ON THE TRANSLATION FUNCTORS 
FOR A SEMISIMPLE ALGEBRAIC GROUP

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Introduction.

Let $G$ be a simply connected semisimple algebraic group over an algebraically closed field of characteristic $p \neq 0$. Assume that the root system of $G$ is irreducible. In [11] J. C. Jantzen introduced the functors $T^n_\lambda$, called the translation functors in [2], between certain categories of rational $G$ modules. Here $\lambda$ and $\mu$ are weights in the closure of the bottom alcove. In [11] and [2], the effect of $T^n_\lambda$ on the simple modules was given under the assumption that $\lambda$ is inside the bottom alcove; this result was then used, for example, to prove the translation principle ([11, § 3], [2, 2.5]).

In this paper we extend the result on the simple modules to the cases, where $\lambda$ is in a facetApp and $\mu$ in its closure (Theorem 2.5). As corollaries we obtain generalizations of (a part of) the translation principle and [8, Theorem 2]; we also find some composition multiplicities of Weyl modules. Moreover, we derive analogous results for $u_n\cdot T$ modules (cf. [12]).

As an application of the translation functors we generalize [10, Satz 5] as follows. Let $C_n$ be the alcove with $(p^n-1)\varrho$ in its upper closure. Here $\varrho$ is the sum of the fundamental dominant weights. We show that if $\lambda$ lies in the closure of $C_n$ or in the alcove immediately below it, then the indecomposable projective $u_n$ module $Q(n,\lambda)$ can be lifted to a $G$ module.

1. Preliminaries.

Let $G$ be a simply connected semisimple algebraic group over an algebraically closed field of characteristic $p \neq 0$. Let $T$ be a fixed maximal torus of $G$ and let $u_n$ be the hyperalgebra of the nth infinitesimal subgroup of $G$ (cf. [4, 3.2], [14, 2.1]). We shall freely use the well-known properties of the categories of (rational) $G$ modules, $u_n$ modules and $u_n\cdot T$ modules (cf. [4], [6], [7], [12], and [14]). Moreover, the reader may consult for instance [3], [5], [6], and [8] for preliminaries concerning the root system $R$ of $G$, the character group $X = X(T)$ of $T$, the hyperplanes, facets and alcoves in the Euclidean space.

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R ⊗ X, as well as the Weyl group W and the affine Weyl group \( W_p \) operating in \( R ⊗ X \). We assume that \( R \) is irreducible. Let \( X^+ \) be the set of dominant weights with respect to a fixed basis of \( R \).

For a weight \( v \) we let \( \mathcal{M}_v \) be the category of the finite dimensional \( G \) modules for which the highest weights of their composition factors lie in \( W_p v \). Let \( C \) be an alcove. Each finite dimensional \( G \) module \( V \) has for any \( v ∈ C \) a unique maximal submodule \( V_v \) contained in \( \mathcal{M}_v \), and \( V \) is the direct sum of these submodules. Fix two weights \( λ, μ \) in \( C \). We define a functor \( T^μ_λ : \mathcal{M}_λ → \mathcal{M}_μ \) as follows: if \( V ∈ \mathcal{M}_λ \), then \( T^μ_λ V = (V ⊗ E)_μ \), where \( E \) is the simple \( G \) module with the highest weight in \( W(μ − λ) \). The difference between this definition and the one in [2, § 2], [11, § 3], and [14, 5.2] is only notational; we allow \( C \) to be any alcove. Clearly \( T^μ_λ = T^{w_1 μ}_{w_λ} \) for \( w ∈ W_p \).

Analogous functors \( T^μ_λ \) for \( u_μ − T \) modules were defined in [14, 5.2]. We extend the notation as above to encompass weights \( λ, μ \) in the closure of any alcove.

