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BASS NUMBERS AND GOLOD RINGS

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Two series — one homological and the other cohomological — of numerical invariants often carry considerable information on a commutative unitary noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field $k=R/\mathfrak{m}$. On the *homological* side, much attention has been paid to the Poincaré series

$$P_R(t) = \sum b_i t^i \in \mathbf{Z}[[t]]$$

where the Betti number b_i of R is defined to be the rank of the ith module in a minimal free resolution X_* of the R-module k (it is well-known that X_* is unique up to isomorphism, and that $b_i = \dim_k \operatorname{Tor}_i^R(k,k)$). In particular, Serre has remarked that there always is a coefficientwise inequality of formal power series (denoted by the symbol $\ll$):

$$(0.1) \qquad P_R(t) \ll \frac{(1+t)^n}{1-\sum_{i=1}^{n} c_i t^{i+1}},$$

where $n=\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ is the embedding dimension of R, and c_i is the k-dimension of the ith homology group of the Koszul complex $K=K^R$ constructed on a minimal set of generators of $\mathfrak{m}$; (up to isomorphism, the graded skew-commutative k-algebra $H(K)$ is independent of the choice of the generating system). In 1962, E. S. Golod [6] published the proof of the following:

THEOREM. *Equality holds in* (0.1) *if and only if for every $k \geqq 2$ and every system $h_1,\ldots,h_k$ of homogeneous elements of $H(K)$ of positive degree, the Massey product $\langle h_1,\ldots,h_k\rangle$ is defined.*

Rings satisfying the condition of the theorem are called Golod rings; for the definition of Massey products cf. [7].

Our purpose in this paper is to establish in general a coefficientwise upper bound on the *cohomological* series

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$$I_R(t) = \sum \mu_i t^i \in \mathbf{Z}[[t]] ,$$

and to show that, somewhat unexpectedly, it is precisely the Golod rings that are characterized by the extremal property in this context also. Here μ_i, the ith Bass number of R, is defined as being the number of direct summands isomorphic to the injective envelope $E = E_R(k)$, forming the $\mathfrak{m}$-primary component of the ith module in a minimal injective resolution I^* of the R-module R; (it has been proved by Bass [3], extending the results of Matlis [8], that I^* is unique up to isomorphism, and that $\mu_i = \dim_k \operatorname{Ext}_R^i(k, R)$).

We can now state our

THEOREM. *Every non regular local ring satisfies the coefficientwise inequality*

$$I_R(t) \ll \frac{\sum_{i=0}^{n-1} c_{n-i} t^i - t^{n+1}}{1 - \sum_{i=1}^{n} c_i t^{i+1}} . \tag{0.2}$$

Equality holds in (0.2) *if and only if R is a Golod ring, which is not regular.*

The plan of the paper is as follows. In the first section we substitute a homological problem to the cohomological one, and explicit the $H(K)$-module structure of $H(K^E)$, $K^E = K \otimes_R E$; this material is "well-known", but unavailable for direct reference. Section 2 contains a proof of the inequality, very much in the spirit of Serre's argument for (0.1). In section 3 the equality is established for Golod rings, using Golod's R-free resolution of k, and in section 4 it is proved that equality implies the Golod condition, by using a spectral sequence introduced in [1]. In a short final section we show how our result yields the Poincaré series of the canonical module of some rings with determinantal relations, and compare it to the previously available information on the Bass numbers.

All the notation introduced to this point is kept for the rest of the paper.

1. Preliminaries.

LEMMA (1.1). *For all $i \in \mathbf{Z}$, one has natural isomorphisms*

$$\operatorname{Tor}_i^R(k, E) = \operatorname{Ext}_R^i(k, R)^*$$

*where * denotes k-vector space dual.*

PROOF. Since $\operatorname{Ext}_R^i(k, R)$ ($i \in \mathbf{Z}$) is finite dimensional over k, it suffices to prove that $\operatorname{Ext}_R^i(k, R) \cong \operatorname{Tor}_i^R(k, E)^*$. This follows from the sequence of isomorphisms, due to the injectivity of E and the equivalence of $\mathrm{Hom}_k(-,k)$ and $\mathrm{Hom}_R(-,E)$ on k-modules:

$$\begin{aligned}\mathrm{Tor}_i^R(k,E)^* &\cong \mathrm{Hom}_R(\mathrm{Tor}_i^R(k,E),E)\\ &\cong \mathrm{Hom}_R(H_i(X\otimes_R E),E)\\ &\cong H_i(\mathrm{Hom}_R(X\otimes_R E,E))\\ &\cong H_i(\mathrm{Hom}_R(X,\hat{R}))\\ &\cong \mathrm{Ext}_R^i(k,\hat{R})\\ &\cong \mathrm{Ext}_R^i(k,R)\end{aligned}$$

