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LITTLE DIMENSION AND THE IMPROVED NEW INTERSECTION THEOREM

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Abstract

Let R be a commutative noetherian local ring. We define a new invariant for R-modules which we call the little dimension. Using it, we extend the improved new intersection theorem.

1. Introduction

Throughout this paper, we assume that R is a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. An R-module B is called a *balanced big Cohen-Macaulay module* if $B \neq \mathfrak{m}B$ and any system of parameters of R is a regular sequence on B; see [2, §8.5]. By a recent work of André [1] (see also [9]), any commutative noetherian local ring admits a balanced big Cohen-Macaulay module, and thus the following *improved new intersection theorem* holds (see [10, p. 509] and [11, p. 153]).

THEOREM 1.1 (André [1], Evans-Griffith [4]). *Let* $F = (0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0)$ *be a complex of free* R*-modules of finite rank. Assume that*

(1) $\mathsf{H}_i(F)$ *has finite length for all* $i > 0$*, and*

(2) *there is an element* $x \in \mathsf{H}_0(F) \setminus \mathfrak{m}\mathsf{H}_0(F)$ *such that* Rx *has finite length.*

Then $\dim R \leq n$.

In this paper, we extend this theorem by using a new invariant for modules; we define the *little dimension* of an R-module M as

$$\operatorname{ldim}_R M = \inf\{\dim_R Rx \mid x \in M \setminus \mathfrak{m}M\}.$$

Note that we have $\operatorname{ldim}_R M \leq \dim_R M$ if $M \neq \mathfrak{m}M$, and $\operatorname{ldim}_R M = \infty$ otherwise.

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To state our main theorem, we introduce some notation. Let $X = (\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots)$ be a complex of R-modules. The *supremum*, *infimum* and *amplitude* of X are defined by

$$\begin{aligned} \sup X &= \sup\{n \mid \mathsf{H}_n(X) \neq 0\}, \\ \inf X &= \inf\{n \mid \mathsf{H}_n(X) \neq 0\}, \\ \operatorname{amp} X &= \sup X - \inf X. \end{aligned}$$

The *support* of an R-module M, denoted by $\operatorname{Supp}_R M$, is defined as the set of prime ideals $\mathfrak{p}$ of R with $M_\mathfrak{p} \neq 0$. The following theorem is the main result of this paper.

THEOREM 1.2. *Let $F = (0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0)$ be a complex of flat R-modules such that $k \otimes_R F$ has nontrivial homology. Suppose that either*

(a) $\operatorname{amp}(B \otimes_R F) = 0$ *for some balanced big Cohen-Macaulay module B, or*

(b) $\operatorname{Supp} \mathsf{H}_i(F) \subseteq \{\mathfrak{m}\}$ *for all $i > 0$.*

Then there is an inequality

$$\dim R \leq \operatorname{ldim}_R \mathsf{H}_0(F) + \sup(k \otimes_R F).$$

Under the assumption of Theorem 1.1, one has $\operatorname{ldim}_R \mathsf{H}_0(F) = 0$ and $\sup(k \otimes_R F) \leq n$, and the condition (b) in Theorem 1.2 is satisfied. Thus Theorem 1.2 extends Theorem 1.1 based on the existence of balanced Cohen-Macaulay modules. We also emphasize that the main theorem can treat complexes of infinitely generated flat R-modules.

In §2, we prove our main Theorem 1.2. In §3, we discuss the relationship between little dimensions and Cohen-Macaulay modules. Section 4 contains some examples concerning little dimensions. We also observe that the inequality of Theorem 1.2 can be an equality or a strict inequality.

2. Proof of the main theorem

Recall that the (*Krull*) *dimension* of an R-module M, denoted by $\dim_R M$, is defined as the supremum of the lengths of chains of prime ideals in $\operatorname{Supp}_R M$. Also, the *depth* of an R-complex X is defined by

$$\operatorname{depth}_R X = n - \sup(\mathsf{K}(\mathbf{x}) \otimes_R X) = -\sup \mathbf{R}\operatorname{Hom}_R(k, X) = -\sup \mathbf{R}\Gamma_\mathfrak{m}(X),$$

where $\mathbf{x} = x_1, \ldots, x_n$ is a system of generators of $\mathfrak{m}$, and $\mathsf{K}(\mathbf{x})$ stands for the Koszul complex on $\mathbf{x}$; we refer the reader to [6, Theorem I] for details. We give a couple of properties of the little dimension.

PROPOSITION 2.1. *For each R-module M satisfying $M \neq \mathfrak{m}M$ one has* $\operatorname{depth}_R M \leq \operatorname{ldim}_R M \leq \dim_R M$.