Finally, we mention two important formulas. For a weight \( v \) set

\[
S_v = \{ w ∈ W_p \mid w.v = v \}
\]
as in [2]. Let \( C \) be an alcove and \( λ, μ ∈ C ∩ X \). If \( w ∈ W_p \) and \( w_λ ∈ X^+ \), then

\[
(1) \quad \text{ch } T^μ_λ V(w_λ) = \sum_{w_1 ∈ W_1} \chi (ww_1, μ) ,
\]

and for any \( w ∈ W_p \)

\[
(2) \quad \text{ch } T^μ_λ \hat{Z}(n, w_λ) = \sum_{w_1 ∈ W_1} \text{ch } \hat{Z}(n, ww_1, μ) ,
\]

where \( W_1 \) is a system of representatives for \( S_λ/S_λ ∩ S_μ \). These are the formulas [14; 5.2(6), (7)], when \( C \) is the bottom alcove, and the general case follows easily from this.

2. \( T^μ_λ \) for \( L(λ) \) and \( \hat{L}(n, λ) \).

First we generalize the character formula (1) of the previous section.

**Lemma 2.1.** Let \( λ, μ ∈ X \) be in the closure of the same alcove and let \( W_1 \) be a system of representatives for \( S_λ/S_λ ∩ S_μ \). If \( V \) is a finite dimensional \( G \) module and

\[
\text{ch } V = \sum_{w ∈ W_p} a_w \chi (w_λ)
\]

with \( a_w ∈ Z \), then

\[
\text{ch } T^μ_λ V = \sum_{w ∈ W_p} \sum_{w_1 ∈ W_1} a_w \chi (ww_1, μ) .
\]
PROOF. Suppose \( \chi(w, \lambda) = 0 \). Then \( s_w \lambda = w \lambda \) for some \( \lambda \in R \). Hence \( w^{-1} s_w w \in S_\lambda \). For each \( w_1 \) in \( W_1 \), there is a unique \( w_2 \) in \( W_1 \) with \( w^{-1} s_w ww_1 \mu = w_2 \mu \). Putting \( \zeta(w_1) = w_2 \), we get a map \( \zeta : W_1 \rightarrow W_1 \) that is injective and hence bijective. Now

\[
\sum_{w_1 \in W_1} \chi(ww_1, \mu) = \sum_{w_1 \in W_1} \chi(w \zeta(w_1), \mu)
\]

\[
= \sum_{w_1 \in W_1} \chi(s_w ww_1, \mu) = -\sum_{w_1 \in W_1} \chi(ww_1, \mu).
\]

Hence this sum is zero. Therefore we may assume that \( \chi(w, \lambda) \neq 0 \), whenever \( a_w \neq 0 \). Then for each \( w \) with \( a_w \neq 0 \), there is a \( \sigma_w \in W \) with \( \sigma_w w, \lambda \in X^+ \). Now we have

\[
\text{ch } V = \sum_{w \in W_p} a_w \det(\sigma_w) \text{ch } V(\sigma_w w, \lambda).
\]

Let \( E \) be the simple module with the highest weight in \( W(\mu - \lambda) \). Then

\[
\sum_{v \in C} \text{ch } (V \otimes E)_v = \sum_{v \in C} \sum_{w \in W_p} a_w \det(\sigma_w) \text{ch } (V(\sigma_w w, \lambda) \otimes E)_v.
\]

Clearly the characters of modules belonging to different categories \( \mathcal{M}_v \), \( v \in C \), are linearly independent. Hence

\[
\text{ch } T^\mu_\lambda V = \sum_{w \in W_p} a_w \det(\sigma_w) \text{ch } T^\mu_\lambda V(\sigma_w w, \lambda)
\]

\[
= \sum_{w \in W_p} \sum_{w_1 \in W_1} a_w \det(\sigma_w) \chi(\sigma_w ww_1, \mu)
\]

\[
= \sum_{w \in W_p} \sum_{w_1 \in W_1} a_w \chi(ww_1, \mu)
\]

by (1) of section 1.

Let \( \lambda, \mu \in X \) be in the closure of the same alcove. In [14, 5.3], Jantzen showed that if a \( G \) module \( V \in \mathcal{M}_\lambda \) has a filtration by Weyl modules, then so has \( T^\mu_\lambda V \). Moreover, [14, 5.1] gives the corresponding result for \( \mu - T \) modules and \( Z \) filtrations (see [12, 3.3]). Hence the formulas (1) and (2) of section 1 give immediately the following two lemmas (cf. [11, § 3]). Lemma 2.2 is contained in [2, 2.1c], too.