(here $\hat{R}$ is the $\mathfrak{m}$-adic completion of R, and we have used the natural isomorphism $\hat{R}\cong\mathrm{Hom}_R(E,E)$, cf. [8]).

Lemma (1.2) [2]. *For every R-module M and every $i\in\mathbf{Z}$, there is an exact sequence*

$$\begin{aligned}0 &\to \mathrm{Ext}_R^1(K_{i-1}/B_{i-1},K_n^M)\to H_{n-i}(K^M)\xrightarrow{\Delta}\mathrm{Hom}_k(H_i(K),H_n(K^M))\to\\ &\to \mathrm{Ext}_R^1(B_{i-1},K_n^M)\to\mathrm{Ext}_R^1(K_i/B_i,K_n^M)\to\mathrm{Ext}_R^1(H_i(K),K_n^M)\to\\ &\to \mathrm{Ext}_R^2(B_{i-1},K_n^M)\to\ \dots,\end{aligned}$$

where $B_{i-1}=d(K_i)$, and Δ is induced by the multiplication map:

$$H_{n-i}(K^M)\times H_i(K)\to H_n(K^M)\,.$$

Proof. The one given for [2, Proposition 2] for the case $M=R$ works with notational changes only.

Corollary (1.3). *For all $i\in\mathbf{Z}$, the pairings*

$$H_{n-i}(K^E)\times H_i(K)\to H_n(K^E)$$

give rise to isomorphisms

$$H_{n-i}(K^E)\cong H_i(K)^*$$

through the identification $H_n(K^E)=(0:\mathfrak{m})_E=k$.

Remark (1.4). The corollary can be restated by saying that there is an isomorphism

$$H(K^E)\cong E_{H(K)}(s^n k)$$

of graded $H(K)$-modules, where for any graded module N the suspension functor s is defined by setting

$$(sN)_i = N_{i-1} \qquad (i \in \mathbf{Z}) .$$

COROLLARY (1.5). *The following conditions are equivalent, for* $IH(K) = \operatorname{Ker}(H(K) \to k)$:

(a) $(IH(K))^2 = 0$; *and*
(b) $H_j(K^E).H_i(K) = 0$ *for* $i+j \neq n$.

PROOF. Assuming (a), suppose there exist $f \in H_j(K^E)$, $h \in H_i(K)$, such that $fh \neq 0$. By (1.3) we can choose $h' \in H_{n-i-j}(K) \subset IH(K)$, such that $(fh)h' \neq 0$, contradicting the assumption that $hh' = 0$. Reversing the argument we see that (b) implies (a).

For ease of reference we also quote the well-known:

LEMMA (1.6). *Set* $d = \operatorname{depth} R$. *Then* $H_i(K) \neq 0$ *if and only if* $0 \leqq i \leqq n-d$, *and* $H_{n-d}(K) \cong \operatorname{Ext}_R^d(k, R)$.

2. The inequality.

Filtering the double complex $K \otimes_R L$, where L denotes an R-free resolution of E, one obtains a first-quadrant homological spectral sequence with

$$E^2_{p,q} = \operatorname{Tor}_p^R(H_q(K), E) \Rightarrow H_{p+q}(K^E) .$$

From (1.1) and (1.6) it follows that $E^2_{p,q} = 0$ when either $p < d = \operatorname{depth} R$, or $q < 0$, or $q > n-d$. Combining with (1.3) one also gets the natural isomorphisms.

$$\operatorname{Tor}_d^R(k, E) \cong \operatorname{Ext}_R^d(k, R)^* \cong H_{n-d}(K)^* \cong H_d(K^E) .$$

In particular, they give rise to a commutative (at least—up to sign) diagram:

$$\begin{array}{ccc} H_q(K) \otimes \operatorname{Tor}_d^R(k, E) \cong E^2_{d,q} & \xrightarrow{e} & H_{d+q}(K^E) \\ \downarrow \cong & & \| \\ H_q(K) \otimes H_d(K^E) & \xrightarrow{m} & H_{d+q}(K^E) \end{array}$$

where e is the edge homomorphism $E^2_{d,q} \twoheadrightarrow E^\infty_{d,q} \hookrightarrow H_{d+q}(K^E)$, and m is the product map.

