PROJECTIVE MODULES
WITH KRULL DIMENSION

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In this note we consider projective modules with Krull dimension and if $M$ is
a right module with Krull dimension, then we denote the Krull dimension of $M$
by $|M|$. We prove for a commutative ring $A$ and a projective module $M$ that if
$M$ has Krull dimension, then $M$ is finitely generated. We also obtain a positive
result for non-commutative rings.

By examples of Jategaonkar we know that projective modules over right
noetherian right fully bounded rings can have all submodules projective and
have arbitrary Krull dimension [4] and [3, Example 10.3]. We prove that if $A$
is a left and right noetherian ring and $P$ a projective module with Krull
dimension having all submodules projective, then the Krull dimension of $P$ is
at most one.

We start with an easy result.

**Proposition 1.** Let $P$ be a projective module with Krull dimension. If the ring
$A$ modulo the prime radical is left goldie, then $P$ is finitely generated.

**Proof.** We let $N$ denote the prime-radical of $A$. If we can prove that $P/NP$
is finitely generated, then $P$ is finitely generated by [6, Proposition 2.1]. Since $|P|$ exists, also $|P/NP|$ exists, and moreover $P/NP$ is a projective $A/N$-module, thus
without loss of generality we let $A$ be a left goldie ring, which is semiprime and
$P$ a projective left $A$-module with Krull-dimension. Let $Q$ denote the
semisimple left quotient ring of $A$. Since $P$ has Krull-dimension, $P$ has finite
uniform dimension. Thus $P$ has a finitely generated essential submodule, $P_0$,
it is easily seen that $Q \otimes_A P_0$ is an essential submodule of $Q \otimes_A P$, since $Q$ is
semisimple we conclude that $Q \otimes_A P$ is finitely generated. If we write $P$ as a
direct summand of a free module, then it follows that $P$ is a direct summand of
a finitely generated free module, so $P$ itself is finitely generated and the result is
proved.

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For commutative rings we don’t have to assume any sort of chain conditions on the ring $A/N$. Before stating and proving the commutative result we recall a few results concerning the trace ideal of a projective module.

Let $P$ be a projective module over a commutative ring, then the trace ideal of $P$ is denoted by $t(P)$, $A/t(P)$ is a flat module, since all its localizations are 0 or $A$. Note that for all multiplicatively closed sets $S$, $t(P_S) = t(P)_S$. Because $A/t(P)$ is flat, $t(P)$ is a so called pure ideal in particular $t(P)$ has the property that for each element $a$ there exists an $a'$ in $t(P)$ such that $aa' = a$ and also for each finite set of elements $a_1, \ldots, a_m$ in $t(P)$ there exists an element $a'$ in $t(P)$ such that $a_j a' = a_j$ for all $j$. It now follows that if $t(P)$ is not finitely generated there exists a strictly ascending chain of principal ideals $(a_1) \subseteq \ldots \subseteq (a_n) \subseteq \ldots \subseteq t(P)$ such that $a_i = a_i a_{i+1}$ for all $i$, for a more detailed discussion see [5].

**Theorem 1.** Let $R$ be a commutative ring and $P$ a projective module with Krull dimension, then $P$ is finitely generated.

**Proof.** Let $I$ denote the annihilator of $P$, $P$ is still projective as an $R/I$-module and with zero annihilator, thus without loss of generality we assume that $\text{Ann} (P) = 0$. If $t(P)$ is a finitely generated ideal, then we get from our earlier discussion that $R/t(P)$ is flat and finitely presented, hence projective so $t(P)$ is generated by an idempotent, now since $P = t(P)P$ and $P$ has zero annihilator this idempotent must be 1, hence we have an epimorphism from $P^n$ to $R$, hence $R$ has Krull-dimension. Since a semiprime ring with Krull-dimension is a semiprime goldie ring [3], we can use Proposition 1 to conclude that $P$ must be finitely generated.

We do now assume that $t(P)$ is not finitely generated and we prove that $P$ has infinite uniform dimension. We have a strictly ascending chain of principal left ideals in $t(P)$

$$(a_1) \subseteq \ldots \subseteq (a_n) \subseteq \ldots,$$

where $a_n a_{n+1} = a_n$ for all $n$. From our assumptions we get $(a_{n+1} - a_n)P \neq 0$ for all $n$, we claim

$$(a_2 - a_1)P + (a_3 - a_2)P + \ldots +$$

is an infinite direct sum inside $P$. If not we have

$$0 = (a_2 - a_1)p_1 + (a_3 - a_4)p_2 + \ldots + (a_{2+3n} - a_{1+3n} - a_{1+3n})p_{n+1}.$$

We multiply this equation by $a_3$ and get $(a_2 - a_1)p_1 = 0$, next multiply by $a_6$ and so on, thus the sum is direct. The proof of Theorem 1 is now completed.

