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WEIGHTED COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES INDUCED BY DOUBLING WEIGHTS

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Abstract

In this paper, we investigate the boundedness, compactness, essential norm and the Schatten class of weighted composition operators uC_φ on Bergman type spaces A^p_ω induced by a doubling weight ω. Let $X = \{u \in H(\mathbb{D}) : uC_\varphi : A^p_\omega \to A^p_\omega \text{ is bounded}\}$. For some regular weights ω, we obtain that $X = H^\infty$ if and only if φ is a finite Blaschke product.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane, and $H(\mathbb{D})$ the class of all functions analytic on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined by

$$(uC_\varphi f)(z) = u(z) f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

For $0 < p < \infty$, H^p denotes the Hardy space, which consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{H^p} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$

As usual, H^∞ is the space of all bounded analytic functions in $\mathbb{D}$.

We say that μ is a weight, when μ is radial and positive on $\mathbb{D}$. Suppose that ω is an integrable weight on $(0, 1)$. Let $\hat{\omega}(r) = \int_r^1 \omega(s)\, ds$ for $r \in (0, 1)$. We say that ω is regular, denoted by $\omega \in \mathscr{R}$, if there are constants $C > 0$ and $\delta \in (0, 1)$ depending on ω, such that

$$\frac{1}{C} < \frac{\hat{\omega}(r)}{(1-r)\omega(r)} < C, \quad \text{when } \delta < r < 1.$$

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We say that ω is rapidly increasing, denoted by $\omega \in \mathscr{I}$, if

$$\lim_{r\to 1} \frac{\hat{\omega}(r)}{(1-r)\omega(r)} = \infty.$$

Let

$$v_{\alpha,\beta}(r) = (1-r)^{\alpha}\left(\log \frac{e}{1-r}\right)^{\beta}.$$

After a calculation, we have the following typical examples of regular and rapidly increasing weights, see [9], for example.

(i) When $\alpha > -1$ and $\beta \in \mathbb{R}$, $v_{\alpha,\beta} \in \mathscr{R}$.

(ii) When $\alpha = -1$ and $\beta < -1$, $v_{\alpha,\beta} \in \mathscr{I}$ and $|\sin(-\log(1-r))|v_{\alpha,\beta}(r) + 1 \in \mathscr{I}$.

In [8], Peláez introduced the set of doubling weights, denoted by $\hat{\mathscr{D}}$, which includes the set $\mathscr{I} \cup \mathscr{R}$. We say that $\omega \in \hat{\mathscr{D}}$ if there is a constant $C > 0$ such that $\hat{\omega}(r) < C\hat{\omega}((1+r)/2)$, when $0 < r < 1$. We should remark that most of the results in [9], presented in the context of regular and rapidly increasing weights, continue to hold for the wider class $\hat{\mathscr{D}}$. More details about $\mathscr{I}$, $\mathscr{R}$ and $\hat{\mathscr{D}}$ can be found in [8], [9], [11].

For $0 < p < \infty$ and $\omega \in \hat{\mathscr{D}}$, the weighted Bergman space A^p_ω is the space of $f \in H(\mathbb{D})$ for which

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure on $\mathbb{D}$. When $\omega(t) = (1-t)^{\alpha}(\alpha > -1)$, the space A^p_ω becomes the classical weighted Bergman space A^p_α. For classical Bergman space A^p_α, we refer to [3], [7], [18] and references therein. In many respects, the Hardy space H^p is the limit of A^p_α as $\alpha \to -1$. But it is a rough estimate since some of the finer function-theoretic properties of A^p_α do not carry over to H^p. As we know, A^p_ω induced by regular weights have similar properties to A^p_α. But many results in [8], [9], [10], [11], [12], [14] show that spaces A^p_ω induced by rapidly increasing weights, lie "closer" to H^p than any A^p_α.

In [4], Čučkovič and Zhao characterized the boundedness and compactness of weighted composition operators on A^p_α by using the Berezin transform. In [5], they investigated weighted composition operators between different Bergman spaces and Hardy spaces. In [11], Peláez and Rättyä characterized the Schatten class of Toeplitz operators induced by a positive Borel measure on $\mathbb{D}$ and the reproducing kernel of the Bergman space A^2_ω when $\omega \in \hat{\mathscr{D}}$. Let

$\Omega = \{u \in H(\mathbb{D}) : uC_\varphi\colon A^p_\alpha \to A^p_\alpha \text{ is bounded}\}$. In [16], Zhao and Hou proved that $\Omega = H^\infty$ if and only if φ is a finite Blaschke product. The similar result for Hardy space H^p can be seen in [2].

Motivated by [4], [5], [11], under the assumption that $\omega \in \hat{\mathscr{D}}$ and μ is a positive Borel measure, we investigate the boundedness, compactness and essential norm of $uC_\varphi\colon A^p_\omega \to L^q_\mu$ and the Schatten class of $uC_\varphi\colon A^2_\omega \to A^2_\omega$. Motivated by [16], we get that, for some $\omega \in \mathscr{R}$, $X = H^\infty$ if and only if φ is a finite Blaschke product. Here

$$X = \{u : u \in H(\mathbb{D}) \text{ and } uC_\varphi\colon A^p_\omega \to A^p_\omega \text{ is bounded}\}.$$

Throughout this paper, the letter C will denote constants which may differ from one occurrence to another. The notation $A \lesssim B$ means that there is a positive constant C such that $A \le CB$. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. Auxiliary results

In this section we formulate and prove several auxiliary results which will be used in the proofs of main results in this paper.

LEMMA 2.1. *Assume that $\omega \in \hat{\mathscr{D}}$, $r \in (0, 1]$ and*

$$\omega_*(r) = \int_r^1 s\omega(s)\log(s/r)\,ds.$$

Then the following statements hold.

(i) $\omega_* \in \mathscr{R}$ *and* $\omega_*(r) \approx (1-r)\hat{\omega}(r)$ *as* $r \to 1$.

(ii) *There are* $1 < a < b < +\infty$ *and* $\delta \in [0, 1)$, *such that*

$$\frac{\omega_*(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1}\frac{\omega_*(r)}{(1-r)^a} = 0,$$
$$\frac{\omega_*(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1}\frac{\omega_*(r)}{(1-r)^b} = \infty.$$

(iii) $\omega_*(r)$ *is decreasing on* $[\delta, 1)$ *and* $\lim_{r\to 1}\omega_*(r) = 0$.

PROOF. By [11, Lemmas A and 9] and (1.19) in [9], (i) and (ii) hold. (iii) follows from (ii) and $\omega_*(r) = \frac{\omega_*(r)}{(1-r)^a}(1-r)^a$.

REMARK 2.2. We observe that $z = 0$ is the logarithmic singular point of ω_*. So, for any fixed $r_0 \in (0, 1)$, we have $\omega_*(r) \approx (1-r)\hat{\omega}(r)$ for $r_0 \le r < 1$. For simplicity, suppose ω_* and $\hat{\omega}$ are radial, that is, $\omega_*(z) = \omega_*(|z|)$ and $\hat{\omega}(z) = \hat{\omega}(|z|)$ for all $z \in \mathbb{D}$.