Hence, by (1.3) once more:

$$E^\infty_{p,n-p} = \begin{cases} H_n(K^E) \cong k, & \text{when } p = d \\ 0 & \text{otherwise} . \end{cases} \tag{2.1}$$

Setting

$$u_p^r = \operatorname{rank}(d^r\colon E_{p,0}^r \to E_{p-r,r-1}^r)\,,$$

when $r \geqq 2$, and $u_p^r = 0$ for $r = 0, 1$, one has for every $p \geqq 0$ the equality

$$\dim E_{p,0}^2 = \sum_{r=0}^{p} u_p^r + \dim E_{p,0}^{\infty}\,. \tag{2.2}$$

According to (1.1), the left-hand side equals μ_p for all $p \in \mathbf{Z}$. When $p \neq n, n+1$, to obtain an upper bound for the right-hand side we use the trivial inequality

$$u_p^r \leqq \dim E_{p-r,r-1}^2 = \mu_{p-r}c_{r-1} \qquad (r \geqq 2)\,,$$

the fact that by (1.1) and (1.6)

$$\sum_{r=2}^{p} \mu_{p-r}c_{r-1} = \sum_{r=2}^{n+1} \mu_{p-r}c_{r-1}\,,$$

and the relation

$$\dim E_{p,0}^{\infty} \leqq \dim H_p(K^E) = c_{n-p}\,,$$

implied by (1.3). Altogether we get:

$$\mu_p \leqq \sum_{r=2}^{n+1} \mu_{p-r}c_{r-1} + c_{n-p} \qquad \text{for } p \neq n,\ n+1\,.$$

On the other hand, since R is not regular, one has $d \neq n$, hence by (2.1) $\dim E_{n,0}^{\infty} = 0$, and

$$u_{n+1}^{n-d+1} \leqq \dim E_{d,n-d}^{n-d+1} - 1 \leqq \dim E_{d,n-d}^2 - 1 = \mu_d c_{n-d} - 1\,.$$

Bounding u_p^r as above for $(r,p) \neq (n-d+1, n+1)$, we obtain from (2.2):

$$\mu_n \leqq \sum_{r=2}^{n+1} \mu_{n-r}c_{r-1}$$

$$\mu_{n+1} \leqq \sum_{r=2}^{n+1} \mu_{n+1-r}c_{r-1} - 1\,.$$

Forming the power series $\sum \mu_p t^p$ gives

$$I_R(t) \ll I_R(t)\left(\sum_{i=1}^{n} c_i t^{i+1}\right) + \sum_{i=0}^{n-1} c_{n-i}t^i - t^{n+1}$$

which is just another way to write (0.2).

3. Golod rings.

A Golod ring is usually introduced by the requirement that the Massey product $\langle h_1, \ldots, h_r \rangle$ be defined [7] for every system of homogeneous elements in $IH(K)$. However, it is easily seen, and noted already in [6], that this condition can be put in the following stronger form:

(3.0). There exists a basis $S = \{h_1, \ldots, h_t\}$ of $IH(K)$, consting of homogeneous elements, and a function γ defined on the disjoint union $\coprod_{n=1}^{\infty} S^n$ and taking values in the set of homongeneous elements of IK, such that:

(i) $\gamma(h_i) = z_i$, where z_i is a cycle representing h_i;

(ii) $$d\gamma(h_{i_1}, \ldots, h_{i_m}) = \sum_{j=1}^{m-1} \bar{\gamma}(h_{i_1}, \ldots, h_{i_j}) . \gamma(h_{i_{j+1}}, \ldots, h_{i_m}) .$$

Here and below $IK = \operatorname{Ker}(\varepsilon\colon K \to k)$, and $\bar{a} = (-1)^{\deg(a)+1} a$ for any homogeneous element a.