Next we consider projective modules with Krull-dimension having all submodules projective.
Theorem 2. Suppose $A$ is a left and right noetherian ring and $P$ is a module with Krull dimension having all submodules projective, then $|P| \leq 1$.

Proof. By Proposition 1, $P$ must be finitely generated. Using [3, 1.1 (i)], and induction on the number of generators, we may reduce the proof to the case in which $P$ is a cyclic left $A$-module, so $P$ is isomorphic to $Ae$, where $e$ is idempotent. Let us also notice that for modules $Q_1$ and $Q_2$ both having all submodules projective, then also all submodules of $Q_1 \oplus Q_2$ are projective. By noetherian induction we can assume the result for all proper factor rings. If $\text{Ann}(P) \neq 0$, then every submodule of $P/\text{Ann}(P)P$ is $A/\text{Ann}(P)$-projective, hence $|P| \leq 1$. Thus we assume without loss of generality that $\text{Ann}(P) = 0$.

We will prove that $Ae$ as a right $eAe$-module is noetherian.

Suppose we are given an ascending chain of $eAe$ submodules of $Ae$

$$I_1 \subseteq \ldots \subseteq I_k \subseteq \ldots,$$

then

$$I_1A \subseteq \ldots \subseteq I_kA \subseteq \ldots,$$

is an ascending chain of right ideals of $A$, hence it terminates. If we multiply each term in the last chain by $e$ on the right hand side and note that for all $k$, $I_k = I_k e = I_k eAe = I_k Ae$, then our claim follows.

Let $T = eAe$ and write $P = p_1 T + \ldots + p_n T$, then

$$\text{Ann}_A(P) = \bigcap_{j=1}^{n} \text{Ann}(p_j) = 0,$$

hence we have an embedding of $A$ into

$$A/\text{Ann}(p_1) \oplus \ldots \oplus A/\text{Ann}(p_n),$$

which is isomorphic to $Ap_1 \oplus \ldots \oplus Ap_n$, which is a submodule of $P^n$. Combining this with our earlier remarks we get that $A$ is left hereditary. It now follows from a result of Chatters [2, Theorem 2.2] $A$ has Krull dimension at most one and hence $|P| \leq 1$.

One might also notice that in general $A/\text{Ann}(P)$ is hereditary.

The author will like to thank T. Lenagan for showing how to remove the assumption that $A$ was left fully bounded from our first version of the paper. In fact the last part of the argument is due to him.

In case our ring was a commutative ring, then Proposition 1 follows immediately from a result by Bass [1], which says that a projective module over a commutative noetherian ring is a direct sum of finitely generated submodules. We do not know whether or not Bass's result also holds in the
non-commutative case. Moreover in the commutative case we need no chain condition to prove Theorem 2, in fact one can easily prove the following:

**Proposition 2.** Let \( R \) be a commutative ring and \( P \) a projective module with Krull-dimension having all submodules projective. Then \( |P| \leq 1 \).

The following is also easy to prove.

**Proposition 3.** Let \( M \) be a right module with Krull-dimension and \( S \) a right Ore set. If \( M_S \) denotes the module of right quotients of \( M \), then \( |M_S| \) exists and \( |M_S| \leq |M| \).

For commutative rings most "dimensions" can be computed as supremum over local ones. Clearly this is not true for Krull-dimension. But if our module has Krull-dimension, then we have a positive result.

**Theorem 3.** Let \( A \) be a commutative ring and \( M \) a module with Krull-dimension, then \( |M| = \sup_m |M_m| \), where \( m \) runs through all maximal ideals.

**Proof.** By [7, corollary] we may assume that \( M \) is a cyclic module and consequently we can assume \( M \) is the ring. Now by [3, Corollary 7.5], we can take \( A \) to be an integral domain. We now use induction on \( |A| \). Using [3, Proposition 6.1] and the induction we get for all ideals \( I \) (\( I \neq 0 \))

\[
|A| = \sup_I \{|A/I| + 1\} \leq \sup_m \{|A_m/I_m| + 1\} \leq \sup_m |A_m|.
\]

The result is now proved. The last argument could also be replaced by a reference to [3, Theorem 8.12].

One might also note that Proposition 2 is a corollary of Theorem 3.

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**References**


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