Suppose $\mathbb{T}$ is the boundary of $\mathbb{D}$ and $I \subset \mathbb{T}$ is an interval. The Carleson square $S(I)$ can be defined as

$$S(I) = \{re^{it} : e^{it} \in I,\ 1 - |I| \le r < 1\},$$

where $|I|$ denotes the normalized Lebesgue measure of I. For convenience, for each $a \in \mathbb{D} \setminus \{0\}$, we define

$$I_a = \left\{e^{i\theta} : |\arg(ae^{-i\theta})| \le \frac{1 - |a|}{2}\right\}$$

and write $S(a) = S(I_a)$. By (26) in [8], when $\omega \in \hat{\mathscr{D}}$, we have

$$\omega(S(a)) \approx \omega_*(a), \quad \text{for all } a \in \mathbb{D} \text{ and } |a| \ge \tfrac{1}{2}. \tag{2.1}$$

The following lemma is a straightforward result of [8, Lemma 3.1] (or [9, Lemma 2.4]).

LEMMA 2.3. *Suppose $\omega \in \hat{\mathscr{D}}$ and $0 < p < \infty$. Then there exists $\gamma_0 > 0$, such that, for all $\gamma > \gamma_0$,*

$$|F_{a,p,\gamma}(z)| \approx \frac{1}{\omega(S(a))^{1/p}}, \quad \|F_{a,p,\gamma}\|_{A^p_\omega} \approx 1, \quad \text{when } a \in \mathbb{D},\ z \in S(a),$$

and $\lim_{|a|\to 1} \sup_{|z|\le r} |F_{a,p,\gamma}(z)| = 0$, *when* $r \in (0, 1)$. *Here and henceforth,*

$$F_{a,p,\gamma}(z) = \left(\frac{1 - |a|^2}{1 - \overline{a}z}\right)^{(\gamma+1)/p} \frac{1}{(\omega(S(a)))^{1/p}}.$$

For simplicity, in the rest of this paper, we always assume that γ is large enough so that Lemma 2.3 holds whenever we mention the function $F_{a,p,\gamma}$.

For a given Banach space X of analytic functions on $\mathbb{D}$, a positive Borel measure μ on $\mathbb{D}$ is called a *q-Carleson measure* for X, if the identity operator Id: $X \to L^q_\mu$ is bounded. By [8, Theorem 3.3], when $\omega \in \hat{\mathscr{D}}$ and $0 < p \le q < \infty$, a Borel measure μ on $\mathbb{D}$ is a q-Carleson measure for A^p_ω if and only if

$$\sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{(\omega(S(a)))^{q/p}} < \infty.$$

Moreover, $\|\mathrm{Id}\|^q_{A^p_\omega \to L^q_\mu} \approx \sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{(\omega(S(a)))^{q/p}}$. Then we have the following lemma.

LEMMA 2.4. *Let* $0 < p \le q < \infty$. *Suppose* $\omega \in \hat{\mathcal{D}}$, μ *is a positive Borel measure on* $\mathbb{D}$. *Then for some* (*equivalently for all*) large enough γ,

$$\|\mathrm{Id}\|^q_{A^p_\omega \to L^q_\mu} \approx \sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{q/p}} \approx \sup_{a\in\mathbb{D}} \int_{S(a)} |F_{a,p,\gamma}(z)|^q \, d\mu(z)$$
$$\approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |F_{a,p,\gamma}(z)|^q \, d\mu(z).$$

PROOF. By Lemma 2.3, we have

$$\frac{\mu(S(a))}{\omega(S(a))^{q/p}} \approx \int_{S(a)} |F_{a,p,\gamma}(z)|^q \, d\mu(z) \le \int_{\mathbb{D}} |F_{a,p,\gamma}(z)|^q \, d\mu(z), \tag{2.2}$$

when $a \in \mathbb{D}$. So,

$$\sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{q/p}} \approx \sup_{a\in\mathbb{D}} \int_{S(a)} |F_{a,p,\gamma}(z)|^q \, d\mu(z) \lesssim \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |F_{a,p,\gamma}(z)|^q \, d\mu(z).$$

By Lemma 2.3 and [8, Theorem 3.3] (also see [9, Theorem 2.1]), we obtain

$$\int_{\mathbb{D}} |F_{a,p,\gamma}(z)|^q \, d\mu(z) \lesssim \|\mathrm{Id}\|^q_{A^p_\omega \to L^q_\mu} \approx \sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{q/p}}.$$

The proof is complete.

LEMMA 2.5. *Suppose* $0 < p \le q < \infty$, $\omega \in \hat{\mathcal{D}}$, μ *is a positive Borel measure on* $\mathbb{D}$, *and* γ *is large enough. Let* $\frac{1}{2} < r < 1$ *and*

$$N_r^* = \sup_{|a|>r} \int_{\mathbb{D}} |F_{a,p,\gamma}(z)|^q \, d\mu(z).$$

If μ *is a* q*-Carleson measure for* A^p_ω, *then* $\mu_r = \mu|_{\mathbb{D}\setminus r\mathbb{D}}$ *is also a* q*-Carleson measure for* A^p_ω, *where* $r\mathbb{D} = \{z \in \mathbb{D} : |z| < r\}$. *Moreover, there is a* $C > 0$, *such that*

$$\sup_{a\in\mathbb{D}} \frac{\mu_r(S(a))}{(\omega(S(a)))^{q/p}} \le C N_r^*. \tag{2.3}$$

PROOF. It is obvious that μ_r is a q-Carleson measure for A^p_ω. Let

$$N_r = \sup_{|a|\ge r} \frac{\mu(S(a))}{(\omega(S(a)))^{q/p}}.$$

When $|a| \geq r$, $\frac{\mu_r(S(a))}{(\omega(S(a)))^{q/p}} < N_r$ is obvious. When $|a| < r$, letting $k = \operatorname{int}\left(\frac{1-|a|}{1-r}\right) + 1$, there exists $a_1, a_2, \ldots, a_k \in \mathbb{D}$ such that $S(a) \cap \mathbb{D} \setminus r\mathbb{D} \subset \bigcup_{i=1}^{k} S(a_i)$ and $|a_i| = r$ for $i = 1, 2, \ldots, k$. By Lemma 2.1 and (2.1), we have

$$\begin{aligned}
\mu_r(S(a)) &\leq \sum_{i=1}^{k} \mu(S(a_i)) \leq N_r \sum_{i=1}^{k} (\omega(S(a_i)))^{q/p} \\
&\lesssim N_r\left(\frac{1-|a|}{1-r} + 1\right)\omega_*(r)^{q/p} \\
&\approx N_r\left(\frac{1-|a|}{1-r} + 1\right)(1-r)^{q/p}\hat{\omega}(r)^{q/p} \\
&\leq N_r\left(\left(\frac{1-r}{1-|a|}\right)^{(q/p)-1} + \left(\frac{1-r}{1-|a|}\right)^{q/p}\right)(1-|a|)^{q/p}\hat{\omega}(a)^{q/p} \\
&\lesssim N_r \omega(S(a))^{q/p}.
\end{aligned}$$

So, there exists $C > 0$, such that

$$\frac{\mu_r(S(a))}{\omega(S(a))^{q/p}} \leq C N_r.$$

By (2.2), we have $N_r \lesssim N_r^*$. Therefore, (2.3) holds. The proof is complete.