**Lemma 2.2.** Let \( F \) be a facette, \( \lambda \in F \cap X^+ \) and \( \mu \in F \cap X \). Then \( T^\mu_\lambda V(\lambda) \cong V(\mu) \) for \( \mu \in X^+ \), while \( T^\mu_\lambda V(\lambda) = 0 \) for \( \mu \notin X^+ \).

**Lemma 2.2'.** Let \( F \) be a facette, \( \lambda \in F \cap X \) and \( \mu \in F \cap X \). Then \( T^\mu_\lambda Z(n, \lambda) \cong Z(n, \mu) \).
We use the sign $\uparrow$ for the strong linkage relation as in [9, § 6]. From [15] we get the following result.

**Lemma 2.3.** Let $F, F'$ be facettes, $\lambda \in F \cap X^+$ and $\xi \in F' \cap X^+$. Assume that $\overline{F \cap F'} \neq \emptyset$, $\xi \uparrow \lambda$ and that $\xi \uparrow \tau \uparrow \lambda$ implies $\tau \in X^+$. Then $[V(\lambda): L(\xi)] \neq 0$.

Set $\mathcal{C}_0 = \{x \in \mathbb{R} \otimes X \mid \langle x + q, \alpha \rangle > 0 \ \forall \alpha \in R^+\}$. For a facette $F$ let $\hat{F}$ be its upper closure (cf. [8]).

**Corollary 2.4.** If $F$ is a facette, $w \in W_p$, $\lambda \in F \cap X^+$ and $\overline{F \cap (w.F) \cap \mathcal{C}_0} \neq \emptyset$, then $[V(\lambda): L(w.\lambda)] \neq 0$.

**Proof.** Let $F_0$ be a facette in $\overline{F \cap (w.F) \cap \mathcal{C}_0}$ and $C$ an alcove with $w.F \subseteq \hat{C}$ ([8, Satz 4]). Clearly $F_0 \subseteq \hat{C}$. In the notation of [9, p. 137], $C$ and $w^{-1}.C$ belong to $\mathcal{R}(F_0)$. By [9, Lemma 6] $C \uparrow w^{-1}.C$; hence $w.\lambda \uparrow \lambda$.

Now let $w.\lambda \uparrow \tau \uparrow \lambda$. There is a chain $w.\lambda < s_1 w.\lambda < \ldots < s_k w.\lambda = \tau$, where each $s_i$ is a reflection, $k \geq 0$. The chain gives $C \uparrow C'$, where $C' = s_k \ldots s_1.C$ and $\tau \in C'$. Similarly, the relation $\uparrow \lambda$ gives an alcove $C''$ with $C' \uparrow C''$ and $\lambda \in C''$. Then $C, C'' \in \mathcal{R}(F_0)$. By [9, Lemma 6], $C' \in \mathcal{R}(F_0)$. Hence $F_0 \subseteq C'$ and $s_k \ldots s_1 F_0 \subseteq C'$. Therefore $F_0 = s_k \ldots s_1 F_0 \subseteq F'$, where $F' = s_k \ldots s_1 w.F$. Hence $\tau \in F' \subseteq \mathcal{C}_0$. Now the assertion follows from lemma 2.3.

We prove analogous results for $u_n - T$ modules.

**Lemma 2.3'.** Let $F, F'$ be facettes and $\lambda \in F \cap X$, $\xi \in F' \cap X$. If $\overline{F \cap F'} \neq \emptyset$ and $\xi \uparrow \lambda$, then $[\hat{Z}(n, \lambda): \hat{L}(n, \xi)] \neq 0$.