PROOF. The second inequality is clear, which already appeared in the previous section. To show the first one, choose an element $x \in M \setminus \mathfrak{m}M$ with $\operatorname{ldim}_R M = \dim_R Rx$. Then there is an ideal I of R such that $Rx \cong R/I$ and $Ix = 0$. Thus the first inequality follows from [12, Lemma 3.3].

LEMMA 2.2. *Let M and N be R-modules such that $M \neq \mathfrak{m}M$ and $N \neq \mathfrak{m}N$.*

(1) *Let $x \in M \setminus \mathfrak{m}M$ and $y \in N \setminus \mathfrak{m}N$. Then $x \otimes y \in (M \otimes_R N) \setminus \mathfrak{m}(M \otimes_R N)$. In particular, $M \otimes_R N \neq 0$.*

(2) *There is an inequality* $\operatorname{ldim}_R(M \otimes_R N) \leq \min\{\operatorname{ldim}_R M, \operatorname{ldim}_R N\}$.

PROOF. (1) The elements $\overline{x} \in M/\mathfrak{m}M$ and $\overline{y} \in N/\mathfrak{m}N$ are nonzero, and so is $\overline{x} \otimes \overline{y} \in (M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N)$.

(2) Choose elements $x \in M \setminus \mathfrak{m}M$ and $y \in N \setminus \mathfrak{m}N$ such that $\operatorname{ldim}_R M = \dim_R Rx$ and $\operatorname{ldim}_R N = \dim_R Ry$. Using (1), we have $x \otimes y \in (M \otimes_R N) \setminus \mathfrak{m}(M \otimes_R N)$, and get the inequality $\operatorname{ldim}_R(M \otimes_R N) \leq \dim_R R(x \otimes y)$. It remains to note that $\dim_R R(x \otimes y) \leq \min\{\dim_R Rx,\ \dim_R Ry\}$ holds.

Denote by $\mathsf{D}(R)$ the unbounded derived category of all R-modules. For an R-complex X we denote by $\operatorname{Td}_R X$ the *restricted Tor-dimension*, which is by definition the supremum of $\sup(T \otimes_R X)$ where T runs through the flat R-modules. Now we can prove our main theorem.

PROOF OF THEOREM 1.2. Let B be a balanced big Cohen-Macaulay R-module. If $\mathsf{H}_0(F) = \mathfrak{m}\mathsf{H}_0(F)$, then $\operatorname{ldim}_R \mathsf{H}_0(F) = \infty$, and there is nothing to prove. Hence, we assume $\mathsf{H}_0(F) \neq \mathfrak{m}\mathsf{H}_0(F)$. It follows from Lemma 2.2(1) that $\mathsf{H}_0(B \otimes_R F) \cong B \otimes_R \mathsf{H}_0(F) \neq 0$. In particular, we have $\inf(B \otimes_R F) = 0$. By the fact that B is big Cohen-Macaulay and [12, Theorem 2.1], we get $\dim R = \operatorname{depth}_R B = \operatorname{depth}_R(B \otimes_R F) + \sup(k \otimes_R F)$. Thus, it suffices to show $\operatorname{depth}_R(B \otimes_R F) \leq \operatorname{ldim}_R \mathsf{H}_0(F)$.

(a) As $\inf(B \otimes_R F) = 0$, it hold that $\sup(B \otimes_R F) = 0$ by assumption. Hence $B \otimes_R F \cong \mathsf{H}_0(B \otimes_R F) \cong B \otimes_R \mathsf{H}_0(F)$ in $\mathsf{D}(R)$. We have $B \otimes_R \mathsf{H}_0(F) \neq \mathfrak{m}(B \otimes_R \mathsf{H}_0(F))$ by Lemma 2.2(1). Therefore $\operatorname{depth}_R(B \otimes_R \mathsf{H}_0(F)) \leq \operatorname{ldim}_R \mathsf{H}_0(F)$ by Proposition 2.1 and Lemma 2.2(2). Now the inequality $\operatorname{depth}_R(B \otimes_R F) \leq \operatorname{ldim}_R \mathsf{H}_0(F)$ follows.

(b) Set $s = \sup(B \otimes_R F)$. If $s = 0$, then $\operatorname{amp}(B \otimes_R F) = 0$ since $\inf(B \otimes_R F) = 0$, and so we can apply (a) to deduce the assertion. Hence, we assume $s > 0$.

