MATH. SCAND. 51 (1982), 227–231

PROJECTIVE MODULES WITH KRULL DIMENSION

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In this note we consider projective modules with Krull dimension and if M is a right module with Krull dimension, then we denote the Krull dimension of M by $|M|$. We prove for a commutative ring A and a projective module M that if M has Krull dimension, then M is finitely generated. We also obtain a positive result for non-commutative rings.

By examples of Jategaonkar we know that projective modules over right noetherian right fully bounded rings can have all submodules projective and have arbitrary Krull dimension [4] and [3, Example 10.3]. We prove that if A is a left and right noetherian ring and P a projective module with Krull dimension having all submodules projective, then the Krull dimension of P is at most one.

We start with an easy result.

PROPOSITION 1. *Let P be a projective module with Krull dimension. If the ring A modulo the prime radical is left goldie, then P is finitely generated.*

PROOF. We let N denote the prime-radical of A. If we can prove that P/NP is finitely generated, then P is finitely generated by [6, Proposition 2.1]. Since $|P|$ exists, also $|P/NP|$ exists, and moreover P/NP is a projective A/N-module, thus without loss of generality we let A be a left goldie ring, which is semiprime and P a projective left A-module with Krull-dimension. Let Q denote the semisimple left quotient ring of A. Since P has Krull-dimension, P has finite uniform dimension. Thus P has a finitely generated essential submodule, P_0, it is easily seen that $Q \otimes_A P_0$ is an essential submodule of $Q \otimes_A P$, since Q is semisimple we conclude that $Q \otimes_A P$ is finitely generated. If we write P as a direct summand of a free module, then it follows that P is a direct summand of a finitely generated free module, so P itself is finitely generated and the result is proved.

Received April 28, 1981; in revised form December 18, 1981.

For commutative rings we don't have to assume any sort of chain conditions on the ring A/N. Before stating and proving the commutative result we recall a few results concerning the trace ideal of a projective module.

Let P be a projective module over a commutative ring, then the trace ideal of P is denoted by $t(P)$, $A/t(P)$ is a flat module, since all its localizations are 0 or A. Note that for all multiplicatively closed sets S, $t(P_s)=t(P)_s$. Because $A/t(P)$ is flat, $t(P)$ is a so called pure ideal in particular $t(P)$ has the property that for each element a there exists an a' in $t(P)$ such that $aa'=a$ and also for each finite set of elements $a_1,\ldots,a_m$ in $t(P)$ there exists an element a' in $t(P)$ such that $a_ja'=a_j$ for all j. It now follows that if $t(P)$ is not finitely generated there exists a strictly ascending chain of principal ideals $(a_1)\subsetneqq\ldots\subsetneqq(a_n)\subsetneqq\ldots$ in $t(P)$ such that $a_i=a_ia_{i+1}$ for all i, for a more detailed discussion see [5].

THEOREM 1. *Let R be a commutative ring and P a projective module with Krull dimension, then P is finitely generated.*

PROOF. Let I denote the annihilator of P, P is still projective as an R/I-module and with zero annihilator, thus without loss of generality we assume that Ann $(P)=0$. If $t(P)$ is a finitely generated ideal, then we get from our earlier discussion that $R/t(P)$ is flat and finitely presented, hence projective so $t(P)$ is generated by an idempotent, now since $P=t(P)P$ and P has zero annihilator this idempotent must be 1, hence we have an epimorphism from P^n to R, hence R has Krull-dimension. Since a semiprime ring with Krull-dimension is a semiprime goldie ring [3], we can use Proposition 1 to conclude that P must be finitely generated.

We do now assume that $t(P)$ is not finitely generated and we prove that P has infinite uniform dimension. We have a strictly ascending chain of principal left ideals in $t(P)$

$$(a_1)\subsetneqq\ldots\subsetneqq(a_n)\subsetneqq,$$

where $a_na_{n+1}=a_n$ for all n. From our assumptions we get $(a_{n+1}-a_n)P\neq 0$ for all n, we claim

$$(a_2-a_1)P+(a_5-a_4)P+\ldots+$$

is an infinite direct sum inside P. If not we have

$$0 = (a_2-a_1)p_1+(a_5-a_4)p_2+\ldots+(a_{2+3n}-a_{1+3n}-a_{1+3n})p_{n+1}\,.$$

We multiply this equation by a_3 and get $(a_2-a_1)p_1=0$, next multiply by a_6 and so on, thus the sum is direct. The proof of Theorem 1 is now completed.