The following lemma can be proved in a standard way (see, for example, Theorem 3.11 in [3]).

LEMMA 2.6. *Let $0 < p, q < \infty$, $\omega \in \hat{\mathscr{D}}$ and let μ be a positive Borel measure. If $T \colon A^p_\omega \to L^q_\mu$ is linear and bounded, then T is compact if and only if whenever $\{f_k\}$ is bounded in A^p_ω and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, $\lim_{k\to\infty} \|T f_k\|_{L^q_\mu} = 0$.*

The following lemma can be found in [16] without a proof. For the benefit of the readers, we will prove it here.

LEMMA 2.7. *Let φ be an analytic self-map of $\mathbb{D}$. Then φ is a finite Blaschke product if and only if $\lim_{|w|\to 1} |\varphi(w)| = 1$.*

PROOF. The sufficiency of the statement is obvious. Next we prove the necessity.

Suppose $\lim_{|w|\to 1} |\varphi(w)| = 1$. Let $E \subset \mathbb{D}$ be compact. Then there exists a constant $r \in (0, 1)$ such that $E \subset r\overline{\mathbb{D}}$, where $r\overline{\mathbb{D}} = \{z \in \mathbb{D} : |z| \leq r\}$. Since $\lim_{|w|\to 1} |\varphi(w)| = 1$, there is a constant $t \in (0, 1)$ such that for all $|z| > t$, we have $|\varphi(z)| > r$. Therefore, $\varphi^{-1}(E) \subset t\overline{\mathbb{D}}$. By the continuity of φ, $\varphi^{-1}(E)$ is closed. So, $\varphi^{-1}(E)$ is compact. By subsection 7.1.3 of [15], φ is proper. By subsection 7.3.1 of [15], φ is a finite Blaschke product. The proof is complete.

3. Main results and proofs

THEOREM 3.1. *Assume $\omega \in \hat{\mathcal{D}}$, $0 < p \le q < \infty$, $u\colon \mathbb{D} \to \mathbb{C}$ is a measurable function, φ is an analytic self-map of $\mathbb{D}$, and μ is a positive Borel measure on $\mathbb{D}$. Then*

$$\|uC_\varphi\|^q_{A^p_\omega \to L^q_\mu} \approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^q |u(z)|^q \, d\mu(z).$$

PROOF. By Lemma 2.3, we have

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^q |u(z)|^q \, d\mu(z) \lesssim \|uC_\varphi\|^q_{A^p_\omega \to L^q_\mu}.$$

Let $\upsilon(E) = \int_{\varphi^{-1}(E)} |u(z)|^q \, d\mu(z)$ for all Borel sets E. For all $f \in A^p_\omega$, letting $w = \varphi(z)$, by Lemma 2.4 we have

$$\begin{aligned} \|uC_\varphi f\|^q_{L^q_\mu} &= \int_{\mathbb{D}} |f(\varphi(z))|^q |u(z)|^q \, d\mu(z) = \int_{\mathbb{D}} |f(w)|^q \, d\upsilon(w) \\ &= \|f\|^q_{L^q_\upsilon} \le \|\mathrm{Id}\|^q_{A^p_\omega \to L^q_\upsilon} \|f\|^q_{A^p_\omega} \\ &\lesssim \|f\|^q_{A^p_\omega} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |F_{a,p,\gamma}(w)|^q \, d\upsilon(w). \end{aligned} \tag{3.1}$$

Making the change of variable $w = \varphi(z)$, we obtain

$$\int_{\mathbb{D}} |F_{a,p,\gamma}(w)|^q \, d\upsilon(w) = \int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^q |u(z)|^q \, d\mu(z).$$

The proof is complete.

Let X and Y be Banach spaces. Recall that the essential norm of a linear operator $T\colon X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

Obviously $T\colon X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$.

THEOREM 3.2. *Assume $\omega \in \hat{\mathcal{D}}$, $1 \le p \le q < \infty$, $u\colon \mathbb{D} \to \mathbb{C}$ is a measurable function, φ is an analytic self-map of $\mathbb{D}$, and μ is a positive Borel measure on $\mathbb{D}$. If $uC_\varphi\colon A^p_\omega \to L^q_\mu$ is bounded, then*

$$\|uC_\varphi\|^q_{e,A^p_\omega \to L^q_\mu} \approx \limsup_{|a|\to 1} \int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^q |u(z)|^q \, d\mu(z).$$

PROOF. Since $uC_\varphi\colon A^p_\omega \to L^q_\mu$ is bounded, $u \in L^q_\mu$ and $\|u\|_{L^q_\mu} \lesssim \|uC_\varphi\|_{A^p_\omega\to L^q_\mu}$.

Upper estimate of $\|uC_\varphi\|_{e,A^p_\omega\to L^q_\mu}$. Suppose $f(z) = \sum_{k=0}^\infty \hat{f}_k z^k \in H(\mathbb{D})$. For $n \in \mathbb{N}$, let

$$K_n f(z) = \begin{cases} \sum\limits_{k=0}^{n} \hat{f}_k z^k, & p > 1; \\ \sum\limits_{k=0}^{n} \big(1 - \frac{k}{n+1}\big) \hat{f}_k z^k, & p = 1, \end{cases}$$

and $R_n = \mathrm{Id} - K_n$. By [17, Proposition 1 and Corollary 3], when $1 < p < \infty$, K_n is bounded uniformly on H^p. By [6], $\|K_n\|_{H^1\to H^1} \le 1$. So, when $p \ge 1$, there is a $C = C(p)$ such that

$$\|K_n f\|^p_{A^p_\omega} \le C \int_0^1 \omega(s)s\,ds \int_0^{2\pi} |f(se^{\mathrm{i}\theta})|^p\,d\theta \le C\|f\|^p_{A^p_\omega},$$

and

$$\|R_n\|_{A^p_\omega\to A^p_\omega} = \|\,\mathrm{Id} - K_n\|_{A^p_\omega\to A^p_\omega} \le 1 + \|K_n\|_{A^p_\omega\to A^p_\omega} \le C^{1/p} + 1. \qquad (3.2)$$

By Lemma 2.6 and Cauchy's estimate, we see that $K_n\colon A^p_\omega \to A^p_\omega$ is compact. So, we get

$$\begin{aligned} \|uC_\varphi\|_{e,A^p_\omega\to L^q_\mu} &= \|uC_\varphi(K_n + R_n)\|_{e,A^p_\omega\to L^q_\mu} \le \|uC_\varphi R_n\|_{e,A^p_\omega\to L^q_\mu} \\ &\le \|uC_\varphi R_n\|_{A^p_\omega\to L^q_\mu}. \end{aligned} \qquad (3.3)$$

For any fixed $r \in (0, 1)$, by (3.1) we have

$$\|uC_\varphi R_n f\|^q_{L^q_\mu} = \int_{\mathbb{D}\setminus r\mathbb{D}} |R_n f(w)|^q\,d\upsilon(w) + \int_{r\mathbb{D}} |R_n f(w)|^q\,d\upsilon(w),$$

where υ is defined in the proof of Theorem 3.1.