The following is proved in [6]:

THEOREM (3.1). *Let R be a Golod ring, and $h_1, \ldots, h_t$ be a basis of $IH(K)$ satisfying the conditions* (3.0). *Let N be a free graded R-module with homogeneous basis $u_1, \ldots, u_t$*: $\deg u_i = \deg h_i + 1$ $(1 \leqq i \leqq t)$. *Then the R-module $X = K \otimes_R T(N)$, with T denoting the tensor algebra, becomes a minimal R-free resolution of k, when it is given the usual grading, and the differential*:

$$d(x \otimes [u_{i_1}| \ldots |u_{i_m}]) = dx \otimes [u_{i_1}| \ldots |u_{i_m}] - \sum_{k=1}^{m} \bar{x}\gamma(h_{i_1}, \ldots, h_{i_k}) \otimes [u_{i_{k+1}}| \ldots |u_{i_m}] .$$

(We write $[u_{i_1}| \ldots |u_{i_m}]$ for $u_{i_1} \otimes \ldots \otimes u_{i_m}$, and $[\cdot]$ for $1 \in T(N)$.)

In this section we shall assume R is a Golod ring which is not regular. In particular, the homology of $X \otimes_R E$ is $\operatorname{Tor}^R(k, E)$. We shall identify this complex with $K^E \otimes_R T(N)$ as graded modules, which for the differential gives the formula:

(3.2) $$d(y \otimes [u_{i_1}| \ldots |u_{i_m}]) = (dy) \otimes [u_{i_1}| \ldots |u_{i_m}] - \sum_{k=1}^{m} \bar{y} \cdot \gamma(h_{i_1}, \ldots, h_{i_k}) \otimes [u_{i_{k+1}}| \ldots |u_{i_m}] .$$

Next we introduce a filtration $\{F^p\}$ by setting:

$$(F^p)_i = \left\{ \sum y \otimes [u_{i_1}| \ldots |u_{i_m}] \;\middle|\; m \leqq p, \deg(y) + \sum_{j=1}^{m} \deg(u_{i_j}) = i \right\}$$

and look at the resulting spectral sequence.

Since $E^0_{p,q}=(F^p/F^{p-1})_{p+q}=(K^E\otimes_R N^{\otimes p})_{p+q}$, formula (3.2), and the freeness of N imply:

$$E^1_{p,q} = (H(K^E)\otimes_R N^{\otimes p})_{p+q} = (H(K^E)\otimes_k IH(K)^{\otimes p})_{p+q}$$
$$d^1(f\otimes[h_{i_1}|\dots|h_{i_p}]) = -\bar{f}h_{i_1}\otimes[h_{i_2}|\dots|h_{i_p}]\,.$$

Since by (3.0 (ii)) for $m=2$ we see that $IH(K)^2=0$, it follows from (1.5) that:

$$dE^1_{p+1,*} = e\otimes IH(K)^{\otimes p}\,, \tag{3.3}$$

e denoting a generator of the one-dimensional vector space $H_n(K^E)$. From the exact sequences of graded vector spaces and degree zero maps

$$0\leftarrow dE^1_{p,*}\leftarrow E^1_{p,*}\leftarrow Z^1_{p,*}\leftarrow 0$$
$$0\leftarrow E^2_{p,*}\leftarrow Z^1_{p,*}\leftarrow dE^1_{p+1,*}\leftarrow 0$$

one gets the equality

$$|E^2_{p,*}|(u) = |H(K^E)|(u)\cdot[|IH(K)|(u)]^p-u^n[|IH(K)|(u)]^{p-1}-u^n[|IH(K)|(u)]^p\,,$$

where the notation $|W|(u)$ stands for the Hilbert series $\sum\dim_k(W_i)u^i$ of the graded vector space W. From (1.3) this can be explicited as

$$|E^2_{p,*}|(u) = \left(\sum_{i=0}^{n-1}c_{n-i}u^i\right)\left(\sum_{i=1}^{n}c_iu^i\right)^p-u^n\left(\sum_{i=1}^{n}c_iu^i\right)^{p-1} \tag{3.4}$$

which gives for the series in two variables $|E^2_{**}|(t,u)=\sum\dim(E^2_{p,q})t^pu^q$ the closed formula

$$|E^2_{**}|(t,u) = \frac{\sum_{i=0}^{n-1}c_{n-i}u^i-tu^n}{1-t\sum_{i=1}^{n}c_iu^i}\,.$$