Let us show that $\operatorname{Supp} \mathsf{H}_i(B \otimes_R F) \subseteq \{\mathfrak{m}\}$ for all $i > 0$. Fix a non-maximal prime ideal $\mathfrak{p}$ of R. It follows from [3, Proposition (5.3.4) and Theorems (5.3.6), (5.3.8)] that

$$\begin{aligned}\sup(B \otimes_R F)_{\mathfrak{p}} - \sup F_{\mathfrak{p}} &\le \operatorname{Td}_R B \\ &= \sup\{\operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{q}}} B_{\mathfrak{q}} \mid \mathfrak{q} \in \operatorname{Spec} R\}. \quad (\diamondsuit)\end{aligned}$$

If $\kappa(\mathfrak{q}) \otimes^{\mathbf{L}}_R B \neq 0$, then we see from [5, Remark 2.9] and [14, Theorems (3.2) and (3.3)] that $\operatorname{depth}_{R_{\mathfrak{q}}} B_{\mathfrak{q}} = \dim R_{\mathfrak{q}}$. Hence $\operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{q}}} B_{\mathfrak{q}} \le 0$. If $\kappa(\mathfrak{q}) \otimes^{\mathbf{L}}_R B = 0$, then $\mathbf{R}\operatorname{Hom}_{R_{\mathfrak{q}}}(\kappa(\mathfrak{q}), B_{\mathfrak{q}}) = 0$; see [5, Proposition 2.8]. In other words, $\operatorname{depth}_{R_{\mathfrak{q}}} B_{\mathfrak{q}} = \infty$. Thus, the last term of $(\diamondsuit)$ is non-positive, which implies $\sup(B \otimes_R F)_{\mathfrak{p}} \le \sup F_{\mathfrak{p}}$. As $\sup F_{\mathfrak{p}} \le 0$ by assumption, we obtain $\sup(B \otimes_R F)_{\mathfrak{p}} \le 0$, which shows $\mathsf{H}_i(B \otimes_R F)_{\mathfrak{p}} = 0$ for all $i > 0$. Consequently, $\operatorname{Supp} \mathsf{H}_i(B \otimes_R F) \subseteq \{\mathfrak{m}\}$ for all $i > 0$, as required.

By the proof of [8, Chapter I, Lemma 4.6(3)], we get an injective resolution $I = (0 \to I_s \to I_{s-1} \to \cdots)$ of the R-complex $B \otimes_R F$ as $\sup(B \otimes_R F) = s$, and

$$\operatorname{depth}_R(B \otimes_R F) = -\sup \Gamma_{\mathfrak{m}}(I) = -\sup(0 \to \Gamma_{\mathfrak{m}}(I_s) \to \Gamma_{\mathfrak{m}}(I_{s-1}) \to \cdots).$$

Since $\operatorname{Supp} \mathsf{H}_s(B \otimes_R F) \subseteq \{\mathfrak{m}\}$, we observe that $\Gamma_{\mathfrak{m}}(I_s) = I_s$, which implies $\mathsf{H}_s(\Gamma_{\mathfrak{m}}(I)) = \mathsf{H}_s(I) = \mathsf{H}_s(B \otimes_R F) \neq 0$. Consequently, we get $\operatorname{depth}_R(B \otimes_R F) = -s < 0 \le \operatorname{ldim}_R \mathsf{H}_0(F)$.

REMARK 2.3. We may wonder whether or not in Theorem 1.2 condition (b) implies condition (a). This implication does not necessarily hold even for a bounded complex of free modules of finite rank. Indeed, suppose that R is not regular. Take a minimal system of generators $\mathbf{x} = x_1, \ldots, x_n$ of the maximal ideal $\mathfrak{m}$, and let $F = \mathsf{K}(\mathbf{x})$ be a Koszul complex. Then, clearly, condition (b) is satisfied. As $\mathsf{H}_0(B \otimes F) = B/\mathfrak{m}B \neq 0$, if (a) is satisfied as well, then $B \otimes F$ is acyclic, and it follows that

$$\operatorname{depth}_R B = n - \sup(\mathsf{K}(\mathbf{x}) \otimes_R B) = n > \dim R \ge \dim_R B,$$

which contradicts Proposition 2.1. Therefore, condition (a) is not satisfied.

It is seen from the proof of Theorem 1.2 and [12, Theorem 2.1] that for a complex F which does not satisfy (a) but satisfies (b), the inequality in Theorem 1.2 is strict. In fact, in the above example we have

$$\dim R < n = 0 + n = \operatorname{ldim} \mathsf{H}_0(F) + \sup(k \otimes_R F).$$

Recall that for an ideal I of R, the *codimension* of I is defined by $\operatorname{codim} I = \dim R - \dim R/I$. For a complex $F = (0 \to F_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} F_0 \to 0)$ of free R-modules of finite rank, the *codimension* of F is defined by $\operatorname{codim} F =$

$\inf_{1 \le i \le n}(\operatorname{codim} \mathrm{I}_{r_i}(d_i) - i)$, where $r_i = \sum_{j=i}^{n}(-1)^{j-i} \operatorname{rank} F_j$ (in [2] this is called the *expected rank* of d_i). Using our Theorem 1.2, we recover [2, Theorem 9.4.1]:

COROLLARY 2.4. *Let* $F = (0 \to F_n \to \cdots \to F_0 \to 0)$ *be a complex of finitely generated free* R*-modules with* $\operatorname{codim} F \ge 0$. *For each* $x \in \mathsf{H}_0(F) \setminus \mathfrak{m}\mathsf{H}_0(F)$ *one has* $\operatorname{codim}(\operatorname{ann}(x)) \le n$.