**Proof.** For $x \in X$, we let $T(x)$ be the translation in $R \otimes X$ by $x$. Choose an integer $m$. According to [12, 2.8]

$$[\hat{Z}(n, \lambda): \hat{L}(n, \xi)] = [\hat{Z}(n, \lambda + mp^n q): \hat{L}(n, \xi + mp^n q)].$$

Hence we may replace $\lambda, \xi, F$ and $F'$ by $\lambda + mp^n q$, $\xi + mp^n q$, $T(mp^n q)(F)$ and $T(mp^n q)(F')$, respectively. Taking $m$ large enough, we can therefore assume that if $\lambda \uparrow \nu \uparrow \xi$ or $[\hat{Z}(n, \lambda): \hat{L}(n, \nu)] \neq 0$, then $\nu \in X^+$. Now 2.3 implies $[V(\lambda): L(\xi)] \neq 0$. On the other hand, by [14, 3.1(5)] we have

$$\text{ch } V(\lambda) = \sum_{\tau \in X^+} \sum_{v \in X_n} [\hat{Z}(n, \lambda): \hat{L}(n, p^n v + \tau)] \text{ch } V(v)^{\hat{F}^{\tau}} \text{ch } L(\tau).$$

Here $X_n = X_n(T)$ (cf. [12, 1.4]). Put $\xi = p^n \eta + \tau$, $\tau \in X_n$. Then
0 = [V(\lambda) : L(\xi)] = \sum_{v \in \mathcal{X}} [\tilde{Z}(n, \lambda) : \tilde{L}(n, p^n v + \tau)][V(v) : L(\eta)].

Let \nu be a weight giving a non-zero term on the right hand side. Then \eta \uparrow \nu and \(p^n v + \tau) \uparrow \lambda by the strong linkage principle [1, cor. 3] and its counterpart [14, 3.3]. In particular \nu \geq \eta. This implies easily \(p^n \eta + \tau) \uparrow \(p^n v + \tau). Hence \xi \uparrow \(p^n v + \tau) \uparrow \lambda.

Let \mathcal{C} be an alcove with \lambda \in \mathcal{C}. As in the proof of 2.4 we can find alcoves \mathcal{C}' and \mathcal{C}'' with \mathcal{C} \uparrow \mathcal{C}'' \uparrow \mathcal{C}, \(p^n v + \tau) \in \mathcal{C}'', and \xi \in \mathcal{C}''. Let \mathcal{F}_0 be a facet in \bar{\mathcal{F}} \cap \bar{\mathcal{F}}'. Then \mathcal{C}, \mathcal{C}' and \mathcal{C}'' are in \mathcal{A}(\mathcal{F}_0) by [9, Lemma 6]. Set \mathcal{F}' = w.F', where w is the element of \mathcal{W}_p with \mathcal{C}'' = w.C'. Now \mathcal{F}_0 \subseteq \mathcal{C}' \cap \mathcal{C}'' . Therefore \mathcal{W}_p.F_0 = F_0. On the other hand \(p^n v + \tau) \in \mathcal{C}'' \cap (\mathcal{W}_p, \xi) and \mathcal{W}_p, \xi \in \mathcal{C}'' ; so

\[ T(p^n(v - \eta), \xi) = p^n v + \tau = w.\xi. \]

This implies that \(T(p^n(v - \eta)) \text{ and w coincide on } F', \text{ and therefore on } F_0. \text{ Hence} \]

\[ T(p^n(v - \eta)).F_0 = w.F_0 = F_0. \]

So we actually have \nu = \eta. This proves the lemma.

**Corollary 2.4'.** If \mathcal{F} is a facet, \(w \in \mathcal{W}_p, \lambda \in \mathcal{F} \cap X \text{ and } \bar{\mathcal{F}} \cap (w.F) \neq \emptyset, \text{ then } \tilde{Z}(n, \lambda) : \tilde{L}(n, w.\lambda) \neq 0. \)

Now we are ready to prove the main results of the paper.

**Theorem 2.5.** Let \mathcal{F} be a facet, \lambda \in \mathcal{F} \cap X^+ and \mu \in \mathcal{F} \cap X. If \mu \in \mathcal{F}'', then \(T^\mu_\lambda L(\lambda) \simeq L(\mu); \text{ otherwise } T^\mu_\lambda L(\lambda) = 0.

**Theorem 2.5'.** Let \mathcal{F} be a facet, \lambda \in \mathcal{F} \cap X and \mu \in \mathcal{F} \cap X. If \mu \in \mathcal{F}'', then \(T^\mu_\lambda \tilde{L}(n, \lambda) \simeq \tilde{L}(n, \mu); \text{ otherwise } T^\mu_\lambda \tilde{L}(n, \lambda) = 0.

