{ 1, 2, \ldots, n(j) \} \) be regarded as a discrete topological space and let

\[ \alpha_j : C(K_j) \to D_{n(j)} M_{n(j)} \]

be defined by

\[ \alpha_j(u) = \text{diag}(u(1), u(2), \ldots, u(n(j))), \quad u \in C(K_j). \]

Then \( DA \) is C*-isomorphic to \( \lim_{\rightarrow} \{ C(K_j), g_j \}_{j=1}^{\infty} \), where \( g_j = \alpha_{j+1}^{-1} f_j \circ \alpha_j \). In the UHF (i.e., unital) case there exist quotient maps \( g_j^* : K_{j+1} \to K_j \) so that

\[ u(g_j^*(i)) = (g_j(u))(i), \quad u \in C(K_j), \quad i \in K_{j+1}. \]

The inverse limit of the sequence

\[ \ldots \leftarrow K_j \xleftarrow{g_j^*} K_{j+1} \leftarrow \ldots, \]

namely

\[ K = \lim_{\leftarrow} \{ K_j, g_j^* \}_{j=1}^{\infty} \]

is homeomorphic to \( \prod_{j=1}^{\infty} K_{j+1}/K_j \), and thus to \( \Delta \). So \( \lim_{\rightarrow} \{ C(K_j), g_j \}_{j=1}^{\infty} \) is C*-isomorphic to \( C(K) \) and thus to \( C(\Delta) \).

In the non-unital case we let \( \tilde{K} = K_j \cup \{0\} = \{0, 1, 2, \ldots, n(j)\} \), and we identify \( C(K_j) \) with

\[ C^{(0)}(\tilde{K}_j) = \{ u \in C(\tilde{K}_j); \quad u(0) = 0 \}. \]

The inverse system of quotient maps is now

\[ \ldots \leftarrow \tilde{K}_j \xleftarrow{g_j^*} \tilde{K}_{j+1} \leftarrow \ldots, \]

where \( g_j^*(0) = 0 \) and \( u(g_j^*(i)) = (g_j(u))(i) \) for all \( i \in \tilde{K}_{j+1} \) and \( u \in C^{(0)}(\tilde{K}_j) \). Let

\[ \tilde{\Delta} = (0, 0, \ldots) \in \prod_{j=1}^{\infty} \tilde{K}_j \quad \text{and} \quad \tilde{K} = \lim_{\leftarrow} \{ K_j, g_j \}_{j=1}^{\infty}. \]

Then
\[ \lim_{\nu} \{ C(K_j), g_{jj} \}_{j=1}^\infty = \lim_{\nu} \{ C^{(0)}(\tilde{K}), g_{jj} \}_{j=1}^\infty = C^{(0)}(\tilde{K}) = \{ u \in C(\tilde{K}); u(\bar{0}) = 0 \} . \]

If \( A \cong \text{LC}(l_2) \), i.e., \( \lim \sup_{k \to \infty} r(f_k) = 1 \), then \( \tilde{K} \) is homeomorphic to a sequence converging to \( \bar{0} \). So \( DA \) is \( C^* \)-isomorphic to \( c_0 \), the space of all numerical sequences converging to zero with the "sup" norm. If \( \lim \sup_{k \to \infty} r(f_k) \geq 2 \) then \( \tilde{K} \) does not have isolated points. Thus, being zero-dimensional, compact and metrizable (as a closed subspace of \( \prod_{j=1}^\infty \tilde{K}_j \)), \( \tilde{K} \) is homeomorphic to \( A \). Thus, \( DA \) is \( C^* \)-isomorphic to
\[ C^{(0)}(A) = \{ u \in C(A); u(\bar{0}) = 0 \} = C_0(A \setminus \{ \bar{0} \}) . \]

It follows that \( DA \) is (linearly) isomorphic to \( C(A) \).

Let us summarize this discussion formally:

**Theorem 3.7.** Let \( A \) be a matroid \( C^* \)-algebra.

1. *Up to a \( C^* \)-isomorphism, the above definition of the diagonal \( DA \) of \( A \) is independent of the particular representation
   \[ A = \lim_{\nu} \{ M_{n(j)} f_{jj} \}_{j=1}^\infty ; \]

2. *If \( A \) is a UHF algebra, then \( DA \) is \( C^* \)-isomorphic to \( C(A) \);*

3. *If \( A \) is non-unital and \( A \cong \text{LC}(l_2) \), that is \( \lim \sup_{j \to \infty} r(f_j) \geq 2 \), then \( DA \) is \( C^* \)-isomorphic to \( C_0(A \setminus \{ \bar{0} \}) \), and thus linearly isomorphic to \( C(A) \);*

4. *If \( \lim \sup_{j \to \infty} r(f_j) = 1 \), that is \( A \cong \text{LC}(l_2) \), then \( DA \) is \( C^* \)-isomorphic to \( c_0 \);*

5. *In all cases, \( DA \) is a 1-complemented \( C^* \)-subalgebra of \( A \).*

We conclude the paper by suggesting the following problem:

**Problem.** Characterize, up to a linear-topological isomorphism, all AF-algebras (i.e., direct limits of finite dimensional \( C^* \)-algebras).

Our methods and results might be helpful in studying this general problem, since every finite-dimensional \( C^* \)-algebra is the finite direct sum of matrix algebras.
REFERENCES


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