Our claim that for the Golod ring R equality holds in (0.2) is now seen to be an immediate consequence of the following

FACT (3.5). In the spectral sequence introduced above, the differentials

$$d^r\colon E^r_{p,q}\to E^r_{p-r,q+r-1}$$

are trivial for $r\geqq 2$.

Indeed, once this is established, by (1.1) we have $I_R(t)=|\mathrm{Tor}^R(k,E)|(t)$, while from the convergence of the spectral sequence to $H(X\otimes_R E)=\mathrm{Tor}^R(k,E)$ follow the equalities:

$$|\mathrm{Tor}^R(k,E)|(t) = |E^{\infty}_{**}|(t,t) = |E^2_{**}|(t,t) = \frac{\sum_{0=i}^{n-1} c_{n-i}t^i - t^{n+1}}{1-\sum_{i=1}^{n} c_i t^{i+1}}.$$

Fixing in $JH(K^E) = \mathrm{Ker}\,(H(K^E) \to H_n(K^E))$ a homogeneous basis $f_1,\ldots,f_t$, such that $-\bar{f}_i h_j = \delta_{ij}e$ (cf. (1.3) and (1.5)), we first note that a homogeneous basis of $E^2_{p,*}$ is given by the classes of the following elements of $E^1_{p,*}$:

$$f_j \otimes [h_{i_1}|\ldots|h_{i_p}], \qquad 1 \leqq j \leqq t,\ j \neq i_1$$

$$f_j \otimes [h_j\,|\,h_{i_2}|\ldots|h_{i_p}] - f_{j+1} \otimes [h_{j+1}\,|\,h_{i_2}|\ldots|h_{i_p}], \qquad 1 \leqq j \leqq t-1\ ,$$

where in both cases $1 \leqq i_s \leqq t$ for $s = 1, 2, \ldots, p$. Indeed, by (1.5) they are cycles, and they are linearly independent modulo boundaries by (3.3). Hence their classes span in $E^2_{p,*}$ a graded linear subspace, whose Hilbert series is seen by direct count to equal the right-hand side of (3.4), which establishes our claim.

By the very definition of a spectral sequence associated to a filtered complex, (3.5) is now seen to be equivalent to the following statement (For details cf. e.g. the treatment of spectal sequences by R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1958, or pp. 138–139 in D. Kraines and C. Schochet, *Differentials in the Eilenberg–Moore spectral sequence*, J. Pure Appl. Algebra 2 (1972), 131–148.)

Let x_p be any of the elements:

$$y_j \otimes [z_{i_1}|\ldots|z_{i_p}]$$

$$y_j \otimes [z_j\,|\,z_{i_2}|\ldots|z_{i_p}] - y_{j+1} \otimes [z_{j+1}\,|\,z_{i_2}|\ldots|z_{i_p}]$$

with y_j a homogeneous cycle in the class of f_j, z_{i_s} as in (3.0 (i)), and the same restrictions on j and the i_s as above. Then there exist $x_k \in K^E \otimes T(N)^k$, $0 \leqq k \leqq p-1$, such that $d(x_p + x_{p-1} + \ldots + x_0) = 0$.

No problem arises for elements of the second type. In fact:

$$\begin{aligned} d(x_p) = &-(\bar{y}_j z_j - \bar{y}_{j+1} z_{j+1}) \otimes [z_{i_2}|\ldots|z_{i_p}] \\ &- \sum_{k=2}^{p} (\bar{y}_j \gamma(h_j, h_{i_2}, \ldots, h_{i_k}) - \bar{y}_{j+1}\gamma(h_{j+1}, h_{i_2}, \ldots h_{i_k})) \otimes [z_{i_{k+1}}|\ldots|z_{i_p}]\ . \end{aligned}$$