PROOF. Let B be a balanced big Cohen-Macaulay R-module. Lemma 2.2(1) implies $\mathsf{H}_0(B \otimes_R F) \cong B \otimes_R H_0(F) \ne 0$. By [2, Lemma 9.1.8] we have $\inf(B \otimes_R F) = 0$, whence $\operatorname{amp}(B \otimes_R F) = 0$. Theorem 1.2 yields $\dim R \le \operatorname{ldim}_R H_0(F) + \sup(k \otimes_R F) \le \operatorname{ldim}_R H_0(F) + n \le \dim_R Rx + n$, which shows $\operatorname{codim}(\operatorname{ann}(x)) \le n$.

3. Little Cohen-Macaulay modules

Let M be a finitely generated R-module. Recall that M is called *Cohen-Macaulay* if $\operatorname{depth}_R M = \dim_R M$. Following this, we say that M is little Cohen-Macaulay if $\operatorname{depth}_R M = \operatorname{ldim}_R M$. Also, recall that the *Cohen-Macaulay defect* of M is defined by

$$\operatorname{cmd}_R M = \dim_R M - \operatorname{depth}_R M.$$

Following this, we define the *little Cohen-Macaulay defect* of M by

$$\operatorname{lcmd}_R M = \operatorname{ldim}_R M - \operatorname{depth}_R M.$$

On the other hand, we denote by $\mathrm{G}^*\text{-}\dim_R M$ the *upper Gorenstein dimension* of M, that is, the infimum of $\operatorname{pd}_S(M \otimes_R R') - \operatorname{pd}_S R'$, where $R \to R'$ runs over the faithfully flat homomorphisms and $S \to R'$ runs over those surjective homomorphisms which satisfy $\mathbf{R}\mathrm{Hom}_S(R', S) \cong R'[-g]$ with $g = \operatorname{pd}_S R' < \infty$.

REMARK 3.1. Let M be a finitely generated R-module.

(1) Assume $M \ne 0$. Then $\operatorname{ldim}_R M \le \dim_R M$ by Proposition 2.1. Hence $\operatorname{lcmd}_R M \le \operatorname{cmd}_R M$.

(2) There is an inequality $\mathrm{G}^*\text{-}\dim_R M \le \operatorname{pd}_R M$, and the equality holds if the right-hand side is finite. We refer the reader to [15] for details.

Using the little Cohen-Macaulay defect and our Theorem 1.2, we can improve a theorem of Sharif and Yassemi [13] concerning the Cohen-Macaulay defect.

THEOREM 3.2. *Let* $M \ne 0$ *be a finitely generated* R*-module of finite upper Gorenstein dimension. Then*

$$\operatorname{cmd} R \le \operatorname{lcmd}_R M, \qquad \dim R \le \operatorname{ldim}_R M + \mathrm{G}^*\text{-}\dim_R M.$$

PROOF. The assertion follows by replacing $\operatorname{cmd}_R M$ and $\dim_R M$ in the proof of [13, Theorem 2.1] with $\operatorname{lcmd}_R M$ and $\operatorname{ldim}_R M$ respectively, and using Theorem 1.2 instead of the new intersection theorem.

REMARK 3.3. In view of Remark 3.1(1), Theorem 3.2 gives a refinement of [13, Theorem 2.1].

We obtain a couple of corollaries of Theorem 3.2.

COROLLARY 3.4. *Let $M \neq 0$ be a finitely generated R-module of finite projective dimension. Then*

$$\operatorname{cmd} R \leq \operatorname{lcmd}_R M, \qquad \dim R \leq \operatorname{ldim}_R M + \operatorname{pd}_R M.$$

PROOF. The first inequality follows from Theorem 3.2 and Remark 3.1(2). The second inequality is an immediate consequence of the first one and the Auslander-Buchsbaum formula.

COROLLARY 3.5. *The following are equivalent.*

(1) *The local ring R is Cohen-Macaulay.*
(2) *There exists a Cohen-Macaulay R-module of finite projective dimension.*
(3) *There exists a Cohen-Macaulay R-module of finite upper Gorenstein dimension.*
(4) *There exists a little Cohen-Macaulay R-module of finite projective dimension.*
(5) *There exists a little Cohen-Macaulay R-module of finite upper Gorenstein dimension.*

PROOF. The implications (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are shown by Remark 3.1(1), while the implications (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) follow from Remark 3.1(2). If R is Cohen-Macaulay, then the R-module R/Q with Q a parameter ideal has finite length and finite projective dimension. This shows (1) $\Rightarrow$ (2). It is immediate from the first inequality in Theorem 3.2 that (5) $\Rightarrow$ (1) holds. Now the five conditions are proved to be equivalent.