Let $\omega_n = \int_0^1 s^n \omega(s)\,ds$. Since $B^\omega_z(\zeta) = \sum_{n=0}^\infty \frac{(\zeta\bar{z})^n}{2\omega_{2n+1}}$ is the reproducing kernel of A^p_ω (see [12], [14]), we have

$$|R_n f(w)| = \big|\langle R_n f, B^\omega_w\rangle_{A^2_\omega}\big| = \big|\langle f, R_n B^\omega_w\rangle_{A^2_\omega}\big| \lesssim \|f\|_{A^p_\omega}\|R_n B^\omega_w\|_{H^\infty}.$$

Here $\langle\cdot,\cdot\rangle_{A^2_\omega}$ is the inner product induced by $\|\cdot\|_{A^2_\omega}$ and $\|\cdot\|_{H^\infty}$ denotes the norm of a function in H^∞. When $|w| \le r$, we obtain

$$\|R_n B^\omega_w\|_{H^\infty} \le \frac{1}{n}\sum_{k=1}^\infty \frac{kr^{k-1}}{2\omega_{2k+1}} + \sum_{k=n+1}^\infty \frac{r^k}{2\omega_{2k+1}}.$$

By [14, Lemma 6], $\sum_{k=1}^{\infty} \frac{kr^{k-1}}{2\omega_{2k+1}}$ is convergent and $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} \frac{r^k}{2\omega_{2k+1}} = 0$. So, for any $\varepsilon > 0$, there is a $N = N(\varepsilon, \omega, r)$, such that

$$|R_n B_w^{\omega}| < \varepsilon, \quad \text{for all } |w| \le r \text{ and } n > N.$$

So, for all $n > N$,

$$\int_{r\mathbb{D}} |R_n f(w)|^q \, d\upsilon(w) \le \varepsilon^q \|u\|_{L_\mu^q}^q \|f\|_{A_\omega^p}^q.$$

Let $\upsilon_r = \upsilon|_{\mathbb{D}\setminus r\mathbb{D}}$. By (3.2), Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
&\int_{\mathbb{D}\setminus r\mathbb{D}} |R_n f(w)|^q \, d\upsilon(w) \\
&\quad = \int_{\mathbb{D}} |R_n f(w)|^q \, d\upsilon_r(w) \lesssim \|R_n f\|_{A_\omega^p}^q \sup_{a\in\mathbb{D}} \frac{\upsilon_r(S(a))}{\omega(S(a))^{q/p}} \\
&\quad \lesssim \|R_n f\|_{A_\omega^p}^q \sup_{|a|>r} \int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^q |u(z)|^q \, d\mu(z) \\
&\quad \lesssim \|f\|_{A_\omega^p}^q \sup_{|a|>r} \int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^q |u(z)|^q \, d\mu(z).
\end{aligned} \tag{3.4}$$

Letting $n \to \infty$, by (3.3)-(3.4), we get

$$\|uC_\varphi\|_{e,A_\omega^p\to L_\mu^q}^q \lesssim \sup_{|a|>r} \int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^q |u(z)|^q \, d\mu(z) + \varepsilon^q \|u\|_{L_\mu^q}^q.$$

Since ε is arbitrary, by letting $r \to 1$, we obtain

$$\|uC_\varphi\|_{e,A_\omega^p\to L_\mu^q}^q \lesssim \limsup_{|a|\to 1} \int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^q |u(z)|^q \, d\mu(z).$$

Lower estimate of $\|uC_\varphi\|_{e,A_\omega^p\to L_\mu^q}$. Assume that $K\colon A_\omega^p \to L_\mu^q$ is compact. By Lemmas 2.3 and 2.6,

$$\lim_{|a|\to 1} \|KF_{a,p,\gamma}\|_{L_\mu^q} = 0.$$

Then

$$\begin{aligned}
\|uC_\varphi - K\|_{A_\omega^p\to L_\mu^q} &\gtrsim \limsup_{|a|\to 1} \|(uC_\varphi - K)F_{a,p,\gamma}\|_{L_\mu^q} \\
&\ge \limsup_{|a|\to 1} \|uC_\varphi F_{a,p,\gamma}\|_{L_\mu^q} - \limsup_{|a|\to 1} \|KF_{a,p,\gamma}\|_{L_\mu^q} \\
&= \limsup_{|a|\to 1} \|uC_\varphi F_{a,p,\gamma}\|_{L_\mu^q}.
\end{aligned}$$

Therefore, we get

$$\|uC_\varphi\|^q_{e,A^p_\omega\to L^q_\mu} \gtrsim \limsup_{|a|\to 1}\int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^q |u(z)|^q \, d\mu(z),$$

as desired. The proof is complete.

THEOREM 3.3. *Assume $\omega \in \hat{\mathscr{D}}$, $0 < q < p < \infty$, $u\colon \mathbb{D} \to \mathbb{C}$ is a measurable function, φ is an analytic self-map of $\mathbb{D}$, and μ is a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) *$uC_\varphi\colon A^p_\omega \to L^q_\mu$ is bounded;*

(ii) *$uC_\varphi\colon A^p_\omega \to L^q_\mu$ is compact;*

(iii) *$\Psi^\gamma_{u,\varphi}(a) \in L^{p/(p-q)}_\omega$ for all γ large enough;*

(iv) *$\Psi^\gamma_{u,\varphi}(a) \in L^{p/(p-q)}_\omega$ for some γ large enough.*

Moreover, if γ is fixed,

$$\|uC_\varphi\|^q_{A^p_\omega\to L^q_\mu} \approx \|\Psi^\gamma_{u,\varphi}\|_{L^{p/(p-q)}_\omega}. \tag{3.5}$$

Here,

$$\Psi^\gamma_{u,\varphi}(a) := \int_{\mathbb{D}} |F_{a,p,\gamma}(\varphi(z))|^p |u(z)|^q \, d\mu(z).$$

PROOF. Let $\upsilon(E) = \int_{\varphi^{-1}(E)} |u(z)|^q \, d\mu(z)$ for all Borel set E. By (3.1), we have

$$\|uC_\varphi f\|^q_{L^q_\mu} = \|f\|^q_{L^q_\upsilon}.$$

So, $uC_\varphi\colon A^p_\omega \to L^q_\mu$ is bounded (compact) if and only if $\mathrm{Id}\colon A^p_\omega \to L^q_\upsilon$ is bounded (compact). By (26) in [8] and Theorem 3 in [13], we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (3.5). Since $1-|a| \le |1-\overline{a}z|$ holds for all $a, z \in \mathbb{D}$, we get (iii) $\Leftrightarrow$ (iv). The proof is complete.