Now

$$\mathrm{cls}\,(\bar{y}_j z_j - \bar{y}_{j+1} z_{j+1}) = (\bar{f}_j h_j - \bar{f}_{j+1} h_{j+1}) = 0\ ,$$

hence $\bar{y}_j z_j - \bar{y}_{j+1} z_{j+1} \in dK^E_{n+1} = 0$. On the other hand, since $k \geqq 2$,

$$\deg(\bar{y}_j \gamma(h_j, h_{i_2}, \ldots, h_{i_k}) = \deg(y_j) + \deg(h_j) + \sum \deg(h_{i_l}) + k - 1 > \deg y_j + \deg h_j = n,$$

hence all the elements in the sum are equal to zero. We have shown that x_p is a cycle in $K^E \otimes_R T(N)$, hence we can set $x_k = 0$ for $k = 0, 1, \ldots, p-1$.

Given an element $y \otimes [z_{i_1}|\ldots|z_{i_p}]$ of the first type, we show there always exist homogeneous elements $v_1, v_2, \ldots, v_p \in K^E$, such that

$$(3.6) \qquad dv_k = \sum_{j=0}^{k-1} \bar{v}_j \gamma(h_{i_{j+1}}, \ldots, h_{i_k}) \quad \text{for } 1 \leqq k \leqq p,\ v_0 = y.$$

By induction, we can assume chosen elements $v'_1, \ldots, v'_{l-1}$ satisfying (3.6) (with $v'_0 = y$). Clearly,

$$z = \sum_{j=0}^{l-1} \bar{v}'_j \gamma(h_{i_{j+1}}, \ldots, h_{i_l})$$

is a cycle, and cls (z) belongs to the Massey product $\langle f, h_{i_1}, \ldots, h_{i_l} \rangle$ (cf. [7]). We want to prove that, changing the v'_i-s if necessary, one can find a v_l with $dv_l = z$.

If $\deg(z) > n$, then $z = 0$. If $\deg(z) = n$, then $z = \alpha e$ for some $\alpha \in k$. Let w be a cycle in $K^E_{n-\deg(h_{i_l})}$ such that cls (w). $h_{i_l} = e$. Then with $v_{l-1} = v'_{l-1} = v'_{l-1} - \alpha w$, $v_k = v'_k$ for $1 \leqq k \leqq l-2$, and $v_l = 0$, (3.6) is obviously satisfied for $k = 1, 2, \ldots, l$. Finally, supposing $\deg(z) < n$, we shall prove by induction on l that $\langle f, h_{i_1}, \ldots, h_{i_l} \rangle$ contains only zero, the case $l = 1$ being handled by (1.5). So let $l > 1$ and assume cls $(z) = g$ is a non-zero element. The inductive assumption implies this product is strictly defined. On the other hand, the Golod condition gives that $\langle h_{i_1}, \ldots, h_{i_l}, h \rangle$ is a strictly defined and trivial product in $H(K)$, where $gh = e \neq 0 \in H_n(K^E)$. Now applying [7, (3.2.iii)] we obtain the contradiction

$$0 \neq gh \in \langle f, h_{i_1}, \ldots, h_{i_l} \rangle h = \bar{f} \langle \bar{h}_{i_1}, \ldots, \bar{h}_{i_l}, h \rangle = 0.$$

Hence we can set $v_i = v'_i$ $(1 \leqq i \leqq l-1)$ and choose v_l such that $dv_l = z$.

Using formulas (3.2) and (3.6), it is now a straightfoward formal computation to see that with

$$x_{p-k} = v_k \otimes (z_{i_{k+1}}|\ldots|z_{i_p}), \qquad 1 \leqq k \leqq p,$$

one has $d(x_p + x_{p-1} + \ldots + x_0) = 0$, hence (3.5) is proved.

4. The equality.

In this section we assume R is a ring for which equality holds in (0.2). It is trivial to observe that R cannot be regular, i.e. that $IH(K) \neq 0$, since supposing the contrary we get the absurd $I_R(t) = -t^{n+1}$.