**Proof.** Let \mathcal{F} be a facet, \lambda \in \mathcal{F} \cap X^+ and \mu \in \mathcal{F} \cap X. We get the first assertion of 2.5 as in [11]: Write

\[ \chi_\mu(\lambda) = \sum_{w \in \mathcal{W}_p} a(w, \lambda) \chi(w.\lambda) \]

as in [8, p. 130]. Then [8, Theorem 1] and Lemma 2.1 imply

\[ \text{ch } L(\mu) = \sum_{w \in \mathcal{W}_p} a(w, \lambda) \chi(w.\mu) = \text{ch } T^\mu_\lambda L(\lambda). \]

Hence \(T^\mu_\lambda L(\lambda) \simeq L(\mu). \)
Next let $F$ and $\lambda$ be as above and $\mu \in (\hat{F} \setminus \hat{F}) \cap X$. If $\mu \notin X^+$, then $T_\mu^\alpha L(\lambda)$ is zero by 2.2, since it is a quotient of $T_\mu^\alpha V(\lambda)$. Therefore we may assume that $\mu \in X^+$. From [8, Satz 4] one easily sees that $\mu$ is in the upper closure of $w\cdot F$ for some $w \in W_p$. Put $x = [V(\lambda): L(w, \lambda)]$. By Corollary 2.4, $x \neq 0$. Then $w \cdot \lambda \neq \lambda$, and $V(\lambda)$ has $L(\lambda)$ once and $L(w, \lambda)$ $x$ times as a composition factor. Operating with $T_\mu^\alpha$ to a composition series of $V(\lambda)$, we get a filtration for $T_\mu^\alpha V(\lambda) \cong V(\mu)$ in which $T_\mu^\alpha L(\lambda)$ occurs once, and $T_\mu^\alpha L(w, \lambda)$ $x$ times as a quotient. By the first part of the proof $T_\mu^\alpha L(w, \lambda) \cong L(\mu)$. Now $T_\mu^\alpha L(\lambda)$ is a quotient of $T_\mu^\alpha V(\lambda) \cong V(\mu)$; so if it was not zero, it would have $L(\mu)$ as a quotient. This would imply $[V(\mu): L(\mu)] \geq 1 + x \geq 2$. Hence $T_\mu^\alpha L(\lambda) = 0$. Note too that we actually get $x = 1$. This gives a part of Corollary 2.6.

Now let us consider Theorem 2.5'. We prove only the first assertion; the rest of 2.5' can be derived in a way entirely analogous to the proof of 2.5. So let $F$ be a facet, $\lambda \in F \cap X$ and $\mu \in \hat{F} \cap X$. If $\lambda \in X$, then $\mu \in X$, and by 2.5 we get (see [14, 5.2])

$$T_\mu^\alpha \hat{L}(n, \lambda) \cong T_\mu^\alpha (L(\lambda)|_{w^{-1}T}) \cong (T_\mu^\alpha L(\lambda)|_{w^{-1}T}) \cong \hat{L}(n, \mu).$$

Next let $\lambda' = \mu' - p^n v$ with $\lambda' \in X$, $\mu' = \mu - p^n v$ and $F' = T(-p^n v)(F)$. Then $F'$ is a facet, $\lambda' \in F' \cap X$ and $\mu' \in \hat{F'} \cap X$. Using a result analogous to [14, 5.2(8)] we have

$$T_\mu^\alpha \hat{L}(n, \lambda) \cong T_\mu^\alpha (\hat{L}(n, p^n v) \otimes \hat{L}(n, \lambda'))$$

$$\cong \hat{L}(n, p^n v) \otimes T_\mu^\alpha \hat{L}(n, \lambda') \cong \hat{L}(n, \mu).$$

**Corollary 2.6.** Let $F, \lambda$ and $w$ be as in 2.4 (respectively 2.4'). Then $[V(\lambda): L(w, \lambda)]$ (respectively $[\hat{L}(n, \lambda): \hat{L}(n, w, \lambda)]$) equals 1.