REMARK 3.4. Suppose all of the $p, q, \mu, \omega, u, \varphi$ meet the conditions of the Theorem 3.3 and υ is defined in the proof of Theorem 3.3. For all $a \in \mathbb{D}\setminus\{0\}$ and $r \in (0,1)$, let

$$\Gamma(a) = \left\{z \in \mathbb{D} : |\arg z - \arg a| < \frac{1}{2}\left(1 - \frac{|z|}{|a|}\right)\right\},$$

$$T(a) = \{z \in \mathbb{D} : a \in \Gamma(z)\}, \quad \Delta(a,r) = \left\{z \in \mathbb{D} : \left|\frac{a-z}{1-\overline{a}z}\right| < r\right\},$$

and let

$$Q(z) = \int_{\Gamma(z)} \frac{d\upsilon(\xi)}{\omega(T(\xi))}, \quad M_\omega(\upsilon)(z) = \sup_{z \in S(a)} \frac{\upsilon(S(a))}{\omega(S(a))},$$
$$\Phi_r(z) = \int_{\Gamma(z)} \frac{\upsilon(\Delta(a,r))}{\omega(T(z))} \frac{dA(z)}{(1-|z|)^2}.$$

By [13, Theorem 3], for any fixed γ and r, we have

$$\begin{aligned} \|uC_\varphi\|^q_{A^p_\omega \to L^q_\mu} &\approx \|\mathrm{Id}\|^q_{A^p_\omega \to L^q_\upsilon} \approx \|\Psi^\gamma_{u,\varphi}\|_{L^{p/(p-q)}_\omega} \approx \|M_\omega(\upsilon)\|_{L^{p/(p-q)}_\omega} \\ &\approx \|Q\|_{L^{p/(p-q)}_\omega} + \upsilon(\{0\}) \approx \|\Phi_r\|_{L^{p/(p-q)}_\omega} + \upsilon(\{0\}). \end{aligned} \tag{3.6}$$

Using (26) in [8], we know that all the $\omega(S(a))$ and $\omega(T(a))$ in (3.5) and (3.6) can be exchanged.

THEOREM 3.5. *Assume $\omega \in \hat{\mathscr{D}}$ satisfying $\int_0^1 (\log(e/(1-t))^2 \omega(t)\,dt < \infty$, $0 < p < \infty$, $u \in H(\mathbb{D})$, and φ is an analytic self-map of $\mathbb{D}$. If $uC_\varphi\colon A^2_\omega \to A^2_\omega$ is compact, then $uC_\varphi \in S_p(A^2_\omega)$ if and only if*

$$\int_{\mathbb{D}} \left(\frac{\sigma(\Delta(z,r))}{\omega_*(z)}\right)^{p/2} \frac{dA(z)}{(1-|z|^2)^2} < \infty$$

for some (equivalently for all) $0 < r < 1$. Moreover, we have

$$|uC_\varphi|^p_p \approx \int_{\mathbb{D}} \left(\frac{\sigma(\Delta(z,r))}{\omega_*(z)}\right)^{p/2} \frac{dA(z)}{(1-|z|^2)^2}.$$

Here $\sigma(E) = \int_{\varphi^{-1}(E)} |u(z)|^2 \omega(z)\,dA(z)$ for all Borel sets $E \subset \mathbb{D}$, and $|\cdot|_p$ is the norm of Schatten p-class of A^2_ω.

PROOF. For all $f, g \in A^2_\omega$, we have

$$\begin{aligned} \langle (uC_\varphi)^* uC_\varphi f, g\rangle_{A^2_\omega} &= \langle uC_\varphi f, uC_\varphi g\rangle_{A^2_\omega} \\ &= \int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}|u(z)|^2\omega(z)\,dA(z) \\ &= \int_{\mathbb{D}} f(\zeta)\overline{g(\zeta)}\,d\sigma(\zeta). \end{aligned} \tag{3.7}$$

Suppose $B^\omega_z(\zeta)$ is the reproducing kernel of A^2_ω, that is,

$$f(z) = \langle f, B^\omega_z\rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\omega(\zeta)\,dA(\zeta).$$

Consider the Toeplitz operator as follows.

$$T_\sigma f(z) = \int_{\mathbb{D}} f(\eta)\overline{B_z^\omega(\eta)}\, d\sigma(\eta).$$

Since ω is radial, by [9, §4.1], polynomials are dense in A^p_ω for all $0 < p < \infty$. So, if $f, g \in A^2_\omega$, then there are two polynomial sequences $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \|f - f_n\|_{A^2_\omega} = 0, \quad \text{and} \quad \lim_{n\to\infty} \|g - g_n\|_{A^2_\omega} = 0.$$

Since $uC_\varphi \colon A^2_\omega \to A^2_\omega$ is compact, by Theorem 3.1 and Lemma 2.4, we have

$$\|uC_\varphi\|^2_{A^2_\omega \to A^2_\omega} \approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |F_{a,2,\gamma}(z)|^2\, d\sigma(z) \approx \sup_{a\in\mathbb{D}} \frac{\sigma(S(a))}{\omega(S(a))} < \infty.$$

Then, Id: $A^2_\omega \to A^2_\sigma$ and Id: $A^1_\omega \to A^1_\sigma$ are bounded by Lemma 2.4. So, we have $\lim_{k\to\infty} \|g - g_k\|_{A^2_\sigma} = 0$.

For any $h \in H^\infty \subset A^1_\omega$, by [14, Theorem C] (see also [12, Theorem 1]), there exists a constant $C = C(h, u, \varphi, \omega)$ such that

$$\begin{aligned} |T_\sigma h(z)| &\le \|h\|_{H^\infty} \int_{\mathbb{D}} |B_z^\omega(\eta)|\, d\sigma(\eta) \\ &\le \|h\|_{H^\infty} \|\mathrm{Id}\|_{A^1_\omega \to A^1_\sigma} \|B_z^\omega\|_{A^1_\omega} \le C \log \frac{e}{1-|z|}. \end{aligned}$$

Therefore,

$$\|T_\sigma h\|^2_{A^2_\omega} \lesssim \int_{\mathbb{D}} \left(\log \frac{e}{1-|z|}\right)^2 \omega(z)\, dA(z) < \infty.$$

That is to say $T_\sigma h \in A^2_\omega$.

Since $g_n \in A^2_\omega = (A^2_\omega)^*$, for any $n \in \mathbb{N}$, by Lemma 11 of [14],

$$\begin{aligned} \langle T_\sigma f_n, g\rangle_{A^2_\omega} &= \lim_{k\to\infty} \langle T_\sigma f_n, g_k\rangle_{A^2_\omega} = \lim_{k\to\infty} \langle f_n, g_k\rangle_{A^2_\sigma} \\ &= \int_{\mathbb{D}} f_n(\eta)\overline{g(\eta)}\, d\sigma(\eta). \end{aligned} \tag{3.8}$$

Since g is arbitrary, by (3.7) and (3.8), we have

$$T_\sigma f_n = (uC_\varphi)^*(uC_\varphi) f_n. \tag{3.9}$$

For any fixed $z_0 \in \mathbb{D}$, when $|z - z_0| < (1 - |z_0|)/2$, by Hölder's inequality and [14, Lemma 6], we have

$$|T_\sigma f(z) - T_\sigma f_n(z)| \le \int_{\mathbb{D}} |f(\eta) - f_n(\eta)||\overline{B_z^\omega(\eta)}|\, d\sigma(\eta)$$
$$\lesssim \sup_{|z|<(|z_0|+1)/2} \frac{1}{\omega(S(z))}\sqrt{\sigma(\mathbb{D})}\|f - f_n\|_{A^2_\sigma}.$$

So, we obtain $\lim_{n\to\infty} T_\sigma f_n(z) = T_\sigma f(z)$ for all $z \in \mathbb{D}$. Using (3.9), we have

$$T_\sigma = (uC_\varphi)^* uC_\varphi.$$

By [18, Theorem 1.26], when $p > 0$, $uC_\varphi \in S_p(A^2_\omega)$ if and only if $(uC_\varphi)^* uC_\varphi \in S_{\frac{p}{2}}(A^2_\omega)$. By [14, Theorem 3] (or [11, Theorem 1]), $T_\sigma \in S_{\frac{p}{2}}(A^2_\omega)$ if and only if

$$\int_{\mathbb{D}} \left(\frac{\sigma(\Delta(z,r))}{\omega_*(z)}\right)^{p/2} \frac{dA(z)}{(1-|z|^2)^2} < \infty,$$

for some (equivalently for all) $r \in (0, 1)$. Moreover,

$$|uC_\varphi|_p^p = |T_\sigma|_{\frac{p}{2}}^{\frac{p}{2}} \approx \int_{\mathbb{D}} \left(\frac{\sigma(\Delta(z,r))}{\omega_*(z)}\right)^{p/2} \frac{dA(z)}{(1-|z|^2)^2}.$$

The proof is complete.