Let R be a Cohen-Macaulay local ring. In view of Corollary 3.5, it is natural to ask if there exists a non-Cohen-Macaulay, little Cohen-Macaulay R-module of finite projective dimension. Evidently, we have to assume that R has positive dimension, and then the question is actually affirmative: Let Q be a parameter ideal of R and put $M = R \oplus R/Q$. Then $\operatorname{depth}_R M = \operatorname{ldim}_R M = 0$, $\dim_R M = \dim R > 0$ and $\operatorname{pd}_R M < \infty$. Hence it may be more meaningful to look for an indecomposable one. However, we do not have such an example even in the case that R is regular. For example, if R is a discrete valuation ring with uniformizer x, then every indecomposable R-module M is isomorphic to either R or $R/(x^n)$ for some $n > 0$, and hence M is Cohen-Macaulay. This leads us to the following modified question.

QUESTION 3.6. Let R be a regular local ring of dimension at least two. Does there exist an indecomposable non-Cohen-Macaulay, little Cohen-Macaulay R-module?

REMARK 3.7. Let $S \to R$ be a surjective homomorphism of (commutative noetherian) local rings. Let M be a (possibly infinitely generated) R-module. Then $\operatorname{depth}_S M = \operatorname{depth}_R M$ and $\operatorname{ldim}_S M = \operatorname{ldim}_R M$. Indeed, the first equality follows from the description of a depth by a Koszul complex. The second one holds since $\dim_S Sx = \dim_R Rx$ for any $x \in M$ and $\mathfrak{n}M = \mathfrak{m}M$, where $\mathfrak{n}$ is the maximal ideal of S. These equalities would help us extend the above question to a homomorphic image of a regular local ring.

We make observations that give some restrictions to construct a module as in Question 3.6. Recall that a finitely generated R-module M is called *unmixed* if $\operatorname{Ass}_R M = \operatorname{Assh}_R M$, where $\operatorname{Assh}_R M$ stands for the set of prime ideals $\mathfrak{p}$ in $\operatorname{Supp}_R M$ such that $\dim R/\mathfrak{p} = \dim_R M$.

PROPOSITION 3.8. *Let $M \neq 0$ be an R-module. If M is cyclic or unmixed, then* $\operatorname{ldim}_R M = \dim_R M$.

PROOF. First, we consider the case where M is cyclic. Then $M \cong R/I$ for some ideal I of R, and we have

$$\begin{aligned}\operatorname{ldim}_R R/I &= \inf\{\dim_R R\overline{x} \mid \overline{x} \in (R/I) \setminus (\mathfrak{m}/I)\}\\ &= \inf\{\dim_R R\overline{x} \mid x \text{ is a unit of } R\} = \dim_R R\overline{1} = \dim_R R/I.\end{aligned}$$

Next, we consider the case where M is unmixed. Take an element $x \in M \setminus \mathfrak{m}M$ satisfying $\operatorname{ldim}_R M = \dim_R Rx = \dim R/\operatorname{ann}(x)$. Suppose $\operatorname{ldim}_R M < \dim_R M$. Then for all $\mathfrak{p} \in \operatorname{Assh}_R M$, the ideal $\operatorname{ann}(x)$ is not contained in $\mathfrak{p}$. Using the assumption $\operatorname{Ass}_R M = \operatorname{Assh}_R M$ and prime avoidance, we find an element $y \in \operatorname{ann}(x)$ which is M-regular. Then $yx = 0$, which implies $x = 0$. This contradicts the choice of x.

Recall that R is called *coprimary* if R has a unique associated prime. A typical example of a coprimary ring is an integral domain. Here is a direct consequence of the above proposition.

COROLLARY 3.9. *Suppose that R is coprimary. Let I be a nonzero ideal of R. Regarding I as an R-module, one has* $\operatorname{ldim}_R I = \dim_R I = \dim R$.

PROOF. Note that there are inclusions $\emptyset \neq \mathrm{Assh}_R\, I \subseteq \mathrm{Ass}_R\, I \subseteq \mathrm{Ass}\, R$. Since $\mathrm{Ass}\, R$ consists only of one element, one has $\mathrm{Assh}_R\, I = \mathrm{Ass}_R\, I = \mathrm{Ass}\, R$. The assertion now follows from Proposition 3.8(2).

4. Several examples illustrating our results

In this section, we make observations on our results obtained in the previous sections, by presenting various examples. In the following two examples, we consider the inequality given in Theorem 1.2. As we see, it is sometimes an equality, and is sometimes a strict inequality.