Now 2.2 and 2.5 imply the following generalization of (a part of) the translation principle (cf. [2, 2.5], [10, Satz 7], [11, §3]):

**Corollary 2.7.** Let $F$ be a facet, $\lambda \in F \cap X^+$, $\mu \in \hat{F} \cap X^+$, $w \in W_p$ and assume that $w \cdot \lambda \in X^+$. Then $[V(w, \lambda): L(\lambda)]$ equals $[V(w, \mu): L(\mu)]$ for $w \cdot \mu \in X^+$ and 0 for $w \cdot \mu \notin X^+$.

**Corollary 2.7'.** Let $F$ be a facet, $\lambda \in F \cap X$, $\mu \in \hat{F} \cap X$, and $w \in W_p$. Then $[\hat{L}(n, w, \lambda): \hat{L}(n, \lambda)]$ equals $[\hat{L}(n, w, \mu): \hat{L}(n, \mu)]$.

Theorem 2.5 can also be used to generalize [8, Theorem 2] in the following way (see also [2, 2.4]).

**Corollary 2.8.** Let $F$ be a facet, $\lambda \in F \cap X^+$ and $\mu \in (\hat{F} \setminus \hat{F}) \cap X^+$. Write
\[
\chi_p(\lambda) = \sum_{w \in W_p} a(w, \lambda) \chi(w, \lambda),
\]
where \( w, \lambda \in X^+ \) if \( a(w, \lambda) \neq 0 \). Then
\[
\sum_{w \in W_p} a(w, \lambda) \chi(w, \mu) = 0.
\]
If \( w \in W_p \) with \( \chi(w, \mu) \neq 0 \), then
\[
\sum_{w' \in S_\mu} a(ww', \lambda) = 0.
\]

**Proof.** The first claim follows from 2.1 and 2.5.
If \( w \in W_p \) with \( a(w, \lambda) \neq 0 \), then \( w, \mu \in X^+ \) or \( \chi(w, \mu) = 0 \). Omitting the terms with \( \chi(w, \mu) = 0 \) in the sum \( \sum_w a(w, \lambda) \chi(w, \mu) \) and combining equal characters, we get a linear combination of linearly independent characters. Hence the first part of 2.8 implies the second one.

3. \( T_M^\mu \) for \( \hat{Q}(n, \lambda) \).

In this section we use the translation functors to generalize [10, Satz 5]. The following lemma is analogous to [13, Satz 2.24].

**Lemma 3.1.** Let \( F \) be a facette, \( \lambda \in \hat{F} \cap X \) and \( \mu \in F \cap X \). Then \( T_M^\mu \hat{Q}(n, \lambda) \cong \hat{Q}(n, \mu) \).

**Proof.** The \( u_n - T \) module \( T_M^\mu \hat{Q}(n, \lambda) \) is projective (see [14, 5.4]), and hence it is a direct sum of modules of the form \( \hat{Q}(n, w, \mu) \), \( w \in W_p \). The multiplicity of \( \hat{Q}(n, w, \mu) \) in the sum equals the dimension of
\[
\text{Hom}_{u_n - T}(\hat{L}(n, w, \mu), T_M^\mu \hat{Q}(n, \lambda)) \cong \text{Hom}_{u_n - T}(T_M^\mu \hat{L}(n, w, \mu), \hat{Q}(n, \lambda))
\]
(cf. [2, §2]) By 2.5' this dimension is 1 if \( w, \mu = \mu \), and 0 otherwise.

**Corollary 3.2.** Let \( F \) be a facette, \( \lambda \in \hat{F} \cap X \) and \( \mu \in F \cap X \). If \( \hat{Q}(n, \lambda) \) extends to a \( G \) module, then \( \hat{Q}(n, \mu) \) extends, too.

**Proof.** If \( Q \) is a \( G \) module with \( Q|_{u_n - T} \cong \hat{Q}(n, \lambda) \), then \( T_M^\mu Q \) is a \( G \) module with \( (T_M^\mu Q)|_{u_n - T} \cong \hat{Q}(n, \mu) \).

The following theorem shows that the restriction \( p \ell f \) in [10, Satz 5] is superfluous (see also [12; 4.2, 6.1]). For \( v \in X \) set
\[
S(v) = \sum_{\tau \in W(v)} e(\tau) \quad \text{and} \quad \nu^{(n)} = w_0(v) + (p^n - 1)q.
\]
Here $w_0$ is the longest element of $W$. Let $C_n$ be the alcove with $(p^n-1)q$ in its upper closure. Finally, we denote the Steinberg module $V((p^n-1)q)$ by $St_n$.