THEOREM 3.6. *Suppose $1 < p < \infty$, $\omega \in \mathscr{R}$, φ is a finite Blaschke product. If uC_φ is bounded on A^p_ω, then $u \in H^\infty(\mathbb{D})$.*

PROOF. By [12, Corollary 7], we have $(A^p_\omega)^* \simeq A^{p'}_\omega$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Let B_z^ω be the reproducing kernel of A^2_ω. By [14, Lemma 6], $B_z^\omega \in H^\infty \subset A^{p'}_\omega$. Since uC_φ is bounded on A^p_ω, so is $(uC_\varphi)^*$ on $A^{p'}_\omega$. For all $f \in A^p_\omega$, we have,

$$\langle (uC_\varphi)^* B_z^\omega, f\rangle_{A^2_\omega} = \langle B_z^\omega, uC_\varphi f\rangle_{A^2_\omega} = \overline{u(z) f(\varphi(z))} = \overline{u(z)}\langle B^\omega_{\varphi(z)}, f\rangle_{A^2_\omega}.$$

Therefore,

$$(uC_\varphi)^* B_z^\omega = \overline{u(z)} B^\omega_{\varphi(z)}.$$

So,

$$|u(z)|\|B^\omega_{\varphi(z)}\|_{A^{p'}_\omega} = \|\overline{u(z)} B^\omega_{\varphi(z)}\|_{A^{p'}_\omega} = \|(uC_\varphi)^* B_z^\omega\|_{A^{p'}_\omega}$$
$$\le \|(uC_\varphi)^*\|_{A^{p'}_\omega \to A^{p'}_\omega} \|B_z^\omega\|_{A^{p'}_\omega}.$$

Let $M = \|(uC_\varphi)^*\|_{A^{p'}_\omega \to A^{p'}_\omega}$. By [14, Theorem C], we have

$$|u(z)|^{p'} \le M^{p'} \left(\frac{\|B^\omega_z\|_{A^{p'}_\omega}}{\|B^\omega_{\varphi(z)}\|_{A^{p'}_\omega}} \right)^{p'} \approx M^{p'} \left(\frac{\omega(S(\varphi(z)))}{\omega(S(z))} \right)^{p'-1}. \tag{3.10}$$

Suppose $\varphi(z) = z^m \prod_{k=1}^n \frac{|a_k|}{a_k} \frac{a_k - z}{1-\overline{a_k} z}$. Let

$$c = \max\{|a_k| : k = 1, 2, \dots, n\}, \quad d = \min\{|a_k| : k = 1, 2, \dots, n\}.$$

As in the proof of [2, Lemma 2.1], for $c < |z| < 1$, we have

$$\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \le m + 2n \frac{1+d}{1-d}.$$

By Lemma 2.1, there are $1 < a < b < \infty$ and $\delta \in (0, 1)$, such that $\frac{\omega_*(r)}{(1-r)^b}$ is increasing on $[\delta, 1)$. Let

$$r_0 = \inf\{r : r > \max\{c, \delta\} \text{ and } |\varphi(z)| \ge \delta \text{ for all } |z| = r\}.$$

Then $0 < r_0 < 1$. Obviously, we have

$$\sup_{|z| \le r_0} \frac{\omega(S(\varphi(z)))}{\omega(S(z))} < \infty.$$

When $|z| > r_0$, by (2.1), we have

$$\frac{\omega(S(\varphi(z)))}{\omega(S(z))} \approx \frac{\omega_*(\varphi(z))}{\omega_*(z)}.$$

So, if $|\varphi(z)| < |z|$, we obtain

$$\frac{\omega_*(\varphi(z))}{\omega_*(z)} = \frac{\frac{\omega_*(\varphi(z))}{(1-|\varphi(z)|)^b}}{\frac{\omega_*(z)}{(1-|z|)^b}} \frac{(1 - |\varphi(z)|)^b}{(1 - |z|)^b} \lesssim \left(m + 2n \frac{1+d}{1-d} \right)^b.$$

If $|z| \le |\varphi(z)| < 1$, by Lemma 2.1, we get

$$\frac{\omega_*(\varphi(z))}{\omega_*(z)} \approx \frac{(1 - |\varphi(z)|) \int_{|\varphi(z)|}^1 \omega(t)\, dt}{(1 - |z|) \int_{|z|}^1 \omega(t)\, dt} \lesssim m + 2n \frac{1+d}{1-d}. \tag{3.11}$$

Therefore, by (3.10)–(3.11), we obtain that $u \in H^\infty$. The proof is complete.

To state and prove the next result, we introduce a set. Let $1 < p < \infty$, $\omega \in \mathscr{R}$ and let φ be an analytic self-map of $\mathbb{D}$. We define

$$X := \{u \in H(\mathbb{D}) : uC_\varphi(A^p_\omega) \subset A^p_\omega\}.$$

THEOREM 3.7. *Let $\omega \in \mathscr{R}$, $1 < p < \infty$ and φ be an analytic self-map of $\mathbb{D}$. Suppose*

(i) $\hat{\omega}(\varphi_t(r))\hat{\omega}(r) \lesssim \hat{\omega}(t)$, *for all* $0 \le r \le t < 1$, *here* $\varphi_t(r) = \frac{t-r}{1-tr}$;

(ii) $2A + AB - B > 0$, *where*

$$A = \liminf_{t \to 1} \frac{\int_t^1 \omega(s)\,ds}{(1-t)\omega(t)} \quad \text{and} \quad B = \limsup_{t \to 1} \frac{\int_t^1 \omega(s)\,ds}{(1-t)\omega(t)}.$$

If $X = H^\infty$, then φ is a finite Blaschke product.

PROOF. By [11, Proposition 18], we see that $C_\varphi : A^p_\omega \to A^p_\omega$ is bounded. So, for any $u \in X$, we can define $\|u\|_X = \|uC_\varphi\|_{A^p_\omega \to A^p_\omega}$. Next, we will prove that X is complete under the norm $\|\cdot\|_X$.