EXAMPLE 4.1. (1) Let $F = \mathsf{K}(x)$ be the Koszul complex of an element $x \in \mathfrak{m}$. Then one has $\dim R = \mathrm{ldim}_R\, \mathsf{H}_0(F) + \sup(k \otimes_R F)$ if and only if x is a subsystem of parameters of R. Indeed, it is clear that $\sup(k \otimes_R F) = 1$, while $\mathrm{ldim}_R\, \mathsf{H}_0(F) = \dim R/(x)$ by Proposition 3.8. Moreover, F satisfies the condition (a) of Theorem 1.2 when x is a subsystem of parameters.

(2) Let R be a Cohen-Macaulay local ring. Let $M \neq 0$ be an R-module which is either cyclic or unmixed. Assume that M has finite projective dimension, and let F be a minimal free resolution of M. Then, by the Auslander-Buchsbaum formula, $\dim R = \mathrm{ldim}_R\, \mathsf{H}_0(F) + \sup(k \otimes_R F)$ if and only if M is Cohen-Macaulay. In fact, $\sup(k \otimes_R F) = \mathrm{pd}_R\, M$ and $\mathrm{ldim}_R\, \mathsf{H}_0(F) = \dim_R M$ by Proposition 3.8. Moreover, F satisfies the condition (b) of Theorem 1.2 because $\mathrm{Supp}_R\, \mathsf{H}_i(F) = \emptyset \subseteq \{\mathfrak{m}\}$ for $i > 0$.

In the next example, we treat complexes of infinitely generated flat R-modules.

EXAMPLE 4.2. (1) Let $0 \to \bigoplus_{\mathbb{N}} R \to \prod_{\mathbb{N}} R \to C \to 0$ be the natural exact sequence, where $\mathbb{N}$ denotes the set of positive integers. Then $F = (0 \to \bigoplus_{\mathbb{N}} R \to \prod_{\mathbb{N}} R \to 0)$ is a flat resolution of C, and $\mathrm{Supp}\, \mathsf{H}_1(F) = \emptyset \subseteq \{\mathfrak{m}\}$. There are natural isomorphisms $k \otimes_R (\bigoplus_{\mathbb{N}} R) \cong \bigoplus_{\mathbb{N}} k$ and $k \otimes_R (\prod_{\mathbb{N}} R) \cong \prod_{\mathbb{N}} k$, where the latter holds since k is finitely presented. Hence $\sup(k \otimes_R F) = 0$. This also yields $C/\mathfrak{m}C \neq 0$, which implies $\mathrm{ldim}_R\, C \leq \dim R$. On the other hand, Theorem 1.2 implies $\dim R \leq \mathrm{ldim}_R\, \mathsf{H}_0(F) + \sup(k \otimes_R F) = \mathrm{ldim}_R\, C$. Therefore the equality $\dim R = \mathrm{ldim}_R\, \mathsf{H}_0(F) + \sup(k \otimes_R F)$ holds.

(2) Let $x \in \mathfrak{m}$ and $F = \prod_{n \in \mathbb{N}} \mathsf{K}(x^n)$. Note that F satisfies the condition (b) of Theorem 1.2 when x is a non-zerodivisor of R. We have $\mathsf{H}_0(F) = \prod_{n \in \mathbb{N}} R/(x^n)$, and $\sup(k \otimes_R F) = 1$ as $k \otimes_R F = \prod_{n \in \mathbb{N}} \mathsf{K}(x^n, k)$. Also, it is seen that $\mathrm{ldim}_R(\prod_{n \in \mathbb{N}} R/(x^n)) = \inf\{\dim_R R/(x^n) \mid n \in \mathbb{N}\} = \dim R/(x)$. Hence, as with Example 4.1(1), one has $\dim R = \mathrm{ldim}_R\, \mathsf{H}_0(F) + \sup(k \otimes_R F)$ if and only if x is a subsystem of parameters of R. See also Example 4.5(3).

Next we consider the two inequalities given in Proposition 2.1.

EXAMPLE 4.3. (1) Let N be a finitely generated R-module with $\dim_R N > 0$, and set $M = k \oplus N$. One then has $\operatorname{depth}_R M = \operatorname{ldim}_R M = 0 < \dim_R M$.

(2) Let $R = k[\![x, y]\!]/(x^2, xy)$ with k a field. Let $\mathfrak{m} = (x, y)$ be the maximal ideal of R. Then we have $\operatorname{depth}_R \mathfrak{m} = \operatorname{ldim}_R \mathfrak{m} = 0 < 1 = \dim_R \mathfrak{m}$, and $\operatorname{depth} R = 0 < 1 = \operatorname{ldim} R = \dim R$. In fact, note that $\mathfrak{m} = (x) \oplus (y)$ and $(x) \cong k$. Hence $\operatorname{ldim}_R \mathfrak{m} = 0$ by (1). Proposition 3.8 implies $\operatorname{ldim} R = \dim R$.