The following result (valid without restrictions on R or E) is a particular case of [1, Theorem (3.1.1) and Proposition (5.1.1)]:

(4.1). THEOREM. *There exists a first-quadrant homological spectral sequence with*

$$E^2_{p,q} = \operatorname{Tor}^{H(K)}_{p,q}(k, H(K^E)) \Rightarrow \operatorname{Tor}^R_{p+q}(k, E) .$$

Moreover, let p denote the natural projection

$$H(K^E) \to H(K^E)/IH(K).H(K^E) = E^2_{0,*}$$

and let

$$e\colon E^2_{0,*} \to E^\infty_{0,*} \hookrightarrow \operatorname{Tor}^R_*(k, E)$$

be the edge homomorphism. Then the kernel of the composition $\sigma = ep$: $H(K^E) \to \operatorname{Tor}^R(k, E)$ *is the set of all elements of* $H(K^E)$, *decomposable in terms of matric Massey products.*

Writing $|W|(t,u)|_{p+q<n}$ for the polynomial $\sum_{p+q<n} \dim_k (W_{p,q}) t^p u^q$, a trivial majoration of the E^2-term of this spectral sequence, using the $H(K)$-free resolution provided by the reduced bar-construction, yields:

$$\sum_{i=0}^{n-1} \mu_i t^i \ll |E^2_{**}|(t,t)\big|_{p+q<n} \ll |H(K^E)\otimes \bar{B}(H(K))|(t,t)\big|_{p+q<n}$$

$$= \left.\frac{\sum_{i=0}^{n-1} c_{n-i} t^i - t^{n+1}}{1-\sum_{i=1}^{i=n} c_i t^{i+1}}\right|_{\text{degree } <n} .$$

From our hypothesis on $I_R(t)$ we see that in fact equality holds throughout, which in particular implies that in degrees different from n:

$$H_*(K^E) = E^2_{0,*} = E^3_{0,*} = \ldots = E^\infty_{0,*} .$$

Since $E^2_{0,*} = \operatorname{Coker}(H(K^E)\otimes IH(K) \to H(K^E))$, we conclude from the first equality that:

$$H_i(K^E).H_j(K) = 0 \quad \text{for } i+j \neq n . \tag{4.2}$$

Together with the relation $E^2_{0,*} = E^\infty_{0,*}$, this furthermore implies that $(\operatorname{Ker}\sigma)_i = 0$ for $i \neq n$, hence by (4.1) above we conclude:

(4.3). If the Massey product of $f \in H(K^E)$ and $h_1,\ldots,h_m \in IH(K)$ is defined and $\deg\langle f, h_1,\ldots,h_m\rangle \neq n$, then $\langle f, h_1,\ldots,h_m\rangle = \{0\}$.

We shall now show that every Massey product $\langle h_1,\ldots,h_m\rangle$ is defined and contains only zero. From (4.2) and (1.5) we can assume by induction the statement proved for values smaller than p ($p \geqq 2$). In particular, $\langle h_1,\ldots,h_p\rangle$ is a strictly defined product in the sense of [7, (1.2) and (1.3)]. Taking $h \in \langle h_1,\ldots,h_p\rangle$, we want to show $h=0$, and clearly there is a problem only when $\deg h = i \leqq n-d$ ($d = \operatorname{depth} R < n$). Assuming $h \neq 0$ by (1.3) there exists an $f \in H_j(K^E)$, $j=n-i$, such that $fh \neq 0$. We now note that the Massey products $\langle f, h_1,\ldots,h_m\rangle$ are defined and contain only zero for $1 \leqq m \leqq p-1$. For $m=1$ this is implied by (4.2), hence $\langle f, h_1,\ldots,h_m\rangle$ is strictly defined by induction, and contains only zero by (4.3): indeed,

$$\deg\langle f, h_1,\ldots,h_m\rangle \;=\; n-i+\sum_{i=1}^{m} \deg h_i + m-1 \;<\; n-i+\deg\langle h_1,\ldots,h_p\rangle \;=\; n\,.$$

We are now under the hypotheses of [7, (3.2iii)], hence:

$$fh \in f\langle h_1,\ldots,h_p\rangle \;=\; \langle \bar f, \bar h_1,\ldots,\bar h_{p-1}\rangle h_p \;=\; 0\,,$$

which gives the required contradiction.

We have now proved that $\langle h_1,\ldots,h_p\rangle$ is defined (and contains only zero) for every set of homogeneous elements of $IH(K)$, hence R is Golod. This completes the proof of the theorem.