Let $\{u_n\}$ be a Cauchy sequence in X. Then $\{u_n C_\varphi\}$ is a Cauchy sequence in $B(A^p_\omega)$, which denotes the set of bounded operators on A^p_ω. So, there exists a $T \in B(A^p_\omega)$, such that $\lim_{n\to\infty} u_n C_\varphi = T$. Since $h(z) = 1 \in A^p_\omega$,

$$u := T(1) \in A^p_\omega, \quad \text{and} \quad \lim_{n\to\infty} \|u_n - u\|_{A^p_\omega} = 0.$$

Therefore, for all $f \in A^p_\omega$,

$$\lim_{n\to\infty} u_n(z) f(\varphi(z)) = u(z) f(\varphi(z)).$$

Since $\lim_{n\to\infty} \|u_n C_\varphi f - Tf\|_{A^p_\omega} = 0$, we get

$$\lim_{n\to\infty} u_n(z) f(\varphi(z)) = (Tf)(z).$$

So, we have $Tf = uC_\varphi f$ for all $f \in A^p_\omega$. Therefore, we get $u \in X$, as desired.

Since $X = H^\infty$ and $C_\varphi \in B(A^p_\omega)$, for all $u \in X$, we get

$$\|u\|_X \le \|u\|_{H^\infty} \|C_\varphi\|_{A^p_\omega \to A^p_\omega}.$$

By the Inverse Mapping Theorem, $\|u\|_X \approx \|u\|_{H^\infty}$.

By $\omega \in \mathscr{R}$, we have $AB > 0$. Therefore $\frac{2}{B} + 1 > \frac{1}{A}$. So, there exists a constant $\varepsilon \in (0, A)$ such that $\frac{2}{B+\varepsilon} + 1 - \frac{1}{A-\varepsilon} > 0$.

Let $a = \frac{1}{B+\varepsilon} - 1$ and $b = \frac{1}{A-\varepsilon} - 1$. Then $2a + 2 - b > 0$. By the proof of [9, Lemma 1.1], there is a $\delta \in (0, 1)$ such that

$$\frac{\omega(r)}{(1-r)^a} \text{ is essential decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\omega(r)}{(1-r)^a} = 0;$$

and

$$\frac{\omega(r)}{(1-r)^b} \text{ is essential increasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\omega(r)}{(1-r)^b} = \infty.$$

Let

$$\mu(z) = \begin{cases} \omega(z), & \delta \le |z| < 1; \\ \dfrac{\omega(\delta)(1-|z|)^a}{(1-\delta)^a}, & |z| < \delta. \end{cases}$$

Then it is easy to check that the following statements hold:

(i) $\frac{\mu(r)}{(1-r)^a}$ is essential decreasing on $[0, 1)$ and $\lim_{r\to 1} \frac{\mu(r)}{(1-r)^a} = 0$;

(ii) $\frac{\mu(r)}{(1-r)^b}$ is essential increasing on $[0, 1)$ and $\lim_{r\to 1} \frac{\mu(r)}{(1-r)^b} = \infty$;

(iii) $\mu \in \mathscr{R}$, $A = \liminf_{t\to 1} \frac{\int_t^1 \mu(s)\,ds}{(1-t)\mu(t)}$ and $B = \limsup_{t\to 1} \frac{\int_t^1 \mu(s)\,ds}{(1-t)\mu(t)}$;

(iv) $\|f\|_{A^p_\mu} \approx \|f\|_{A^p_\omega}$ and $\hat{\mu}(\varphi_t(r))\hat{\mu}(r) \lesssim \hat{\mu}(t)$, for all $0 \le r \le t < 1$.

Therefore, without loss of generality, let $\delta = 0$. So, we have

$$\begin{aligned} \frac{\omega(\varphi_w(z))}{\omega(z)} &= \frac{\frac{\omega(\varphi_w(z))}{1-|\varphi_w(z)|^a}(1-|\varphi_w(z)|)^a}{\frac{\omega(z)}{(1-|z|)^a}(1-|z|)^a} \\ &\lesssim \left(\frac{1-|\varphi_w(z)|^2}{1-|z|^2}\right)^a, \end{aligned}$$

when $|\varphi(z)| > |z|$, and

$$\begin{aligned} \frac{\omega(\varphi_w(z))}{\omega(z)} &= \frac{\frac{\omega(\varphi_w(z))}{1-|\varphi_w(z)|^b}(1-|\varphi_w(z)|)^b}{\frac{\omega(z)}{(1-|z|)^b}(1-|z|)^b} \\ &\lesssim \left(\frac{1-|\varphi_w(z)|^2}{1-|z|^2}\right)^b, \end{aligned}$$

when $|\varphi(z)| \le |z|$. Therefore,

$$\begin{aligned} \frac{\omega(\varphi_w(z))}{\omega(z)} &\lesssim \left(\frac{1-|\varphi_w(z)|^2}{1-|z|^2}\right)^a + \left(\frac{1-|\varphi_w(z)|^2}{1-|z|^2}\right)^b \\ &= \left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^a + \left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^b. \end{aligned} \tag{3.12}$$

Let $\alpha = 2(a+2)/p$ and $u_w(z) = (1/(1-\overline{w}z))^\alpha$. Then $\|u_w\|_\infty^p = 1/((1-$

$|w|)^{2a+4})$. For all $f \in A^p_\omega$, by (3.12), we obtain

$$\begin{aligned}
&\|u_w C_\varphi f\|^p_{A^p_\omega} \\
&= \int_{\mathbb{D}} \frac{1}{|1-\overline{w}z|^{p\alpha}} |f\circ\varphi(z)|^p \omega(z)\,dA(z) \\
&= \int_{\mathbb{D}} \frac{1}{|1-\overline{w}\varphi_w(z)|^{p\alpha}} |f\circ\varphi\circ\varphi_w(z)|^p |\varphi_w'(z)|^2 \omega(\varphi_w(z))\,dA(z) \\
&\lesssim \frac{1}{(1-|w|^2)^{a+2}} \int_{\mathbb{D}} |f\circ\varphi\circ\varphi_w(z)|^p \left(1+\left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^{b-a}\right)\omega(z)\,dA(z) \\
&\lesssim \frac{1}{(1-|w|^2)^{b+2}} \int_{\mathbb{D}} |f\circ\varphi\circ\varphi_w(z)|^p \omega(z)\,dA(z) \\
&\lesssim \frac{\|f\|^p_{A^p_\omega}}{(1-|w|^2)^{b+2}(1-|\varphi(w)|)\hat{\omega}(\varphi(w))}.
\end{aligned}$$

The last inequality can be deduced as in [11, Proposition 18]. By $\|u_w\|_\infty \approx \|u_w C_\varphi\|_{A^p_\omega \to A^p_\omega}$, we get

$$\frac{1}{(1-|w|^2)^{2a+2-b}} \lesssim \frac{1}{(1-|\varphi(w)|)\hat{\omega}(\varphi(w))}.$$

By $2a+2-b>0$, we have $|\varphi(w)| \to 1$ as $|w| \to 1$. By Lemma 2.7, we see that φ is a finite Blaschke product. The proof is complete.

By Theorems 3.6 and 3.7, for some regular weight ω, $X = H^\infty$ if and only if φ is a finite Blaschke product. Here, we give two examples.