(3) Suppose $0 < \operatorname{depth} R < \dim R$. Take an R-regular element $x \in \mathfrak{m}$, and set $M = R/(x) \oplus R$. Then $\operatorname{depth}_R M = \operatorname{depth} R/(x) = \operatorname{depth} R - 1$ and $\dim_R M = \dim R$. Also, $\operatorname{ldim}_R M = \operatorname{ldim}_R R/(x) = \dim R/(x) = \dim R - 1$, where the first equality is seen by the definition of the little dimension, while the second equality follows from Proposition 3.8. We conclude that $\operatorname{depth}_R M < \operatorname{ldim}_R M < \dim_R M$.

The third assertion of the above example gives the strict inequalities, assuming that R has positive depth. The following proposition gives the same inequalities in the case where R has depth zero.

PROPOSITION 4.4. *Assume* $\operatorname{depth} R = 0$. *Let x be a minimal generator of $\mathfrak{m}$ such that $\mathfrak{p} = (x)$ is a prime ideal of R,* $\operatorname{ann}(x)$ *is not $\mathfrak{m}$-primary,* $\operatorname{Assh} R = \{\mathfrak{p}\}$ *and $R_\mathfrak{p}$ is a field. Then, regarding $\mathfrak{m}$ as an R-module, we have the strict inequalities* $\operatorname{depth}_R \mathfrak{m} < \operatorname{ldim}_R \mathfrak{m} < \dim_R \mathfrak{m}$.

PROOF. As $\operatorname{depth} R = 0$ and $\mathfrak{m} \neq 0$, we have $0 \neq \operatorname{soc} R \subseteq \mathfrak{m}$. Hence $\operatorname{soc}_R \mathfrak{m} \neq 0$, and $\operatorname{depth}_R \mathfrak{m} = 0$. As $R_\mathfrak{p}$ is a field, we have $\mathfrak{p}R_\mathfrak{p} = 0 = 0R_\mathfrak{p}$, and $\mathfrak{p} = \mathfrak{p}R_\mathfrak{p} \cap R = 0R_\mathfrak{p} \cap R$ is the $\mathfrak{p}$-primary component of the zero ideal 0 of R. Write $0 = \mathfrak{p} \cap I$. Then $I \subseteq \operatorname{ann}(x)$, and we have

$$\operatorname{ldim}_R \mathfrak{m} \leq \dim_R Rx = \dim R/\operatorname{ann}(x) \leq \dim R/I < \dim R = \dim_R \mathfrak{m},$$

where the strict inequality follows from the assumption that $\operatorname{Assh} R = \{\mathfrak{p}\}$.

It follows from the above argument that $\dim R > 0$. If $\mathfrak{p} = \mathfrak{m}$, then $R = R_\mathfrak{p}$ is a field and $\mathfrak{m} = 0$, which is a contradiction. Hence $\mathfrak{p} \neq \mathfrak{m}$. Suppose that there is a minimal generator y of $\mathfrak{m}$ with $\dim_R Ry = 0$. Then $\mathfrak{m}^n y = 0$ for some $n > 0$, and $\mathfrak{m}^n y$ is contained in the prime ideal $\mathfrak{p}$. As $\mathfrak{p} \neq \mathfrak{m}$, we have $y \in \mathfrak{p} = (x)$ and get $y = xz$ for some $z \in R$. Since $y \notin \mathfrak{m}^2$, the element z must be a unit of R. Thus $\mathfrak{m}^n x = 0$ and $\operatorname{ann}(x)$ is $\mathfrak{m}$-primary, this is a contradiction. We conclude that $\operatorname{ldim}_R \mathfrak{m} > 0 = \operatorname{depth}_R \mathfrak{m}$.

EXAMPLE 4.5. Let k be a field.

(1) Let $R = k[\![x, y, z]\!]/(x^2y, xy^2, xz)$. Then $\operatorname{ann}(x) = (xy, y^2, z)$. The zero ideal of R has an irredundant primary decomposition $0 = (x) \cap (y, z) \cap (x^2, y^2, z)$, which shows $\operatorname{Assh} R = \{(x)\}$. Proposition 4.4 yields $\operatorname{depth}_R \mathfrak{m} < \operatorname{ldim}_R \mathfrak{m} < \dim_R \mathfrak{m}$. To be more precise, $\operatorname{depth}_R \mathfrak{m} = 0$, $\operatorname{ldim}_R \mathfrak{m} = 1$ and $\dim_R \mathfrak{m} = 2$.