**Theorem 3.3.** If $\lambda \in \overline{C_n} \cap X^+$, then $\tilde{Q}(n, \lambda)$ can be lifted to a $G$ module. Moreover, $\text{ch} \tilde{Q}(n, \lambda) = S(\lambda^{(n)}) \text{ch} St_n$.

**Proof.** We leave it to the reader to verify that

$$\text{ch} T_{(p^n-1)q}^\lambda St_n = S(\lambda^{(n)}) \text{ch} St_n$$

(see formula (1) of section 1 and [10, p. 447(1)]). For $\mu \in X$, we have

$$\text{ch} \tilde{Z}(n, \mu) = e(\mu - (p^n-1)q) \text{ch} St_n$$

by [12, p. 285]. Hence by [12; 3.7, 3.8] the lowest weight of $\tilde{Q}(n, \tau), \tau \in X$, is the lowest weight of $\tilde{Z}(n, \tau)$, i.e. $\tau - 2(p^n-1)q$. Since the projective $u_n - T$ module $T_{(p^n-1)q}^\lambda St_n$ has $w_0(\lambda^{(n)}) - (p^n-1)q  = \lambda - 2(p^n-1)q$ as its lowest weight, it has $\tilde{Q}(n, \lambda)$ as a direct $u_n - T$ summand. Hence

$$\text{ch} \tilde{Q}(n, \lambda) = S(\lambda^{(n)}) \text{ch} St_n - \sum_{v \in X} a_v e(v)$$

for some non-negative integers $a_v$. On the other hand, by [12, 3.8]

$$\text{ch} \tilde{Q}(n, \lambda) = \Phi \text{ch} St_n ,$$

where

$$\Phi = \sum_{\mu \in X} [\tilde{Z}(n, \mu): \tilde{L}(n, \lambda)] e(\mu - (p^n-1)q) .$$

The coefficients in this sum are non-negative; in particular, the coefficient of $e(\lambda - (p^n-1)q)$ is 1. Now $\text{ch} \tilde{Q}(n, \lambda)$ is invariant under $W$ as $\lambda \in X_n$ ([14, 5.8]). Then also $\Phi$ is invariant under $W$, since $\mathbb{Z}[X]$ is an integral domain. Hence in the expression above for $\Phi$, some terms form $S(\lambda - (p^n-1)q) = S(\lambda^{(n)})$. Thus we have

$$\text{ch} \tilde{Q}(n, \lambda) = S(\lambda^{(n)}) \text{ch} St_n + \sum_{v \in X} b_v e(v) ,$$

where the numbers $b_v$ are non-negative. This implies the asserted character formula. Moreover,

$$(T_{(p^n-1)q}^\lambda St_n)|_{u_n - T} \cong \tilde{Q}(n, \lambda) .$$

Hence $\tilde{Q}(n, \lambda)$ extends to a $G$ module.
Let $C_n$ be as above and let $C'_n$ be the alcove immediately below it. That is, $C'_n = s(C_n)$, where $s$ is the reflection in the hyperplane
\[ \{ x \in \mathbb{R} \otimes X \mid \langle x + \varrho, \alpha_0 \rangle = p(p^{n-1} (h-1) - 1) \} , \]
\(\alpha_0\) the maximal short root.

**Corollary 3.4.** If $\lambda$ is a weight in $\overline{C_n} \cup C'_n$, then $\hat{Q}(n, \lambda)$ can be lifted to a $G$ module.

**Proof.** By 3.3 we may assume that $\lambda \in C'_n$. Then $p \geq h$. Hence there is a weight in the common wall of $C_n$ and $C'_n$. Now 3.2 and 3.3 imply the result.

**Corollary 3.5.** If $G$ is of type $A_2$, then each $\hat{Q}(n, \lambda)$, $\lambda \in X_n$, can be lifted to a $G$ module.

**Proof.** For $n = 1$ one can use 3.4. Then the general case follows from [16].

**References**