COROLLARY 3.8. *Let $1 < p < \infty$ and φ be an analytic self-map of $\mathbb{D}$. Suppose ω is either* (a) *or* (b)*:*

(a) $\omega(r) = (1-r)^\alpha(\log(e/(1-r)))^\beta$, $\alpha > -1$ *and* $\beta \le 0$*;*

(b) $\omega(r) = \exp(-\beta(\log(e/(1-r)))^\alpha)$, $0 < \alpha \le 1$ *and* $\beta > 0$.

Then, $X = H^\infty$ if and only if φ is a finite Blaschke product.

PROOF. By [1, (4.4) and (4.5)], the weights in (a) and (b) are regular.

Suppose φ is a finite Blaschke product. By [11, Proposition 18], $C_\varphi\colon A^p_\omega \to A^p_\omega$ is bounded. So, $H^\infty \subset X$. By Theorem 3.6, $X \subset H^\infty$. Therefore, $X = H^\infty$.

Suppose $X = H^\infty$. By Bernoulli-l'Hôpital theorem, both (a) and (b) meet the condition (ii) of Theorem 3.7. So, if we can prove that (a) and (b) meet the condition (i) of Theorem 3.7, then φ is a finite Blaschke product.

Condition (a). When $0 \le r \le t < 1$, let $\theta = r/t$ and

$$f(\theta, t) = \log \frac{e(1-\theta t^2)}{(1-t)(1+\theta t)} \log \frac{e}{1-\theta t}, \quad (0 \le \theta \le 1, 0 < t < 1).$$

Then

$$f'_\theta(\theta, t) = -\left(\frac{t^2}{1-\theta t^2} + \frac{t}{1+\theta t}\right) \log \frac{e}{1-\theta t} + \frac{t}{1-\theta t} \log \frac{e(1-\theta t^2)}{(1-t)(1+\theta t)}.$$

Suppose $t > 0$, and let

$$\begin{aligned} g(\theta, t) &= \frac{1-\theta t}{t} f'_\theta(\theta, t) \\ &= \log \frac{e(1-\theta t^2)}{(1-t)(1+\theta t)} - \left(\frac{t(1-\theta t)}{1-\theta t^2} + \frac{1-\theta t}{1+\theta t}\right) \log \frac{e}{1-\theta t}. \end{aligned}$$

Then

$$\begin{aligned} g'_\theta(\theta, t) &= -\frac{2t^2}{1-\theta t^2} - \frac{2t}{1+\theta t} - \left(\frac{t^3-t^2}{(1-\theta t^2)^2} - \frac{2t}{(1+\theta t)^2}\right) \log \frac{e}{1-\theta t} \\ &= \frac{1}{(1+\theta t)^2}\left(\frac{(1+\theta t)^2}{(1-\theta t^2)^2} h(\theta, t) + 2t k(\theta, t)\right), \end{aligned}$$

where

$$h(\theta, t) = (t^2 - t^3) \log \frac{e}{1-\theta t} - 2t^2(1-\theta t^2)$$

and

$$k(\theta, t) = \log \frac{e}{1-\theta t} - (1+\theta t).$$

Since

$$h'_\theta(\theta, t) = \frac{t^3 + t^4 - 2\theta t^5}{1-\theta t} > 0, \quad k'_\theta(\theta, t) = t\left(-1 + \frac{1}{1-\theta t}\right) > 0,$$

we have

$$G(\theta, t) := (1+\theta t)^2 g'_\theta(\theta, t)$$

is increasing on $[0, 1]$ about θ. Since $\lim_{t\to 1} G(1, t) = +\infty$, there exists a $\tau \in (0, 1)$ such that $G(1, t) > 0$, for all $t \in (\tau, 1)$. If $t \in (\tau, 1)$, by $G(0, t) < 0$, there is a $\nu(t) \in (0, 1)$, such that

$$G(\theta, t) < 0, \quad \text{when } \theta \in [0, \nu(t)),$$

and

$$G(\theta, t) > 0, \quad \text{when } \theta \in (\nu(t), 1].$$

Since $g'_\theta(\theta,t)=\frac{G(\theta,t)}{(1+\theta t)^2}$, when $t\in(\tau,1)$, $g(\theta,t)$ is decreasing on $[0,\nu(t))$ and increasing on $(\nu(t),1]$. Since

$$g(0,t)=\log\frac{e}{1-t}-(t+1)>0,$$

and

$$g(1,t)=\frac{1}{t+1}\left(t+1-\log\frac{e}{1-t}\right)<0,$$

So there is a $\mu(t)\in(0,1)$ for every $t\in(\tau,1)$, such that, $f(\theta,t)$ is increasing on $[0,\mu(t))$ and decreasing on $(\mu(t),1]$. Since $f(0,t)=f(1,t)=\log(e/(1-t))$,

$$\frac{f(\theta,t)}{\log(e/(1-t))}\geq 1,\quad \text{when } t\in(\tau,1) \text{ and } \theta\in[0,1].$$

It is obvious that

$$\inf_{t\in[0,\tau],\theta\in[0,1]}\frac{f(\theta,t)}{\log(e/(1-t))}>0.$$

Therefore,

$$C_1:=\inf_{0\leq r\leq t<1}\frac{\log\frac{e(1-rt)}{(1-t)(1+r)}\log\frac{e}{(1-r)}}{\log\frac{e}{1-t}}>0.$$

So, when $\alpha>-1$, $\beta\leq 0$ and $\omega(r)=(1-r)^\alpha(\log(e/(1-r)))^\beta$, we have

$$\frac{\omega(\varphi_t(r))\omega(r)}{\omega(t)}\approx\left(\frac{\log\frac{e(1-rt)}{(1-t)(1+r)}\log\frac{e}{(1-r)}}{\log\frac{e}{1-t}}\right)^\beta\leq C_1^\beta.$$

Since $\omega\in\mathscr{R}$, we get

$$\frac{\hat\omega(\varphi_t(r))\hat\omega(r)}{\hat\omega(t)}\approx\frac{\omega(\varphi_t(r))\omega(r)}{\omega(t)}. \tag{3.13}$$

Therefore,

$$\hat\omega(\varphi_t(r))\hat\omega(r)\lesssim\hat\omega(t),\quad \text{when } 0\leq r\leq t<1.$$

Condition (b). Suppose $0\leq r\leq t<1$. Since $\frac{e(1-rt)}{1-r^2}\geq\frac{e(1-r)}{1-r^2}>1$, when $0<\alpha\leq 1$, we have

$$\left(\log\frac{e(1-rt)}{(1-t)(1+r)}\right)^\alpha+\left(\log\frac{e}{(1-r)}\right)^\alpha\geq\left(\log\frac{e^2(1-rt)}{(1-t)(1-r^2)}\right)^\alpha$$
$$\geq\left(\log\frac{e}{1-t}\right)^\alpha.$$

So, when $0 < \alpha \leq 1$, $\beta > 0$ and $\omega(r) = \exp(-\beta(\log(e/(1-r)))^{\alpha})$, we have

$$\omega(\varphi_t(r))\omega(r) \lesssim \omega(t).$$

By (3.13), we get

$$\hat{\omega}(\varphi_t(r))\hat{\omega}(r) \lesssim \hat{\omega}(t), \quad \text{when } 0 \leq r \leq t < 1.$$

The proof is complete.

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