(2) There is also an example of an equidimensional local ring. Let $R = k[[x, y, z]]/(x^2, xy^2, xyz)$. Then $\operatorname{ann}(x) = (x, y^2, yz)$. We have $0 = (x) \cap (y) \cap (y^2, z)$, and $\operatorname{Min} R = \operatorname{Assh} R = \{(x)\}$. Proposition 4.4 implies $\operatorname{depth}_R \mathfrak{m} < \operatorname{ldim}_R \mathfrak{m} < \dim_R \mathfrak{m}$. In fact, we have $\operatorname{depth}_R \mathfrak{m} = 0$, $\operatorname{ldim}_R \mathfrak{m} = 1$ and $\dim_R \mathfrak{m} = 2$.

(3) Let us present an example of an infinitely generated module. Take an element $x \in \mathfrak{m}$, and set $M = \prod_{n\in\mathbb{N}} R/(x^n)$. Then there is an inclusion $\widehat{R} = \varprojlim_{n\in\mathbb{N}} R/(x^n) \hookrightarrow M$, where $\widehat{R}$ denotes the (x)-adic completion of R. The inclusions $R \hookrightarrow \widehat{R} \hookrightarrow M$ yield $\dim R = \dim_R \widehat{R} = \dim_R M$. Now suppose that x is a non-zerodivisor. Then, there is an isomorphism $\mathbf{R}\operatorname{Hom}_R(k, M) \cong \prod_{n\in\mathbb{N}} \mathbf{R}\operatorname{Hom}_R(k, R/(x^n))$, from which we obtain $\operatorname{depth}_R M = \operatorname{depth} R - 1$. We see from Example 4.2(2) that $\operatorname{ldim}_R M = \dim R/(x) = \dim R - 1$. Thus, under the assumption that R is not Cohen-Macaulay, we have $\operatorname{depth}_R M < \operatorname{ldim}_R M < \dim_R M$.

Let M be a nonzero finitely generated R-module of finite projective dimension. Combining the first inequality in Corollary 3.4 with Remark 3.1(1), we have $\operatorname{cmd} R \leq \operatorname{lcmd}_R M \leq \operatorname{cmd}_R M$. We give examples where either/both of these inequalities become strict.

EXAMPLE 4.6. (1) Let R and M be as in Example 4.3(3). Then M has projective dimension one (hence finite), and it holds that $\operatorname{cmd} R = \operatorname{lcmd}_R M < \operatorname{cmd}_R M = \operatorname{cmd} R + 1$.

(2) Suppose that R is regular and $\dim R \geq 2$. Then Corollary 3.9 implies $\operatorname{ldim}_R \mathfrak{m} = \dim_R \mathfrak{m} = \dim R$. As $\operatorname{depth}_R \mathfrak{m} = 1$, we have $\operatorname{cmd} R = 0 < \operatorname{lcmd}_R \mathfrak{m} = \operatorname{cmd}_R \mathfrak{m} = \dim R - 1$. (Here, the regularity of R is needed just to have that $\operatorname{pd}_R \mathfrak{m}$ is finite. More precisely, we have $\operatorname{cmd} R < \operatorname{lcmd}_R \mathfrak{m} = \operatorname{cmd}_R \mathfrak{m} = \dim R - 1$ for any coprimary local ring R with $\operatorname{depth} R \geq 2$.)

(3) Let $S = k[[x, y, z]]$ with k a field, and set $R = S/(x^2y, xy^2, xz)$. Let $\mathfrak{m} = (x, y, z)R$ be the maximal ideal of R. By Example 4.5(1), we have $\operatorname{depth}_R \mathfrak{m} = 0$, $\operatorname{ldim}_R \mathfrak{m} = 1$ and $\dim_R \mathfrak{m} = 2$. Now we regard $\mathfrak{m}$ as an S-module. Then $\operatorname{depth}_S \mathfrak{m} = 0$, $\operatorname{ldim}_S \mathfrak{m} = 1$ and $\dim_S \mathfrak{m} = 2$ by Remark 3.7. It holds that $\operatorname{cmd}_S S = 0$, $\operatorname{lcmd}_S \mathfrak{m} = 1$ and $\operatorname{cmd}_S \mathfrak{m} = 2$. Thus $\operatorname{cmd}_S S < \operatorname{lcmd}_S \mathfrak{m} < \operatorname{cmd}_S \mathfrak{m}$ and $\operatorname{pd}_S \mathfrak{m} = 3 < \infty$. On the other hand, we have $\operatorname{cmd}_R R = 2 > 1 = \operatorname{lcmd}_R \mathfrak{m}$ and $\operatorname{pd}_R \mathfrak{m} = \infty$. This shows that the assumption that M has finite projective dimension is necessary for the first inequality in Corollary 3.4 to hold true.

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