REMARK (4.4). There is an obvious similarity between this argument and the one at the end of the preceding section. One would not be surprosed after noticing that the spectral sequence used there coincides from E^2 on with the one in (4.1).

5. An application and three remarks.

We start with the application.Let

$$S \;=\; k[\{X_{ij}\}_{\substack{1 \leqq i \leqq r \\ 1 \leqq j \leqq s}}]$$

be the polynomial ring in rs indeterminates over the field k, and let $I_r(X)$ be the ideal generated by the maximal mimors of the $r \times s$ matrix $X = (X_{ij})$ $(2 \leqq r \leqq s)$. Denote by T the ring $S/I_r(X)$. It is well known (e.g. [5]) that T is Cohen–Macaulay (of dimension $(r-1)(s+1)$), hence it has a canonical module Ω (isomorphic to $\operatorname{Ext}_S^{s-r+1}(T,S)$). Denote by $b_i(\Omega)$ the rank of the ith free module in a minimal graded T-free resolution of the graded T-module Ω, and let $P^{\Omega}{}_T(t) = \sum b_i(\Omega) t^i \in \mathbf{Z}[\![t]\!]$ denote the Poincaré series of Ω over T.

COROLLARY (5.1). *In the preceding notation, there is equality*:

$$P_R^{\Omega}(t) = \frac{\sum_{i=0}^{s-r} \binom{s-i-1}{r-1}\binom{s}{s-i} t^i - t^{s-r+2}}{1 - \sum_{i=1}^{s-r+1} \binom{i+r-2}{r-1}\binom{s}{i+r-1} t^{i+1}} .$$

PROOF. The linear forms $\{X_{ij}\}$ for $1 \leqq i \leqq r$, $1 \leqq j \leqq s$, $1 \leqq i-j \leqq r-1$ or $s-r+1 \leqq j-i \leqq s-1$, and $\{X_{i,i+k} - X_{i+1,i+k+1}\}$ for $1 \leqq i \leqq s-r+1$, $0 \leqq k \leqq r-1$ form in any order a T-regular sequence of length $(r-1)(s+1)$ [5, (3.9)], hence also an Ω-regular sequence. Denoting by bars the corresponding factor-objects, one has by standard change of rings the equality $P_R^{\Omega}(t) = P_{\bar{R}}^{\bar{\Omega}}(t)$. More or less by the definition of the canonical module, $\bar{\Omega} = E_{\bar{R}}(k)$, hence $P_{\bar{R}}^{\bar{\Omega}}(t) = I_{\bar{R}}(t)$ by (1.1). Now [5, loc. cit.] shows that

$$\bar{R} \cong k[Y_1, \ldots, Y_{s-r+1}]/(Y_1, \ldots, Y_{s-r+1})^r ,$$

and our formula follows from the theorem and the fact that $\bar{R}$ is "the" Golod ring given in the example in [6] with

$$c_i = \binom{i+r-2}{r-1}\binom{s}{i+r-1} .$$

REMARKS (5.2). (i) Bass introduced in [3] the numbers μ_i in order to characterize in terms of these invariants the class of Gorenstein local rings: part of the main result of his paper shows that R is Gorenstein if and only if $I_R(t)$ is a polynomial, in which case $I_R(t) = t^{\dim R}$. Another case, in which the Bass series is known, is given by the rings satisfying $\operatorname{edim} R - \operatorname{depth} R \leqq 2$ [10]. Since this condition implies that R is either a complete intersection (hence Gorenstein) or is Golod, Wiebe's result is contained in our theorem.

(ii) The inequality (0.2) implies in particular that $I_R(t)$ represents the development around the origin of an analytic function whose convergence radius r satisfies $0 < r \leqq 1$ in the non-Gorenstein case. However, this information can also be established directly.

(iii) Another consequence of our result is that Golod rings have rational Bass series, a fact which has already been established by Roos [9] who uses different methods, and does not exhibit an explicit formula. This information has lately been shown to be non-trivial by Bøgvad [4], who has been able, by working with the recent examples of Anick and of Löfwall–Roos of rings with transcendental Poincaré series, to exhibit an artinian ring with transcendental $I_R(t)$.

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