Next we consider projective modules with Krull-dimension having all submodules projective.

THEOREM 2. *Suppose A is a left and right noetherian ring and P is a module with Krull dimension having all submodules projective, then $|P| \leqq 1$.*

PROOF. By Proposition 1, P must be finitely generated. Using [3, 1.1 (i)], and induction on the number of generators, we may reduce the proof to the case in which P is a cyclic left A-module, so P is isomorphic to Ae, where e is idempotent. Let us also notice that for modules Q_1 and Q_2 both having all submodules projective, then also all submodules of $Q_1 \oplus Q_2$ are projective. By noetherian induction we can assume the result for all proper factor rings. If $\operatorname{Ann}(P) \neq 0$, then every submodule of $P/\operatorname{Ann}(P)P$ is $A/\operatorname{Ann}(P)$-projective, hence $|P| \leqq 1$. Thus we assume without loss of generality that $\operatorname{Ann}(P)=0$.

We will prove that Ae as a right eAe-module is noetherian.

Suppose we are given an ascending chain of eAe submodules of Ae

$$I_1 \subseteqq \ldots \subseteqq I_k \subseteqq \ldots,$$

then

$$I_1A \subseteqq \ldots \subseteqq I_kA \subseteqq \ldots,$$

is an ascending chain of right ideals of A, hence it terminates. If we multiply each term in the last chain by e on the right hand side and note that for all k, $I_k = I_k e = I_k eAe = I_k Ae$, then our claim follows.

Let $T = eAe$ and write $P = p_1T + \ldots + p_nT$, then

$$\operatorname{Ann}_A(P) = \bigcap_{j=1}^{n} \operatorname{Ann}(p_j) = 0,$$

hence we have an embedding of A into

$$A/\operatorname{Ann}(p_1) \oplus \ldots \oplus A/\operatorname{Ann}(p_n),$$

which is isomorphic to $Ap_1 \oplus \ldots \oplus Ap_n$, which is a submodule of P^n. Combining this with our earlier remarks we get that A is left hereditary. It now follows from a result of Chatters [2, Theorem 2.2] A has Krull dimension at most one and hence $|P| \leqq 1$.

One might also notice that in general $A/\operatorname{Ann}(P)$ is hereditary.

The author will like to thank T. Lenagan for showing how to remove the assumption that A was left fully bounded from our first version of the paper. In fact the last part of the argument is due to him.

In case our ring was a commutative ring, then Proposition 1 follows immediately from a result by Bass [1], which says that a projective module over a commutative noetherian ring is a direct sum of finitely generated submodules. We do not know whether or not Bass's result also holds in the non-commutative case. Moreover in the commutative case we need no chain condition to prove Theorem 2, in fact one can easily prove the following:

PROPOSITION 2. *Let R be a commutative ring and P a projective module with Krull-dimension having all submodules projective. Then* $|P| \leqq 1$.

The following is also easy to prove.

PROPOSITION 3. *Let M be a right module with Krull-dimension and S a right Ore set. If M_S denotes the module of right quotients of M, then $|M_S|$ exists and* $|M_S| \leqq |M|$.

For commutative rings most "dimensions" can be computed as supremum over local ones. Clearly this is not true for Krull-dimension. But if our module has Krull-dimension, then we have a positive result.

THEOREM 3. *Let A be a commutative ring and M a module with Krull-dimension, then $|M| = \sup_m |M_m|$, where m runs through all maximal ideals.*

PROOF. By [7, corollary] we may assume that M is a cyclic module and consequently we can assume M is the ring. Now by [3, Corollary 7.5], we can take A to be an integral domain. We now use induction on $|A|$. Using [3, Proposition 6.1] and the induction we get for all ideals I $(I \neq 0)$

$$|A| = \sup_I \{|A/I| + 1\} \leqq \sup_{m,I} \{|A_m/I_m| + 1\} \leqq \sup_m |A_m| .$$

The result is now proved. The last argument could also be replaced by a reference to [3, Theorem 8.12].

One might also note that Proposition 2 is a corollary of Theorem 3.

The authors wishes to thank the referee for simplifying the proof of Theorem 3.

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